VOLUME 10, Nº 04
2008

Cubo
A Mathematical Journal

Special Issue: Fixed Point Theory

Guest Editor:
Donal O'Regan

Universidad de La Frontera
Facultad de Ingeniería, Ciencias y Administración
Departamento de Matemática y Estadística
Temuco - Chile

Universidade Federal de Pernambuco
Centro de Ciências Exatas e da Natureza
Departamento de Matemática
Recife - Brazil

CUBO

A Mathematical Journal

EDITORIAL BOARD

The Department of Mathematics of the Universidad de La Frontera Temuco-Chile and the Department of Mathematics of the Universidade Federal de Pernambuco, Recife-Brazil, edit **CUBO,** A Mathematical Journal. **CUBO** appears in three issues per year and is indexed in ZentralBlatt Math., Mathematical Reviews, MathSciNet and Latin Index. **CUBO** publishes original research papers, preferably not longer than 20 printed pages, that contain substantial mathematical results in all branches of pure and applied mathematics.

CUBO (I.S.S.N. 0716.7776) Editorial Office : Departamento de Matemática, Universidade Federal de Pernambuco, CEP 50740-540, Recife-PE., Brazil, cch@dmat.ufpe.br.

CUBO
A Mathematical Journal

EDITORIAL BOARD

Agarwal R.P.
agarwal@fit.edu
Department of Mathematical Sciences
Florida Institute of Technology
Melbourne Florida 32901-6975x – USA

Amann Herbert
herbert.amann@math.unizh.ch
Institute for Mathematics University Zurich
Winterhuerstr 190. CH 8057
Zurich – Switzerland (Suiza)

Ambrosetti Antonio
ambr@sissa.dot.it
Sissa, Via Beirut 2-4
34014 Trieste – Italy

Anastassiou George A.
ganastss@memphis.edu
Department of Mathematical Science
University of Memphis
Memphis TN 38152 – USA

Arad Tzvi
aradtzvi@math.biu.ac.il
Bar-Ilan University
Department of Mathematics & Computer Science
Ramat Gan. 52900 – Israel

Avramov Luchezar
avramov@math.unl.edu
Department of Mathematics
University of Nebraska
Lincoln NE 68588-0323 – USA

Benguria Rafael
rbenguri@fis.puc.cl
Facultad de Física
Pontificia Universidad Católica de Chile
Casilla 306. Santiago – Chile

Berger Marcel
berger@ihes.fr
Institut Des Hautes Etudes Scientifiques (IHES)
35 Route de Chartres
91440 Bures Sur Yvette – France

Bollobás Béla
bollobas@msci.menphis.edu
Department of Mathematical Science
University of Memphis
Memphis TN 38152 – USA

Burton T.A.
taburton@olypen.com
Northwest Research Institute
732 Caroline ST
Port Angeles, WA 98362 – USA

Cardoso Fernando
fernando@dmat.ufpe.br
Departamento de Matemática
Universidade Federal de Pernambuco
Av. Prof. Luiz Freire, s/n, Cid. Universitária
Recife-PE, CEP. 50.540-740 – Brazil

Carlsson Gunnar
gunnar@math.stanford.edu
Department of Mathematics
Stanford University
Stanford. CA 94305-2125 – USA

Cuevas Claudio
cch@dmat.ufpe.br
Departamento de Matemática
Universidade Federal de Pernambuco
Av. Prof. Luiz Freire, s/n, Cid. Universitária
Recife-PE, CEP. 50.540-740 – Brazil

Eckmann Jean Pierre
Jean-pierre.eckmann@physies.unige.ch
Departément de Physique Théorique
Université de Genève
1211 Genève 4 – Switzerland

Elaydi Saber
selaydi@trinity.edu
Department of Mathematics
Trinity University, San Antonio
TX 78212-7200 – USA

Esnault Hélène
esnault@uni.due.de
Mathematick, Universität Essen
FB6 Mathematik 45117
ASSEN – Germany

Fomin Sergey
fomin@umich.edu
Department of Mathematics
University of Michigan
525 East University Ave. Ann Arbor
MI48109 - 1109 – USA

Hale Jack K.
hale@math.gatech.edu
Center for Dynamical Systems and
Nonlinear Studies
Georgia Institute of Technology
Atlanta, Georgia 30332 – USA

Kalai Gil
kalai@math.huji.ac.il
The Hebrew University of Jerusalem
Einstein Institute of Mathematics
Givat ram Jerusalem 91904 – Israel

Kenig Carlos
cek@math.uchicago.edu
Department of Mathematics
University of Chicago
Chicago Illinois 60637 – USA

Kohn Joseph J.
kohn@math.princeton.edu
Mathematics Department, Princeton University
Fine Hall, Washington Road
Princeton, NJ 08544-1000 – USA

Kurylev Yaroslav
y.v.kurylev@lboro.ac.uk
Department of Mathematical Sciences,
Loughborough University. Loughborough Leics
LE 11 3 TU - UK – England

Ladas Gerry
gladas@math.uri.edu
Department of Mathematics
University of Rhode Island
Kingston RI 02881-0816 – USA

Li Peter
pli@math.uci.edu
Department of Mathematics
University of California, Irvine
Irvine CA 92697-3875 – USA

Nagel Rainer
rana@fa.uni-tuebinger.de
Arbeitsbereich Funktionalanalysis,
Mathematisches Institut
AUF DER Morgenstelle 10
D-72076. Tuebingen – Germany

Nirenberg Louis
nirenberg@cims.nyu.edu
Department of Mathematics
Courant Institute of Mathematics Sciences
New York University – USA

Petkov Vesselin
vasselin.petkov@math.u-bordeaux1.fr
Departément de Mathématiques Appliquées
Université Bordeaux I
351 Cours de la Libération 33405
Talence – France

Pinto Manuel
pintoj@uchile.cl
Departamento de Matemática
Facultad de Ciencias, Universidad de Chile
Casilla 653. Santiago – Chile

Ramm Alexander G.
ramm@math.ksu.edu
Mathematics Department,
Kansas State University
Manhattan KS 66506-2602 – USA

Rebolledo Rolando
rrebolle@mat.puc.cl
Departamento de Matemática
Pontificia Universidad Católica de Chile
Santiago - Chile

Robert Didier
didier.robert@univ-nantes.fr
Université de Nantes,
Departément de Mathématiques
UMR 6629 du CNRS,2,
rue de la Houssiniére BP 92208
44072 Nantes Cedex 03 – France

Sá Barreto Antonio
sabarre@math.purdue.edu
Department of Mathematics, Purdue University
West Lafayette, IN 47907-1395 – USA

Shub Michael
michael.shub@utoronto.ca
Mathematics Department, University of Toronto
40 st George Street, Toronto
ON, M5S2E4 – Canada

Simis Aron
aron@dmat.ufpe.br
Departamento de Matemática
Universidade Federal de Pernambuco
Av. Prof. Luiz Freire, s/n, Cid. Universitária
Recife-PE, CEP. 50.540-740 – Brazil

Sjöstrand Johannes
johannes@math.polytechnique.fr
Centre de Mathématiques, École Polytechnique
F-91128, Palaiseau Cedex – France

Stanley Richard
rstan@math.mit.edu
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA. 02139-4307 – USA

Tian Gang
tian@math.mit.edu
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA. 02139-4307 – USA

Uhlmann Gunther
gunther@math.washington.edu
Department of Mathematics,
University of Washington
Box 354350 Seattle WA 98195 – USA

Vainsencher Israel
israel@mat.ufmg.br
Departamento de Matemática
Univ. Federal de Minas Gerais
Av. Antonio Carlos 6627 Caixa Postal 702
CEP 30.123-970, Belo Horizonte, MG – Brazil

CUBO
A MATHEMATICA JOURNAL
Universidad de La Frontera
Universidade Federal de Pernambuco
Volume 10/N$^{\underline{o}}$04 – DECEMBER 2008

Preface

Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. During this time various scientists from different disciplines became interested in fixed point theory and in particular in their application to their own disciplines. Moreover fixed point techniques have been applied in such diverse fields as Biology, Chemistry, Computer Science, Economics and Engineering.

This special issue titled “Fixed Point Theory” contains 11 research articles from mathematicians from all over the world. The central theme throughout this special issue concerns algebraic, geometrical and topological methods in the study of fixed point theory in the applied sciences. The papers selected for this special issue were chosen accordingly. No attempt was made to cover every area in this vast field. Our objective was merely to offer the mathematical community a selection of topics which are of current interest.

We wish to express our appreciation to all the contributors. Without their cooperation this special issue would not have been possible. I would like to dedicate this special issue to Professor Ravi P. Agarwal on the occasion of his 60th birthday.

Donal O’Regan.

CUBO
A MATHEMATICA JOURNAL
Universidad de La Frontera
Universidade Federal de Pernambuco
Volume 10/N$^{\underline{o}}$04 – DECEMBER 2008

SUMMARY

CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$04, (119–135). December 2008

Fixed Point Results for Set-Valued and Single-Valued Mappings in Ordered Spaces

SEPPO HEIKKILÄ
Department of Mathematical Sciences,
University of Oulu,
BOX 3000, FIN-90014 University of Oulu, Finland
email: `sheikki@cc.oulu.fi`

ABSTRACT

In this article we use a recursion principle and generalized iteration methods to prove existence and comparison results for fixed points of set- and single-valued mappings in ordered spaces.

RESUMEN

En este artículo usamos el principio de recurrencia y métodos de iteración generalizados para provar resultados de exitencia y comparación para puntos fijos de aplicaciones conjunto (uni-)valoradas en espacios ordenados.

Key words and phrases: *Poset, set-valued mapping, fixed point, solution, maximal, minimal, sup-center, inf-center, recursion principle, generalized iteration methods, order compact, chain complete.*

Math. Subj. Class.: *47H04, 47H07, 47H10.*

1 Introduction

Let P be a nonempty partially ordered set (poset). As an introductory result we show that a set-valued mapping $\mathcal{F}$ from P to the set $2^P \setminus \emptyset$ of nonempty subsets of P has minimal and maximal fixed points, that is, the set Fix $\mathcal{F} = \{x \in P \mid x \in \mathcal{F}(x)\}$ has minimal and maximal elements, if the following conditions hold.

(c1) $\sup\{c, y\} \in P$ for some $c \in P$ and for every $y \in P$.

(c2) If $x \leq y$ in P, then for every $z \in \mathcal{F}(x)$ there exists a $w \in \mathcal{F}(y)$ such that $z \leq w$, and for every $w \in \mathcal{F}(y)$ there exists a $z \in \mathcal{F}(x)$ such that $z \leq w$.

(c3) Strictly monotone sequences of $\mathcal{F}[P] = \bigcup\{\mathcal{F}(x) : \ x \in P\}$ are finite.

As for the proof, denote $x_0 = c$, and choose y_0 from $\mathcal{F}(x_0)$. If $y_0 \not\leq x_0$, then $x_0 < x_1 := \sup\{c, y_0\}$. Apply then condition (c2) to choose y_1 from $\mathcal{F}(x_1)$ such that $y_0 \leq y_1$. If $y_0 = y_1$, then stop. Otherwise, $y_0 < y_1$, whence $x_1 = \sup\{c, y_0\} \leq x_2 := \sup\{c, y_1\}$, and apply again condition (c2) to choose y_2 from $\mathcal{F}(x_2)$ such that $y_1 \leq y_2$. Continuing in a similar way, condition (c3) ensures that after a finite number of choices we get the situation, where $y_{n-1} = y_n \in \mathcal{F}(x_n)$. In view of the above construction we then have $x_n := \sup\{c, y_{n-1}\} = \sup\{c, y_n\}$.

Denoting $z_0 := x_n$ and $w_0 := y_n$ then $w_0 \in \mathcal{F}(z_0)$ and $w_0 \leq \sup\{c, w_0\} = z_0$. If $w_0 = z_0$, then z_0 is a fixed point of $\mathcal{F}$. Otherwise, denoting $z_1 := w_0$, we have $z_1 < z_0$. In view of condition (c2) there exists a $w_1 \in \mathcal{F}(z_1)$ such that $w_1 \leq w_0$. If equality holds, then $z_1 = w_0 = w_1 \in \mathcal{F}(z_1)$, so that z_1 is a fixed point of $\mathcal{F}$. Otherwise, $w_1 < w_0$, denote $z_2 := w_1$, and choose by (c2) such a $w_2 \in \mathcal{F}(z_2)$ that $w_2 \leq w_1$, and so on. Condition (c3) implies that a finite number of steps yields the situation $z_m := w_{m-1} = w_m \in \mathcal{F}(z_m)$. Thus z_m belongs to Fix $\mathcal{F}$. Being a subset of $\mathcal{F}[P]$, strictly monotone sequences of Fix $\mathcal{F}$ are finite by condition (c3). This property implies in turn that Fix $\mathcal{F}$ has minimal and maximal elements.

The above described result will be generalized in Section 3. For instance, we show that $\mathcal{F}$ has minimal and maximal fixed points when the above condition (c1) holds, condition (c2) is replaced by a stronger monotonicity condition, and (c3) is replaced by a chain completeness of the order closure of the range $\mathcal{F}[P]$. Applications to single-valued mappings are also given. Fixed points of a concrete mapping are approximated by using an algorithmic method developed from the above described reasoning.

The obtained results are used in Section 4 to derive fixed point results in ordered normed spaces and in ordered topological spaces. Existence proofs require several consecutive applications of a recursion principle and generalized iteration methods introduced in [4, 6] and presented in Section 2.

2 Recursions and iterations in posets

Given a nonempty set P, a relation $x < y$ in $P \times P$ is called a *partial ordering*, if $x < y$ implies $y \not< x$, and if $x < y$ and $y < z$ imply $x < z$. Defining $x \le y$ if and only if $x < y$ or $x = y$, we say that $P = (P, \le)$ is a partially ordered set (poset).

An element b of a poset P is called an *upper bound* of a subset A of P if $x \le b$ for each $x \in A$. If $b \in A$, we say that b is the *greatest element* of A, and denote $b = \max A$. A lower bound of A and the least element, $\min A$, of A are defined similarly, replacing $x \le b$ above by $b \le x$. If the set of all upper bounds of A has the least element, we call it a *supremum of* A and denote it by $\sup A$. We say that y is a *maximal element* of A if $y \in A$, and if $z \in A$ and $y \le z$ imply that $y = z$. An infimum of A, $\inf A$, and a minimal element of A are defined similarly. We say that a poset P is a *lattice* if $\inf\{x, y\}$ and $\sup\{x, y\}$ exist for all $x, y \in P$. W is called a *chain* if $x \le y$ or $y \le x$ for all $x, y \in W$. We say that W is *well-ordered* if nonempty subsets of W have least elements, and *inversely well-ordered* if nonempty subsets of W have greatest elements. In both cases W is a chain.

A basis to our considerations is the following recursion principle (cf. [6], Lemma 1.1.1).

Lemma 2.1. *Given a nonempty poset P, a subset $\mathcal{D}$ of $2^P = \{A : A \subseteq P\}$ with $\emptyset \in \mathcal{D}$ and a mapping $f : \mathcal{D} \to P$, there is a unique well-ordered chain C in P such that*

$$x \in C \quad \text{if and only if } x = f(C^{<x}), \text{ where } \; C^{<x} = \{y \in C : y < x\}. \tag{2.1}$$

If $C \in \mathcal{D}$, then $f(C)$ is not a strict upper bound of C.

As an application of Lemma 2.1 we get the following result (cf. [4], Lemma 2).

Lemma 2.2. *Given $G : P \to P$ and $c \in P$, there exists a unique well-ordered chain $C = C(G)$ in P, called* a w-o chain of cG-iterations, *satisfying*

$$x \in C \quad \text{if and only if} \quad x = \sup\{c, G[C^{<x}]\}. \tag{2.2}$$

Proof. Denote $\mathcal{D} = \{W \subseteq P : W \text{ is well-ordered and } \sup\{c, G[W]\} \text{ exists}\}$. Defining $f(W) = \sup\{c, G[W]\}$, $W \in \mathcal{D}$, we get a mapping $f : \mathcal{D} \to P$, and (2.1) is reduced to (2.2). Thus the assertion follows from Lemma 2.1. □

A subset W of a chain C is called an *initial segment* of C if $x \in W$ and $y < x$ imply $y \in W$. The following result is also used in the sequel.

Lemma 2.3. *Given $\mathcal{F} : P \to 2^P \setminus \emptyset$, denote by $\mathcal{G}$ the set of all selections from $\mathcal{F}$, i.e.,*

$$\mathcal{G} := \{G : P \to P : G(x) \in \mathcal{F}(x) \text{ for all } x \in P\}. \tag{2.3}$$

For every $G : P \to P$ denote by C_G the longest such an initial segment of the w-o chain $C(G)$ of cG-iterations that the restriction $G|C_G$ of G to C_G is increasing (i.e., $G(x) \le G(y)$ whenever $x \le y$ in C_G). Define a partial ordering $\prec$ on $\mathcal{G}$ as follows.

(O) $F \prec G$ if and only if C_F is a proper initial segment of C_G and $G|C_F = F|C_F$.

Then $(\mathcal{G}, \preceq)$ has a maximal element.

Proof. Let $\mathcal{C}$ be a chain in $\mathcal{G}$. The definition (O) of $\prec$ implies that the sets C_F, $F \in \mathcal{C}$, form a nested family of well-ordered sets of P. Thus the set $C := \cup\{C_F : F \in \mathcal{C}\}$ is well-ordered. Moreover, it follows from (O) that the functions $F|C_F$, $F \in \mathcal{C}$, considered as relations in $P \times P$, are nested. This ensures that $g := \cup\{F|C_F : F \in \mathcal{C}\}$ is a function from C to P. Since each $F \in \mathcal{C}$ is increasing in C_F, then g is increasing, and $g(x) \in \mathcal{F}(x)$ for each $x \in C$. Let G be such a selection from $\mathcal{F}$ that $G|C = g$. Then $G \in \mathcal{G}$, and G is increasing on C. If $x \in C$, then $x \in C_F$ for some $F \in \mathcal{C}$. The definitions of C and the partial ordering $\prec$ imply that C_F is C or its initial segment, whence $C_F^{<x} = C^{<x}$. Because $F|C_F = g|C_F = G|C_F$, then

$$x = \sup\{c, F[C_F^{<x}]\} = \sup\{c, G[C^{<x}]\}. \tag{2.4}$$

This result implies (cf. the proof of Lemma 2.1) that C is $C(G)$ or its proper initial segment. Since G is increasing on C, then C is C_G or its proper initial segment. Consequently, G is an upper bound of $\mathcal{C}$ in $\mathcal{G}$. This result implies by Zorn's Lemma that $\mathcal{G}$ has a maximal element. □

Let $X = (X, \leq)$ be a poset. When z, $w \in X$, denote

$$[z) = \{x \in X : z \leq x\}, \ (w] = \{x \in X : x \leq w\} \text{ and } [z, w] = [z) \cap (w].$$

A subset A of X is called a *causal set* (causet) if $[z, w] \cap A$ is finite for all z, $w \in A$. We say that A is *bounded from above* if $A \subseteq (z]$, *bounded from below* if $A \subseteq [z)$, and *order bounded* if $A \subseteq [z, w]$ for some z, $w \in X$.

We say that X, equipped with a topology is an *ordered topological space* if the order intervals $[z)$ and $(z]$ are closed for each $z \in X$. If the topology of X is induced by a metric, we say that X is an *ordered metric space.*

Next we define some concepts for sequences and set-valued functions.

Definition 2.1. A sequence $(z_n)_{n=0}^{\infty}$ of a poset is called *increasing* if $z_n \leq z_m$ whenever $n \leq m$, *decreasing* if $z_m \leq z_n$ whenever $n \leq m$, and *monotone* if it is increasing or decreasing. If the above inequalities are strict, the sequence $(z_n)_{n=0}^{\infty}$ is called strictly increasing, strictly decreasing or strictly monotone, respectively.

Definition 2.2. Given posets X and P, we say $\mathcal{F} : X \to 2^P \setminus \emptyset$ is *increasing upward* if $x \leq y$ in X and $z \in \mathcal{F}(x)$ imply that $[z) \cap \mathcal{F}(y)$ is nonempty. $\mathcal{F}$ is *increasing downward* if $x \leq y$ in X and $w \in \mathcal{F}(y)$ imply that $(w] \cap \mathcal{F}(x)$ is nonempty. If $\mathcal{F}$ is increasing upward and downward, we say that $\mathcal{F}$ is *increasing.*

Definition 2.3. Given posets P and X and a set-valued function $\mathcal{F} : X \to 2^P \setminus \emptyset$, consider chains of the form $G[C]$, where C is a nonempty chain in X and G is an increasing selection from $\mathcal{F}|C$. $\mathcal{F}$

is called *chain complete upward* if such chains $G[C]$ have supremums whenever C is well-ordered, *chain complete downward* if such chains $G[C]$ have infimums whenever C is inversely well-ordered, and *chain complete* if both these conditions hold. If every such a chain $G[C]$ has an upper bound in $\mathcal{F}(x)$ whenever x is an upper bound of C in X, we say that $\mathcal{F}$ is called *strongly increasing upward.* If this condition holds with upper bounds replaced by lower bounds, we say that $\mathcal{F}$ is *strongly increasing downward.* If both these conditions hold, then $\mathcal{F}$ is said to be *strongly increasing.*

The following Corollary is an easy consequence of Definitions 2.2 and 2.3.

Corollary 2.1. *(a) $\mathcal{F} : X \to 2^P \setminus \emptyset$ is chain complete upward (respectively downward) if every nonempty chain of $\mathcal{F}[X]$ has a supremum (respectively an infimum).*
(b) If $\mathcal{F}$ is strongly increasing upward (respectively downward), then it is increasing upward (respectively downward).
(c) If $\mathcal{F}$ is increasing upward (respectively downward), and if $\max \mathcal{F}(x)$ (respectively $\min \mathcal{F}(x)$) exists for every $x \in X$, then $\mathcal{F}$ is strongly increasing upward (respectively downward).
(d) An increasing mapping $\mathcal{F}$ is strongly increasing, if chains of X are causets, or if chains of $\mathcal{F}[X]$ are finite, or if the values of $\mathcal{F}$ are finite sets.

In the case when P is an ordered topological space we have the following results.

Lemma 2.4. *Let X be a poset, P an ordered topological space, and $\mathcal{F} : X \to 2^P \setminus \emptyset$.*
(a) If $\mathcal{F}$ is increasing upward (respectively downward), and if its values are compact, then $\mathcal{F}$ is strongly increasing upward (respectively downward).
(b) Assume that (y_n) converges whenever $y_n \in \mathcal{F}(x_n)$ for every n and both (x_n) and (y_n) are increasing (respectively decreasing). If P is second countable or metrizable, then $\mathcal{F}$ is chain complete upward (respectively downward).

Proof. Consider a chain $W = G[C]$, where C is a chain in X and G is an increasing selection from $\mathcal{F}|C$.

(a) Assume that $x \in X$ is an upper bound of C. If $\mathcal{F}$ is increasing upward, then to every $y \in W$ there corresponds a $z \in [y) \cap \mathcal{F}(x)$. Because W is a chain, then the sets $[y) \cap \mathcal{F}(x)$, $y \in W$, satisfy the finite intersection property. Thus their intersection is nonempty if $\mathcal{F}(x)$ is compact, and every element from that intersection is an upper bound of W in $\mathcal{F}(x)$. Similarly one can prove that if $x \in X$ is a lower bound of C, if $\mathcal{F}$ is increasing downward, and if $\mathcal{F}(x)$ is compact, then $\mathcal{F}(x)$ contains a lower bound of W.

(b) If C is well-ordered, and (y_n) is an increasing sequence of $W = G[C]$, then $y_n = G(x_n)$, where $x_n = \min\{x \in C : G(x) = y_n\}$ for every n, and (x_n) is increasing. Thus (y_n) converges by a hypothesis of (b). If P is second countable, then every subset of W is separable. It then follows from [6], Lemma 1.1.7 that W contains an increasing sequence which converges to $\sup W$. This result follows from [6], Proposition 1.1.5 if P is an ordered metric space. These results and their duals imply the conclusions of (b). $\square$

3 Fixed point results in posets

In this section we prove existence and comparison results for fixed points of a set-valued and single-valued functions in a poset P.

3.1 Fixed point results for set-valued functions

As an application of Lemma 2.1 we obtain the following result.

Proposition 3.1. *Assume that $\mathcal{F} : P \to 2^P \setminus \emptyset$ is strongly increasing upward and chain complete upward, and that the set $S_+ = \{x \in P : [x) \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty. Then $\mathcal{F}$ has a maximal fixed point, which is also a maximal element of S_+.*

Proof. Denote $\mathcal{D} = \{W \subset S_+ : W$ is well-ordered and has a strict upper bound in $S_+\}$. Because S_+ is nonempty, then $\emptyset \in \mathcal{D}$. Let $f : \mathcal{D} \to P$ be a function which assigns to each $W \in \mathcal{D}$ an element $y = f(W) \in [x) \cap \mathcal{F}(x)$, where x is a fixed strict upper bound of W in S_+. By Lemma 2.1 there exists exactly one well ordered chain W in P satisfying (2.1). By the above construction and (2.1) each element y of W belongs to $[x) \cap \mathcal{F}(x)$, where x is a fixed strict upper bound of $W^{<y}$ in S_+. Because $\mathcal{F}$ is increasing upward and $x \leq y \in \mathcal{F}(x)$, then $[y) \cap \mathcal{F}(y) \neq \emptyset$, so that $y \in S_+$. It is easy to verify that the set C of these elements x form a well ordered chain in S_+, that the correspondence $x \mapsto y$ defines an increasing selection $G : C \to S_+$ from $\mathcal{F}|C$, and that $W = G[C]$. Because $\mathcal{F}$ is chain complete upward, then $x = \sup W$ exists in P. The above construction implies that x is also an upper bound of C. Since $\mathcal{F}$ is strongly increasing upward, then W has an upper bound y in $\mathcal{F}(x)$. Because $x = \sup W$, then $x \leq y$, so that $y \in [x) \cap \mathcal{F}(x)$, and thus $x = \sup W$ belongs to S_+. $x = \max W$, for otherwise $f(W)$ would exist, and being a strict upper bound of W, would contradict the last conclusion of Lemma 2.1. By the same reason x is a maximal element of S_+.

Because $x \leq y \in F(x)$, then $[y) \cap \mathcal{F}(y) \neq \emptyset$, or equivalently, $y \in S_+$, since $\mathcal{F}$ is increasing upward. Because x is a maximal element of S_+, then $x = y \in \mathcal{F}(x)$, so that x is a fixed point of $\mathcal{F}$. If z is a fixed point of $\mathcal{F}$ and $x \leq z$, then $z \in S_+$, whence $x = z$. Thus x is a maximal fixed point of $\mathcal{F}$. □

The next result is the dual of Proposition 3.1.

Proposition 3.2. *Assume that $\mathcal{F} : P \to 2^P \setminus \emptyset$ is strongly increasing downward and chain complete downward, and that the set $S_- = \{x \in P : (x] \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty. Then $\mathcal{F}$ has a minimal fixed point, which is also a minimal element of S_-.*

If $\mathcal{F}[P]$ has an upper bound (respectively a lower bound) in P, it belongs to S_- (respectively to S_+).

Next we derive other conditions under which the set S_- or the set S_+ is nonempty.

Definition 3.1. Let A be a subset of a poset P. The set $ocl(A)$ of all possible supremums and infimums of chains of A is called an *order closure* of A. If $A = ocl(A)$, then A is *order closed.* We say that a subset A of poset P has a *sup-center* c in P if $c \in P$ and $\sup\{c, x\}$ exists in P for each $x \in A$. If $\inf\{c, x\}$ exists in P for each $x \in A$, we say that c is an *inf-center* of A in P. If c has both these properties it is called an *order center* of A in P. Phrase "in P" is omitted if $A = P$.

If P is an ordered topological space, then the order closure $ocl(A)$ of A is contained in the topological closure of A. If c is the greatest element (respectively the least element) of P, then c is an inf-center, (respectively a sup-center) of P. If P is a lattice, then its every point is an order center of P. If P is a subset of $\mathbb{R}^2$, ordered coordinatewise, a necessary and sufficient condition for a point $c = (c_1, c_2)$ of P to be a sup-center of a subset A of P in P is that whenever a point $y = (y_1, y_2)$ of A and c are unordered, then $(y_1, c_2) \in P$ if $y_2 < c_2$ and $(c_1, y_2) \in P$ if $y_1 < c_1$. No conditions are imposed on other points of A.

The following result is an application of Lemma 2.3.

Proposition 3.3. *Let $\mathcal{F} : P \to 2^P \setminus \emptyset$ be chain complete upward and strongly increasing upward. If $ocl(\mathcal{F}[P])$ has a sup-center in P, then the set $S_- = \{x \in P : (x] \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty.*

Proof. Let c be a sup-center of $ocl(\mathcal{F}[P])$ in P, let $\mathcal{G}$ be defined by (2.3), and let the partial ordering $\prec$ be defined by (O). By Lemma 2.3 $(\mathcal{G}, \preceq)$ has a maximal element G. Let $C(G)$ be the w-o chain of cG-iterations, and let $C = C_G$ be the longest initial segment of $C(G)$ on which G is increasing. Thus C is well-ordered and G is an increasing selection from $\mathcal{F}|C$. Since $\mathcal{F}$ is chain complete upward, then $w = \sup G[C]$ exists. Moreover, $\overline{x} = \sup\{c, w\}$ exists in P by the choice of c, and it is easy to see that $\overline{x} = \sup\{c, G[C]\}$. This result and (2.4) imply that for each $x \in C$,

$$x = \sup\{c, G[C^{<x}]\} \leq \sup\{c, G[C]\} = \overline{x}.$$

This proves that $\overline{x}$ is an upper bound of C. Since $\mathcal{F}$ is strongly increasing upward, then $W = G[C]$ has an upper bound z in $\mathcal{F}(\overline{x})$, and $w = \sup G[C] \leq z$. To show that $\overline{x} = \max C$, assume on the contrary that $\overline{x}$ is a strict upper bound of C. Let F be a selection from $\mathcal{F}$ whose restriction to $C \cup \{\overline{x}\}$ is $G|C \cup \{(\overline{x}, z)\}$. Since G is increasing on C and $F(x) = G(x) \leq w \leq z = F(\overline{x})$ for each $x \in C$, then F is increasing on $C \cup \{\overline{x}\}$. Moreover,

$$\overline{x} = \sup\{c, G[C]\} = \sup\{c, F[C]\} = \sup\{c, F[\{y \in C \cup \{\overline{x}\} : y < \overline{x}\}]\},$$

whence $C \cup \{\overline{x}\}$ is a subset of the longest initial segment C_F of the w-o chain of cF-iterations where F is increasing. Thus $C = C_G$ is a proper subset of C_F, and $F|C_G = F|C_F$. This means by (O) that $G \prec F$. But this is impossible because G is a maximal element of $(\mathcal{G}, \preceq)$. Consequently, $\overline{x} = \max C$. Since G is increasing on C, then $\overline{x} = \sup\{c, G[C]\} = \sup\{c, G(\overline{x})\}$. In particular, $\mathcal{F}(\overline{x}) \ni G(\overline{x}) \leq \overline{x}$, whence $G(\overline{x})$ belongs to the set $(\overline{x}] \cap \mathcal{F}(\overline{x})$. □

As a consequence of Propositions 3.1, 3.2 and 3.3 we obtain the following result.

Theorem 3.1. *Assume that* $\mathcal{F} : P \to 2^P \setminus \emptyset$ *is strongly increasing and chain complete. If* $ocl(\mathcal{F}[P])$ *has a sup-center or an inf-center in* P, *then* $\mathcal{F}$ *has minimal and maximal fixed points.*

Proof. We shall give the proof in the case when $ocl(\mathcal{F}[P])$ has a sup-center in P, the proof in the case of an inf-center being similar. The hypotheses of Proposition 3.3 are then valid, whence there exists a $\overline{x} \in P$ such that $(\overline{x}] \cap \mathcal{F}(\overline{x}) \neq \emptyset$. Thus the hypotheses of Proposition 3.2 hold, whence $\mathcal{F}$ has by Proposition 3.2 a minimal fixed point x_-. In particular $[x_-) \cap \mathcal{F}(x_-) \neq \emptyset$. The hypotheses of Proposition 3.1 are then valid, whence $\mathcal{F}$ has also a maximal fixed point. □

Example 3.1. Assume that $\mathbb{R}^m$ is ordered as follows. For all $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$,

$$x \le y \text{ if and only if } x_i \le y_i,\ i = 1, \ldots, j, \text{ and } x_i \ge y_i,\ i = j+1, \ldots, m, \tag{3.1}$$

where $j \in \{0, \ldots, m\}$. Show that if $\mathcal{F} : \mathbb{R}^m \to 2^{\mathbb{R}^m} \setminus \emptyset$ is increasing, and its values are closed subsets of $\mathbb{R}^m$, and if $\mathcal{F}[\mathbb{R}^m]$ is contained in

$$B_R^p(c) = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : \sum_{i=1}^m |x_i - c_i|^p \le R^p\}$$

for some p, $R \in (0, \infty)$ and $c = (c_1, \ldots, c_m) \in \mathbb{R}^m$, then $\mathcal{F}$ has minimal and maximal fixed points.

Solution. Let $x = (x_1, \ldots, x_m) \in B_R^p(c)$ be given. Since $|\max\{c_i, x_i\} - c_i| \le |x_i - c_i|$ and $|\min\{c_i, x_i\} - c_i| \le |x_i - c_i|$ for each $i = 1, \ldots, m$, it follows that $\sup\{c, x\}$ and $\inf\{c, x\}$ belong to $B_R^p(c)$ for all $x \in B_R^p(c)$. Moreover, every $B_R^p(c)$ is a closed and bounded subset of $\mathbb{R}^m$, whence its monotone sequences converge in $B_R^p(c)$ with respect to the Euclidean metric of $\mathbb{R}^m$. These results, Lemma 2.4 and the given hypotheses imply that $\mathcal{F}$ is chain complete, and strongly increasing, and that c is an order center of $ocl(\mathcal{F}[\mathbb{R}^m])$. Thus the hypotheses of Theorem 3.1 hold, whence $\mathcal{F}$ has minimal and maximal fixed points. □

3.2 Fixed point results for single-valued functions

Next we present existence and comparison results for fixed points of single-valued functions. In the proofs we use the following consequence of Proposition 3.3.

Proposition 3.4. *Assume that* $G : P \to P$ *is increasing, that* $ocl(G[P])$ *has a sup-center* c *in* P, *and that* $\sup G[C]$ *exists whenever* C *is a nonempty chain in* P. *If* C *is the w-o chain of* cG*-iterations, then* $\overline{x} = \max C$ *exists,* $\overline{x} = \sup\{c, G(\overline{x})\} = \sup\{c, G[C]\}$ *and*

$$\overline{x} = \min\{z \in P : \sup\{c, G(z)\} \le z\}. \tag{3.2}$$

Moreover, $\overline{x}$ *is the least solution of the equation* $x = \sup\{c, G(x)\}$ *and is increasing in* G.

Proof. When $\mathcal{F} : P \to 2^P \setminus \emptyset$ is single-valued, it coincides with its unique selection function $G : P \to P$. Moreover, $\mathcal{F}$ is strongly increasing upward if and only if G is increasing, in which case C in Lemma 2.3 is the w-o chain of cG-iterations. The hypotheses given for G imply also that $G = \mathcal{F}$ is chain complete upward, and that c is a sup-center of $ocl(\mathcal{F}[P])$ in P. As a single valued mapping it is also strongly increasing. Thus the proof of Proposition 3.3 implies that $\overline{x} = \max C$ exists and $\overline{x} = \sup\{c, G(\overline{x})\} = \sup\{c, G[C]\}$. To prove (3.2), let $z \in P$ satisfy $\sup\{c, G(z)\} \le z$. Then $c = \min C \le z$. If $x \in C$ and $\sup\{c, G(y)\} \le z$ for each $y \in C^{<x}$, then $x = \sup\{c, G[C^{<x}]\} \le z$. This implies by transfinite induction that $x \le z$ for each $x \in C$. In particular $\overline{x} = \max C \le z$. This result and the fact that $\overline{x} = \sup\{c, G(\overline{x})\}$ imply that $\overline{x} = x$ is the least solution of the equation $x = \sup\{c, G(x)\}$, and that (3.2) holds. The last assertion is an immediate consequence of (3.2). □

The results presented in the next proposition are dual to those of Lemma 2.2 and Proposition 3.4.

Proposition 3.5. *Given $G : P \to P$ and $c \in P$ there exists exactly one inversely well-ordered chain D in P, called an* inversely well-ordered (i.w-o) chain of cG- iterations, *satisfying*

$$x \in D \quad \textit{if and only if } x = \inf\{c, G[\{y \in C : x < y\}]\}. \tag{3.3}$$

Assume that G is increasing, that $ocl(G[P])$ has an inf-center c in P, and that $\inf G[C]$ exists whenever C is a nonempty chain in P. If D is the i.w-o chain of cG-iterations, then $\underline{x} = \min D$ exists, $\underline{x} = \inf\{c, G(\underline{x})\} = \inf\{c, G[D]\}$ and

$$\underline{x} = \max\{z \in P : z \le \inf\{c, G(z)\}\}. \tag{3.4}$$

Moreover, $\underline{x}$ is the greatest solution of the equation $x = \inf\{c, G(x)\}$ and is increasing in G.

Our first fixed point result is a consequence of Propositions 3.4 and 3.5.

Lemma 3.1. *Let P be a poset and $G : P \to P$ an increasing mapping.*
(a) If $P \ni \underline{x} \le G(\underline{x})$, and if $\sup G[C]$ exists whenever C is a chain in $[\underline{x})$, then the w-o chain C of $\underline{x}G$-iterations has a maximum x_* and

$$x_* = \max C = \sup G[C] = \min\{y \in [\underline{x}) : G(y) \le y\}. \tag{3.5}$$

Moreover, x_* is the least fixed point of G in $[\underline{x})$ and is increasing in G.
(b) If $G(\overline{x}) \le \overline{x} \in P$, and if $\inf G[C]$ exists whenever C is a chain $(\overline{x}]$, then the i.w-o chain D of $\overline{x}G$-iterations has a minimum x^* and

$$x^* = \min D = \inf G[D] = \max\{y \in (\overline{x}] : y \le G(y)\}. \tag{3.6}$$

Moreover, x^* is the greatest fixed point of G in $(\overline{x}]$ and is increasing in G.

Proof. (a) Since G is increasing and $\underline{x} \le G(\underline{x})$, then $G[[\underline{x})] \subset [\underline{x})$. Thus the conclusions of (a) are immediate consequences of the conclusion of Proposition 3.4 when $c = \underline{x}$ and G is replaced by its restriction to $[\underline{x})$.

The proof of (b) is dual to that of (a). □

As an application of Propositions 3.4 and 3.5 and Lemma 3.1 we get the following fixed point results.

Theorem 3.2. *Assume that $G : P \to P$ is increasing and that $c \in P$.*
(a) If c is a sup-center of $ocl(G[P])$ in P, and if $\sup G[C]$ and $\inf G[C]$ exist whenever C is a chain in P, then G has minimal and maximal fixed points. Moreover, G has the greatest fixed point x^* in $(\overline{x}]$, where $\overline{x}$ is the least solution of the equation $x = \sup\{c, G(x)\}$. Both $\overline{x}$ and x^* are increasing with respect to G.
(b) If c is an inf-center of $ocl(G[P])$ in P, and if $\sup G[C]$ and $\inf G[C]$ exist whenever C is a chain in P, then G has minimal and maximal fixed points. Moreover, G has the least fixed point x_* in $[\underline{x})$, where $\underline{x}$ is the greatest solution of the equation $x = \inf\{c, G(x)\}$. Both $\underline{x}$ and x_* are increasing with respect to G.

Lemma 3.1, Proposition 3.1 and Proposition 3.2 imply the following results.

Proposition 3.6. *Assume that $G : P \to P$ is increasing.*
(a) If $\sup G[C]$ exists whenever C is a well-ordered chain in P, and if $G[P]$ has a lower bound in P, then G has the least fixed point and a maximal fixed point.
(b) If $\sup G[D]$ exists whenever D is an inversely well-ordered chain in P, and if $G[P]$ has an upper bound in P, then G has the greatest and a minimal fixed point.

Example 3.2. Let $\mathbb{R}^m$ be ordered coordinatewise, and assume that $G : \mathbb{R}^m \to \mathbb{R}^m$ maps increasing sequences of $\mathbb{R}^m$ to bounded and increasing sequences of $\mathbb{R}^m_+$, where $\mathbb{R}_+$ is the set of nonnegative reals. Show that G has the least fixed point and a maximal fixed point.

Solution. Let C be a well-ordered chain in $\mathbb{R}^m$. Since G is increasing, by definition, then $G[C]$ is a well-ordered chain in $\mathbb{R}^m_+$. If (y_n) is an increasing sequence in $G[C]$, and $x_n = \min\{x \in C : G(x) = y_n\}$, then the sequence (x_n) is increasing and $y_n = G(x_n)$ for every n. Thus (y_n) is bounded, by definition of G, and hence converges with respect to the Euclidean metric of $\mathbb{R}^m$. This result implies by Lemma 2.4 that $\sup G[C]$ exists. Moreover, the origin is a lower bound of $G[\mathbb{R}^m]$. Thus the assertions follow from Proposition 3.6. □

3.3 Algorithmic methods

It can be shown that the first elements of the w-o chain C of cG-iterations are: $x_0 = c$, $x_{n+1} = \sup\{c, Gx_n\}$, $n = 0, 1, \ldots$, as long as x_{n+1} exists and $x_n < x_{n+1}$. Assuming that strictly monotone sequences of $G[P]$ are finite, then C is a finite strictly increasing sequence $(x_n)_{n=0}^m$. If $\sup\{c, x\}$ exists for every $x \in G[P]$, then $\overline{x} = \sup\{c, G[C]\} = \max C = x_m$ is the least solution of the equation $x = \sup\{c, G(x)\}$ by the proof of Proposition 3.4. In particular, $G\overline{x} \le \overline{x}$. If $G(\overline{x}) < \overline{x}$, then first elements of the i.w-o chain D of $\overline{x}G$-iterations of $\overline{x}$ are $y_0 = \overline{x} = x_m$, $y_{j+1} = Gy_j$, as long as $y_{j+1} < y_j$. Since strictly monotone sequences of $G[P]$ are finite, D is a finite strictly decreasing sequence $(y_j)_{j=0}^k$, and $x^* = \inf G[D] = y_k$ is the greatest fixed point of G in $(\overline{x}]$ by the proof of Lemma 3.1. This reasoning and its dual imply the following results.

Corollary 3.1. *Conclusions of Theorem 3.2 hold if* $G : P \to P$ *is increasing and strictly monotone sequences of* $G[P]$ *are finite, and if* $\sup\{c, x\}$ *and* $\inf\{c, x\}$ *exist for every* $x \in G[P]$. *Moreover,* x^* *is the last element of the finite sequence determined by the following algorithm:*

(i) $x_0 = c$. For n from 0 while $x_n \neq Gx_n$ do: $x_{n+1} = Gx_n$ if $Gx_n < x_n$ else $x_{n+1} = \sup\{c, Gx_n\}$,

and x_* *is the last element of the finite sequence determined by the following algorithm:*

(ii) $x_0 = c$. For n from 0 while $x_n \neq Gx_n$ do: $x_{n+1} = Gx_n$ if $Gx_n > x_n$ else $x_{n+1} = \inf\{c, Gx_n\}$.

Let $G : P \to P$ satisfy the hypotheses of Theorem 3.2. The result Corollary 3.1 can be applied to approximate the fixed points x^* and x_* of G introduced in Theorem 3.2 in the following manner. Assume that $\underline{G}$, $\overline{G} : P \to P$ satisfy the hypotheses given for G in Corollary 3.1, and that

$$\underline{G}(x) \leq G(x) \leq \overline{G}(x) \quad \text{for all} \quad x \in P. \tag{3.7}$$

Since x^* is increasing with respect to G, it follows from (3.7) that $\underline{x}^* \leq x^* \leq \overline{x}^*$, where $\underline{x}^*$ and $\overline{x}^*$ are obtained by algorithm (i) of Corollary 3.1 with G replaced by $\underline{G}$ and $\overline{G}$, respectively.

Since partial ordering is the only structure needed in the proofs, the above results can be applied to problems where only ordinal scales are available. On the other hand, these results have some practical value also in real analysis. We shall demonstrate this by an example where the above described method is applied to a system of the form

$$x_i = G_i(x_1, \dots, x_m), \quad i = 1, \dots, m, \tag{3.8}$$

where the functions G_i are real valued functions of m real variables.

Example 3.3. In this example we approximate a solution $x^* = (x_1, y_1)$ of the system

$$x = G_1(x, y) := \frac{N_1(x, y)}{2 - |N_1(x, y)|}, \; y = G_2(x, y) := \frac{N_2(x, y)}{3 - |N_2(x, y)|}, \tag{3.9}$$

where

$$N_1(x, y) = \frac{11}{12}x + \frac{12}{13}y + \frac{1}{234} \quad \text{and} \quad N_2(x, y) = \frac{15}{16}x + \frac{14}{15}y - \frac{7}{345}, \tag{3.10}$$

by calculating such upper and lower estimates of (x_1, y_1) whose corresponding coordinates differ less than 10^{-100}.

The mapping $G = (G_1, G_2)$, defined by (3.9), (3.10) maps the set $P = \{(x, y) \in \mathbb{R}^2 : |x| + |y| \leq \frac{1}{2}\}$ into P, and is increasing on P. It follows from Example 2.1 that $c = (0, 0)$ is an order center of P, and that P is chain complete. Thus the results of Theorem 3.2 are valid.

Upper and lower estimates to the fixed point $x^* = (x_1, y_1)$ of G, and hence to a solution (x_1, y_1) of system (3.9), (3.10), can be obtained by applying the algorithm (i) given in Corollary 3.1 to operators $\overline{G}$ and $\underline{G}$, defined by

$$\begin{cases} \overline{G}(x,y) = (10^{-101}\text{ceil}(10^{101}G_1(x,y)), 10^{-101}\text{ceil}(10^{101}G_2(x,y)), \\ \underline{G}(x,y) = (10^{-101}\text{floor}(10^{101}G_1(x,y)), 10^{-101}\text{floor}(10^{101}G_2(x,y)), \end{cases} \tag{3.11}$$

where ceil(x) is the least integer $\geq x$ and floor(x) is the greatest integer $\leq x$. The so defined operators $\underline{G}$, $\overline{G}$ are increasing and map the set $P = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq \frac{1}{2}\}$ into finite subsets of P, and (3.7) holds. We are going to show that the required upper and lower estimates are obtained by algorithm (i) of Corollary 3.1 with G replaced by $\underline{G}$ and $\overline{G}$, respectively. The following Maple program is used in calculations of the upper estimate $\overline{x}^* = (x1, y1)$.

```
(N1, N2):=(11/12*x+12/13*y+1/234,15/16*x+14/15*y-7/345):
(z,w):=(N1/(2-abs(N1)),N2/(3-abs(N2))):(G1,G2):=(ceil(10^101z)/10^101 ,ceil(10^101w)/10^101):
(x0,y0):=(0,0);x:=x0:y:=y0:u:=G1:v:=G2:b[0]:=[x,y]:
for k from 1 while abs(u-x)+abs(v-y)> 0 do:
if u<=x and v<=y then (x,y):=(u,v) else (x,y):=(max(x,u),max(y,v)):fi:
u:=G1:v:=G2:b[k]:=[x,y]:od:n:=k-1: x1:=x;y1=y;
```

The above program yields the following results (n=1246).

$$\begin{cases} x1 = -0.0077531868497808116549106930410370196194714313877471725495045699953 5\\ \hfill 6264082732785848367182252372500 43, \\ y1 : -0.013599615424610901489836719913129280024524254401289927375880599161 78\\ \hfill 38548683927620135569441397855721 \end{cases}$$

In particular, $(x1, y1)$ is the fixed point $\overline{x}^*$ of $\overline{G}$.

Replacing 'ceil' by 'floor' in the above program we obtain components of the fixed point $\underline{x}^* = (x2, y2)$ of $\underline{G}$ (n:=1248).

$$\begin{cases} x2 = -0.00775318684978081165491069304103701961947143138774717254950456999535\\ \hfill 626408273278584836718225237250 05, \\ y2 = -0.01359961542461090148983671991312928002452425440128992737588059916178\\ \hfill 385486839276201355694413978557215 \end{cases}$$

The above calculated components of $\overline{x}^*$ and $\underline{x}^*$ are exact, and their differences are $< 10^{-100}$. According to the above reasoning the exact fixed point x^* of G belongs order interval $[\underline{x}^*, \overline{x}^*]$. In particular, both $(x1, y1)$ and $(x2, y2)$ approximate an exact solution (x_1, y_1) of system (3.9), (3.10) with the required precision. Moreover, $x1 \leq x_1 \leq y1$ and $x2 \leq y_1 \leq y2$. $\square$

Remarks 3.1. The results of Lemma 3.1 and its dual and Corollary 3.1 could be combined to obtain upper and lower estimates also to fixed points of set-valued functions.

4 Special cases

In this section we shall first present existence and comparison results for equations and inclusions in ordered normed spaces. Next we formulate in ordered topological spaces some existence and comparison results derived in section 3.

4.1 Equations and inclusions in ordered normed spaces

Definition 4.1. A closed subset E_+ of a normed space E is called an *order cone* if $E_+ + E_+ \subseteq E_+$, $E_+ \cap (-E_+) = \{0\}$ and $cE_+ \subseteq E_+$ for each $c \geq 0$. The space E, equipped with an order relation '$\leq$', defined by

$$x \leq y \text{ if and only if } y - x \in E_+$$

is called an *ordered normed space.*

It is easy to see that the above defined order relation $\leq$ is a partial ordering in E.

Lemma 4.1. *Let C be a chain in an ordered normed space E, and assume that each monotone sequence of C has a weak limit in E. Then C contains an increasing sequence which converges weakly to* $\sup C$ *and a decreasing sequence which converges weakly to* $\inf C$*. This result holds also when weak convergence is replaced by strong convergence.*

Proof. C has by [6], Lemma 1.1.2 a well-ordered cofinal subchain W. Since all increasing sequences of W have weak limits, there is by [2], Lemma A.3.1 an increasing sequence (x_n) in W which converges weakly to $x = \sup W = \sup C$. Noticing that $-C$ is a chain whose increasing sequences have weak limits, there exists an increasing sequence (x_n) of $-C$ which converges weakly to $\sup(-C) = -\inf C$. Denoting $y_n = -x_n$, we obtain a decreasing sequence (y_n) of C which converges weakly to $\inf C$. In the case of strong convergence the conclusion follows from [6], Proposition 1.1.5.

In what follows, E is an ordered normed space having some of the following properties.

(E0) Bounded and monotone sequences of E have weak limits.

(E1) $x^+ = \sup\{0, x\}$ exists, and $\|x^+\| \leq \|x\|$ for every $x \in E$.

When $c \in E$ and $R \in [0, \infty)$, denote $B_R(c) := \{x \in E : \|x - c\| \leq R\}$. Recall (cf. e.g., [11]) that if a sequence (x_n) of a normed space E converges weakly to x, then (x_n) is bounded, i.e. $\sup_n \|x_n\| < \infty$, and

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|. \tag{4.1}$$

The next auxiliary result is needed in the sequel.

Lemma 4.2. *Let E be an ordered normed space with properties (E0) and (E1). If $c \in E$ and $R \in (0,\infty)$, then c is an order center of $B_R(c)$, and for every chain C of $B_R(c)$ both $\sup C$ and $\inf C$ exist and belong to $B_R(c)$.*

Proof. Since

$$\sup\{c,x\} = (x-c)^+ - c \quad\text{and}\quad \inf\{c,x\} = c-(c-x)^+, \quad \text{for all } \ x \in E, \tag{4.2}$$

then (E1) and (4.2) imply that

$$\|\sup\{c,x\}-c\| = \|\inf\{c,x\}-c\| = \|(x-c)^+\| \le \|x-c\| \le R \ \text{ for every } \ x \in B_R(c).$$

Thus both $\sup\{c,x\}$ and $\inf\{c,x\}$ belong to $B_R(c)$.

Let C be a chain in $B_R(c)$. Since C is bounded there is by (E0) and Lemma 4.1 an increasing sequence (x_n) in C which converges weakly to $x = \sup C$. Since $\|x_n - c\| \le R$ for each n, it follows from (4.1) that

$$\|x-c\| \le \liminf_{n\to\infty} \|x_n - c\| \le R.$$

Thus $x = \sup C$ exists and belongs to $B_R(c)$. Similarly one can show that $\inf G[C]$ exists in E and belongs to $B_R(c)$. □

Applying Theorem 3.2 and Lemmas 4.1 and 4.2 we obtain the following fixed point results.

Theorem 4.1. *Given a subset P of E, assume that $G : P \to P$ is increasing, and that $G[P] \subseteq B_R(c) \subseteq P$ for some $c \in E$ and $R \in (0,\infty)$. Then G has*
(a) minimal and maximal fixed points;
(b) least and greatest fixed points x_* and x^* in the order interval $[\underline{x},\overline{x}]$ of P, where $\underline{x}$ is the greatest solution of $x = \inf\{c, G(x)\}$, and $\overline{x}$ is the least solution of $x = \sup\{c, G(x)\}$.
Moreover, x^*, x_*, $\underline{x}$ and $\overline{x}$ are all increasing with respect to G.

Proof. Let C be a chain in P. Since $G[C]$ is a chain in $B_R(c)$, then both $\sup G[C]$ and $\inf G[C]$ exist in E and belongs to $B_R(c) \subseteq P$ by Lemma 4.2. Because c is an order center of $B_R(c)$ and $ocl(G[P]) \subseteq \overline{G[P]} \subseteq B_R(c) \subseteq P$, then c is an order center of $ocl(G[P])$ in P.

The above proof shows that the hypotheses of Theorem 3.2 are valid. □

In the set-valued case we have the following consequence of Theorem 3.1.

Theorem 4.2. *Assume that P is a subset of E which contains $B_R(c)$ for some $c \in E$ and $R \in (0,\infty)$. Let $\mathcal{F} : P \to 2^P \setminus \emptyset$ be an increasing mapping whose values are weakly compact in E, and whose range $\mathcal{F}[P]$ is contained in $B_R(c)$. Then $\mathcal{F}$ has minimal and maximal fixed points.*

Remarks 4.1. Each of the following spaces has properties (E0) and (E1) (as for the proofs, see, e.g. [1, 2, 3, 5, 6, 7, 8, 10]):

(a) A Sobolev space $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$, $1 < p < \infty$, ordered a.e. pointwise, where Ω is a bounded domain in $\mathbb{R}^N$.
(b) A finite-dimensional normed space ordered by a cone generated by a basis.
(c) l^p, $1 \leq p \leq \infty$, normed by p-norm and ordered coordinatewise.
(d) $L^p(\Omega)$, $1 \leq p \leq \infty$, normed by p-norm and ordered a.e. pointwise, where Ω is a σ-finite measure space.
(e) A separable Hilbert space whose order cone is generated by an orthonormal basis.
(f) A weakly complete Banach lattice or a UMB-lattice (cf.[1]).
(g) $L^p(\Omega, Y)$, $1 \leq p \leq \infty$, normed by p-norm and ordered a.e. pointwise, where Ω is a σ-finite measure space and Y is any of the spaces (b)–(f).
(h) Newtonian spaces $N^{1,p}(Y)$, $1 < p < \infty$, ordered a.e. pointwise, where Y is a metric measure space.

Thus the results of Theorems 4.1–4.2 hold if E is any of the spaces listed in (a)–(h).

4.2 Fixed point results in ordered topological spaces

Let $P = (P, \leq)$ be an ordered topological space, i.e., for each $a \in P$ the order intervals $[a) = \{x \in P : a \leq x\}$ and $(a] = \{x \in P : x \leq a\}$ are closed in the topology of P. In what follows, we assume that P has the following property:

(C) Each well-ordered chain C of P whose increasing sequences converge in P contains an increasing sequence which converges to $\sup C$, and each inversely well-ordered chain C of P whose decreasing sequences converge in P contains a decreasing sequence which converges to $\inf C$.

Corollary 4.1. *The following ordered topological spaces have property (C).*
(a) Ordered metric spaces.
(b) Order closed subsets of ordered normed spaces equipped with a norm topology.
(c) Order closed subsets of ordered normed spaces equipped with a weak topology.
(d) Ordered topological spaces which satisfy the second countability axiom.

Proof. (a) and (b) follow from the result of [6], Proposition 1.1.5 and from its dual.

(c) is a consequence of [2], Appendix, Lemma A.3.1 and its dual.

(d) If P is an ordered topological spaces which satisfies the second countability axiom, then each chain of P is separable, whence P has property (C) by the result of [6], Lemma 1.1.7 and its dual. ☐

The following result is a consequence of Proposition 3.6.

Proposition 4.1. *Given an ordered topological space P with property (C), assume that $G : P \to P$ is an increasing function.*

(a) If $G[P]$ has an upper bound in P, and if G maps decreasing sequences of P to convergent sequences, then G has greatest and minimal fixed points.
(b) If $G[P]$ has a lower bound in P, and if G maps increasing sequences of P to convergent sequences, then G has least and maximal fixed points.

Proof. (a) Let D be an inversely well-ordered chain in P. Since G is increasing, then $G[D]$ is inversely well-ordered. Every decreasing sequence of $G[D]$ is of the form $(G(x_n))$, where (x_n) is a decreasing sequence in D. Thus the hypotheses of (a) and property (C) imply that $x^* = \inf G[D]$ exists and belongs to P. It then follows from Proposition 3.6(b) that G has the greatest fixed point and a minimal fixed point.

The conclusions of (b) is a similar consequence of Proposition 3.6(a). □

The next fixed point result is a consequence of Proposition 4.1 and Lemma 4.1.

Corollary 4.2. *Let P be an order closed subset of an ordered normed space E whose (order) bounded and monotone sequences have weak or strong limits, and let $G : P \to P$ be increasing.*
(a) If $G[P]$ has an upper bound in P, and if G maps decreasing sequences of P to (order) bounded sequences, then G has the greatest and a minimal fixed point.
(b) If $G[P]$ has a lower bound in P, and G maps increasing sequences of P to (order) bounded sequences, then G has the least and a maximal fixed point.

The next result is a consequence of Theorem 3.2.

Theorem 4.3. *Given an ordered topological space P with property (C), assume that $G : P \to P$ is increasing and maps monotone sequences of P to convergent sequences.*
(a) If c is a sup-center of $ocl(G[P])$ in P, then G has minimal and maximal fixed points. Moreover, G has the greatest fixed point x^* in $(\overline{x}]$, where $\overline{x}$ is the least solution of the equation $x = \sup\{c, G(x)\}$. Both $\overline{x}$ and x^* are increasing with respect to G.
(b) If c is an inf-center of $ocl(G[P])$ in P, then G has minimal and maximal fixed points. Moreover, G has the least fixed point x_* in $[\underline{x})$, where $\underline{x}$ is the greatest solution of the equation $x = \inf\{c, G(x)\}$. Both $\underline{x}$ and x_* are increasing with respect to G.

As a consequence of Propositions 3.1 and 3.2 and Theorem 3.1 we get the following result.

Proposition 4.2. *Let P be an ordered topological space with property (C), and let the values of $\mathcal{F} : P \to 2^P \setminus \emptyset$ be compact.*
(a) Assume that following hypothesis holds.

($\mathcal{F}_+$) If (x_n) and (y_n) are increasing and $y_n \in \mathcal{F}(x_n)$ for every n, then (y_n) converges.

If the set $S_+ = \{x \in P : [x) \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty, then $\mathcal{F}$ has a maximal fixed point.
(b) Assume that $\mathcal{F}$ satisfies the following hypothesis.

($\mathcal{F}_-$) If (x_n) and (y_n) are decreasing and $y_n \in \mathcal{F}(x_n)$ for every n, then (y_n) converges.

If the set $S_- = \{x \in P : (x] \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty, then $\mathcal{F}$ has a minimal fixed point.
(b) Assume that the hypotheses ($\mathcal{F}_\pm$) hold. If ocl($\mathcal{F}[P]$) has a sup-center or an inf-center in P, then $\mathcal{F}$ has minimal and maximal fixed points.

Received: April 2008. Revised: April 2008.

References

[1] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Publ. **XXV**, Rhode Island, 1940.

[2] S. Carl and S. Heikkilä, *Nonlinear Differential Equations in Ordered Spaces*, Chapman & Hall/CRC, Boca Raton, 2000.

[3] S. Carl and S. Heikkilä, *Elliptic problems with lack of compactness via a new fixed point theorem*, J. Differential Equations **186** (2002), 122–140.

[4] S. Heikkilä, *A method to solve discontinuous boundary value problems*, Nonlinear Anal., **47**: 2387–2394, 2001.

[5] S. Heikkilä, *Operator equations in ordered function spaces*, in R.P. Agarwal and D. O'Regan (Eds.), Nonlinear Analysis and Applications: To V. Lakshmikantham on his 80:th Birthday, Kluwer Acad. Publ., Dordrecht, ISBN 1-4020-1688-3 (2003), 595–616.

[6] S. Heikkilä and V. Lakshmikantham, *Monotone Iterative Techniques for Discontinuous Nonlinear Differential Equations*, Marcel Dekker, Inc., New York, 1994.

[7] Juha Kinnunen and Olli Martio, *Nonlinear potential theory on metric spaces*, Illinois J. Math. **46**, **3**, 857–883, 2002.

[8] J. Lindenstraus and L. Tzafriri, *Classical Banach Spaces II, Function Spaces*, Springer-Verlag, Berlin, 1979.

[9] H.L. Royden, *Real Analysis*, The MacMillan Company, London, 1968.

[10] N. Shanmugalingam, *Newtonian spaces: An extension of Sobolev spaces to metric measure spaces*, Revista Matemática Iberoamericana, **16**, **2**, (2000), 243–279.

[11] K. Yoshida, *Functional Analysis*, Springer-Verlag, Berlin, 1974.

CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$04, (137–147). December 2008

A New Version of Fan's Theorem and its Applications

M. Fakhar
Department of Mathematics, University of Isfahan,
and Institute for Studies in Theoretical Physics and Mathematics (IPM),
Isfahan, 81745-163, Tehran, Iran
email: `fakhar@math.ui.ac.ir`
and
J. Zafarani
Sheikhbahaee University and University of Isfahan,
Isfahan, 81745-163, Iran
email: `jzaf@zafarini.ir`

ABSTRACT

In this article, using a generalized version of Ky Fan's Theorem, we deduce new proofs for some fixed point theorems and new existence theorems for equilibrium problems.

RESUMEN

Usando una versión generalizada del Teorema de Ky Fan, deducimos nuevas demostraciones para algunos teoremas de punto fijo y nuevos teorema de existencia para problemas de equilibrio.

Key words and phrases: *Fan's Theorem, fixed point theorem, equilibrium problem, Variational inequalities.*

Math. Subj. Class.: *47H10; 49J53; 54H25.*

1 Introduction

Many problems in nonlinear analysis can be solved by showing the nonemptyness of the intersection of certain family of subsets of an underlying set. Each point of the intersection can be a fixed point, a coincidence point, an equilibrium point, a saddle point or an optimal. The first remarkable result on nonempty intersection was the celebrated Knaster, Kuratowski and Mazurkiewicz in 1929 [15], which concerns with certain type of multimaps called the KKM maps later. Fan [11] proved that the assertion of the KKM theorem for infinite dimensional topological vector space. Brézis, Nirenberg and Stampacchia [4] improved Fan's KKM lemma [11] by assuming the closedness condition only on finite dimensional subspaces, with some topological pseudomonotone condition. Chowdhury and Tan [5], replacing finite dimensional subspaces by polytopes, restated the Brézis, Nirenberg and Stampacchia result under weaker assumptions. Ding and Tarafdar [7] obtained the result of Chowdhury and Tan under weaker compactness condition. The Chowdhury and Tan's result was also proved by Kalmoun [14] for transfer closed-valued multi-valued mappings. Our aim here is to derive a new version of Brézis, Nirenberg and Stampacchia's result and then apply it to obtain some fixed point theorems and established the existence solution of equilibrium problems and generalized variational inequalities.

For the reader's convenience, we review a few basic definitions and notations from the fixed point theory. Let X be a Hausdorff topological vector space and K be a nonempty subset of X, then we denote by $< K >$ the family of all nonempty finite subsets of K. Let K_0 be a nonempty subset of K. A set-valued map $\Gamma : K_0 \rightrightarrows K$ is called a KKM map if for each $A \in < K_0 >$, $\text{conv}(A) \subseteq \bigcup_{x\in A} \Gamma(x)$. Let Y be a nonempty set. Then, $\Gamma : Y \rightrightarrows K$ is said to be transfer closed-valued if for any $(y, x) \in Y \times K$ with $x \notin \Gamma(y)$ there exists $y' \in Y$ such that $x \notin \text{cl}_K\Gamma(y')$. If $Y = K$, then we will call Γ transfer closed-valued on K. If $K_0 \subseteq K$, then a map $\Gamma : K \rightrightarrows K$ is called transfer closed-valued on K_0 if the map $y \mapsto \Gamma(y) \bigcap K_0$, $y \in K_0$, is transfer closed-valued. A set-valued map $\Gamma : K \rightrightarrows K$ is called transfer open-valued on K if the set-valued map $\hat{\Gamma} : K \rightrightarrows K$ defined as follows: $\hat{\Gamma}(x) := K \setminus \Gamma(x)$ is transfer closed-valued on K. Let us recall that a set-valued map $\Gamma : K \rightrightarrows K$ has a maximal element, if there exists a point $\bar{x} \in K$ such that $\Gamma(\bar{x}) = \emptyset$.

Suppose that f is a real-valued bifunction on $Y \times K$. Then, we say that f is transfer lower semicontinuous(l.s.c.) in the second variable if for each $(y, x) \in Y \times K$ with $f(y, x) > 0$ there exist $y' \in Y$ and a neighborhood $U(x)$ of x in K such that $f(y', z) > 0$ for all $z \in U(x)$. If $Y = K$ and $A \subseteq K$, then we call f transfer l.s.c. in the second variable on A, if $f\mid_{A\times A}$ is transfer l.s.c. in the second variable.

Definition 1.1 Let $f : K \times K \to \mathbb{R}$. We recall that:

(i) f is pseudomonotone if, for all $(x, y) \in K \times K$, $f(x, y) \geq 0$ implies $f(y, x) \leq 0$;

(ii) f is called 0-segmentary closed if $\forall x, y \in K$, when (y_α) be a net on K converging to y, then the following implication holds, if $f(u, y_\alpha) \leq 0$ for all $u \in [x, y]$, then $f(x, y) \leq 0$.

(iii) $f(.,y)$ is upper sign continuous if the following implication holds for every $x \in K$,

$$f(u,y) \geq 0, \quad \forall u \in \ ]x,y[\Rightarrow f(x,y) \geq 0,$$

We note that if f is hemicontinuous function, then f and $-f$ both are upper sign continuous.

2 Brézis, Nirenberg and Stampacchia type theorem

In [8, 9], the authors refined the Ding and Tarafdar's result [7] and the Kalmoun's result [14]. Based on the Remark 2 in [4], recently the authors obtain a short and direct proof of the following Brézis, Nirenberg and Stampacchia version of Fan's KKM Theorem [10]

Lemma 2.1 *Let K be a nonempty and convex subset of a Hausdorff t.v.s. X. Suppose that $\Gamma : K \rightrightarrows K$ is a set-valued mapping such that the following conditions are satisfied:*

(i) Γ is a KKM map;

(ii) for all $A \in < K >$, Γ is transfer closed-valued on conv(A);

(iii) for all $x, y \in K$, $cl_K(\bigcap_{u \in [x,y]} \Gamma(u)) \cap [x,y] = (\bigcap_{u \in [x,y]} \Gamma(u)) \cap [x,y]$;

(iv) there is a nonempty compact convex set $B \subseteq K$, such that $cl_K(\bigcap_{x \in B} \Gamma(x))$ is compact.

Then, $\bigcap_{x \in K} \Gamma(x) \neq \emptyset$.

Based on the above Lemma, here we obtain another new version of the above result.

Theorem 2.1. *Let K be a nonempty convex subset of a Hausdorff t.v.s. X. Suppose that $\Gamma : K \rightrightarrows K$ is a set-valued mapping such that the following conditions are satisfied:*

(H1) Γ is a KKM map;

(H2) $\forall A \in \langle K \rangle$, Γ is transfer closed-valued on conv(A);

(H3) $\forall x, y \in K$, $cl_K(\bigcap_{u \in [x,y]} \Gamma(u)) \cap [x,y] = (\bigcap_{u \in [x,y]} \Gamma(u)) \cap [x,y]$;

(H4) there exist a nonempty compact convex subset B of K and a nonempty compact subset D of K such that, for each $y \in K \setminus D$ there exists $x \in$ conv$(B \cup \{y\})$ such that $y \notin \Gamma(x)$.

Then, $\bigcap_{x \in K} \Gamma(x) \neq \emptyset$.

Proof. Suppose that $A \in < K >$ and $L_A = \text{conv}(A \bigcup B)$, then L_A is compact. Let $\Gamma_A : L_A \rightrightarrows L_A$ be defined as $\Gamma_A(x) = \Gamma(x) \cap L_A$. Then, from Lemma 2.1, we have

$$\bigcap_{x \in L_A} \Gamma_A(x) \neq \emptyset.$$

Now, we show that

$$\bigcap_{x\in L_A} \Gamma_A(x) \subseteq D.$$

Suppose that this claim is not true, then there exists $y \in \bigcap_{x\in L_A} \Gamma_A(x)$ such that $y \in K \setminus D$. But by assumption (H4) there exists $x \in \text{conv}(B \cup \{y\})$ such that $y \notin \Gamma(x)$. Therefore, $x \notin L_A$. But since $y \in L_A$, then $\text{conv}(B \cup \{y\}) \subseteq L_A$ which contradicts (H4).

Assume that

$$M_A = \bigcap_{x\in L_A} \Gamma(x) \text{ for any } A \in < K >, \tag{1}$$

then

$$M_A \subseteq D \text{ for all } A \in < K > . \tag{2}$$

If $\mathcal{M} = \{M_A : A \in < K >\}$, then by (1) one can see that the class $\mathcal{M}$ has the finite intersection property. Therefore, from (2), we have

$$\bigcap_{A\in<K>} \text{cl}_K M_A \neq \emptyset.$$

If $\bar{x} \in \bigcap_{A\in<K>} \text{cl}_K M_A$, $x \in X$ and $A_0 = \{\bar{x}, x\}$, then $\text{conv}(A_0) = [\bar{x}, x]$ and

$$\bar{x} \in \text{cl}_K M_{A_0} = \text{cl}_K \left(\bigcap_{u\in L_{A_0}} \Gamma(u) \right) \subseteq \text{cl}_K \left(\bigcap_{u\in[\bar{x},x]} \Gamma(u) \right)$$

Hence, by condition (H3)

$$\bar{x} \in \text{cl}_K \left(\bigcap_{u\in[\bar{x},x]} \Gamma(u) \right) \cap [\bar{x}, x] = \left(\bigcap_{u\in[\bar{x},x]} \Gamma(u) \right) \cap [\bar{x}, x].$$

Therefore, $\bar{x} \in \Gamma(x)$ for all $x \in X$ and the proof is complete.

□

Remark 2.2. *(a) By a similar proof as that of the above theorem, we can obtain some other versions of Fan's KKM Theorem.*
Let K_0 be a nonempty subset of K and $\Gamma : K_0 \rightrightarrows K$ satisfying the following conditions:

(i) Γ is a KKM map,

(ii) for each $A \in \mathcal{F}(K_0)$, $\Gamma : A \rightrightarrows conv(A)$ is transfer closed valued,

(iii) for each $x, y \in K_0$,

$$cl_K \left(\bigcap_{u\in[x,y]\cap K_0} \Gamma(u) \right) \cap [x, y] = \left(\bigcap_{u\in[x,y]\cap K_0} \Gamma(u) \right) \cap [x, y],$$

(iv) there exists a nonempty convex compact subset B of K such that for each $y \in K \setminus B$ there exists $x \in conv(B \cup \{y\}) \cap K_0$ such that $y \notin \Gamma(x)$.

Then, $\bigcap_{x\in K_0}\Gamma(x)\neq\emptyset$.

(b) Instead of assumptions (ii) and (iii) in part (a) we can assume that Γ *is transfer closed valued. Furthermore, in this case, condition (iv) of part (a) can replaced by the following condition:*

(iv)′ there exist a nonempty compact convex subset B *of* K *and a nonempty compact subset* D *of* K *such that, for each* $y\in K\setminus D$ *there exists* $x\in conv(B\cup\{y\})\cap K_0$ *such that* $y\notin\Gamma(x)$.

3 Fixed point Theorems

In this section, we deduce slight generalizations of known fixed point theorems from Theorem 2.1.

Theorem 3.1. *Let* K *be a nonempty convex subset of a (t.v.s.)* X *and* $S:K\rightrightarrows K$ *a set-valued map such that:*

(i) for each $A\in<K>$, S^- *is transfer open valued on* $conv(A)$*;*

(ii) for each $x,y\in K$*;*

$$int\left(\bigcup_{z\in[x,y]}S^-(z)\right)\cap[x,y]=\left(\bigcup_{z\in[x,y]}S^-(z)\right)\cap[x,y];$$

(iii) there exist a nonempty convex compact subset B *of* K *and a nonempty compact subset* D *of* K *such that, for each* $y\in K\setminus D$ *there exists* $x\in conv(B\cup\{y\})$ *such that* $x\in S(y)$.

Then, either S *has a maximal element or* $convS$ *has a fixed point.*

Proof. Suppose that S has no maximal element, then $\bigcup_{y\in X}S^-(y)=K$. If $\Gamma(x)=K\setminus S^-(x)$ for all $x\in X$, then

$$\bigcap_{x\in K}\Gamma(x)=\emptyset.$$

Therefore, one of the assumptions of Theorem 2.1 does not hold for Γ. By condition (i), Γ is transfer closed-valued on conv$A$ for any $A\in<K>$. Now suppose that $x,y\in K$, $z\in[x,y]$ and $z\notin\cap_{u\in[x,y]}\Gamma(u)$. Then $z\in(\bigcup_{u\in[x,y]}S^-(u))\cap[x,y]$ and so $z\in int(\bigcup_{u\in[x,y]}S^-(u))\cap[x,y]$. Therefore, there exists an open neighborhood U of z in K such that $U\subseteq\bigcup_{u\in[x,y]}S^-(u)$. Hence,

$$U\bigcap(\cap_{u\in[x,y]}\Gamma(u))=\emptyset.$$

That is $z\notin\mathrm{cl}_K\left(\bigcap_{u\in[x,y]}\Gamma(u)\right)\cap[x,y]$ and so condition (H3) is satisfied. Also condition (iii) implies condition (H4). Therefore, Γ is not KKM map. Thus, there exists a finite subset $A=\{x_1,...,x_n\}$ of K such that

$$\mathrm{conv(A)}\nsubseteq\bigcup_{i=1}^{n}\Gamma(x_i).$$

This implies that there is a point $x \in \mathrm{conv}(A)$ such that $x \in S^-(x_i)$ for all $i = 1, ..., n$. Therefore, for all $i = 1, ..., n$, $x_i \in S(x)$ and $x \in \mathrm{conv}S(x)$.

□

Remark 3.2. *When K is compact, then trivially condition (iii) holds. Furthermore, if S^- is transfer open valued on K, then we can replace $S^-(z)$ in condition (ii) by $intS^-(z)$. Hence, in this case conditions (ii) and (iii) are fulfilled.*

Now we deduce the following version of Theorem 1.2 in Ansari and Yao [1] as a corollary of our Theorem 3.1.

Corollary 3.3. *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Suppose that $S, T : K \rightrightarrows K$ are two set-valued maps with nonempty values such that*

(a) for each $x \in K$, $A \in \langle S(x) \rangle$, $conv(A) \subset T(x)$;

(b) for each $A \in < K >$, S transfer open valued on $conv(A)$;

(c) for each $x, y \in K$,

$$int \left(\bigcup_{z \in [x,y]} S^-(z) \right) \cap [x, y] = \left(\bigcup_{z \in [x,y]} S^-(z) \right) \cap [x, y];$$

(e) there exist a nonempty convex compact subset B of K and a nonempty compact subset D of K such that, for each $y \in K \setminus D$ there exists $x \in conv(B \cup \{y\})$ such that $x \in S(y)$. Then T has a fixed point.

From Theorem 3.1, we also deduce Theorem 1.1 of Djafari-Rouhani, Tarafdar and Watson [6].

Corollary 3.4. *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Suppose that $S, T : K \rightrightarrows K$ are two set-valued maps with nonempty values that*

(a) for each $x \in K$, $A \in \langle S(x) \rangle$, $conv(A) \subset T(x)$.

(b) for each $y \in K$, $S^-(y)$ contains an open set O_y, which may be empty such that $K = \cup\{O_y : y \in K\}$

(c) there exist a nonempty convex compact subset B of K and a points $\{\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\}$ in K such that

$$\bigcap_{x \in B} O_x^c \subseteq \bigcup_{i=1}^{n} O_{\hat{x}_i},$$

where O_x^c is the complement of O_x in K. Then T has a fixed point.

Proof. From (b-c), we obtain that

$$\bigcap_{x \in B} \{K \setminus S^-(x)\} \subseteq \bigcap_{x \in B} O_x^c \subseteq \bigcup_{i=1}^{n} O_{\hat{x}_i} \subseteq \bigcup_{i=1}^{n} S^-(\hat{x}_i).$$

Let $C = B \cup \{\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\}$. Then

$$K = \bigcup_{x \in C} S^-(x).$$

Let $H = \text{conv}(C)$, then H is compact and convex and moreover,

$$H = \bigcup_{x \in C} S^-(x) \cap H \subseteq \bigcup_{x \in H} S^-(x).$$

Now we define the set-valued mapping $\Gamma : H \rightrightarrows H$ as

$$\Gamma(x) = H \setminus S^-(x).$$

Then from Remark 3.2, we conclude the proof.

□

As another consequence of Theorem 3.1, we obtain the following fixed point theorem of Ansari and Lin [2].

Corollary 3.5. *Let K be a nonempty convex subset of a Hausdorff topological vector space X. Suppose that $S, T : K \rightrightarrows K$ are two mutivalued maps such that*

(a) for each $x \in K$, $A \in \langle S(x) \rangle$, $conv(A) \subset T(x)$;

(b) there exist a nonempty convex compact subset B of K and a points $\{\hat{x}_1, \hat{x}_2, ..., \hat{x}_n\}$ in K such that

$$\bigcap_{x \in B} (K \setminus int_K S^-(x)) \subseteq \bigcup_{i=1}^{n} int_K S^-(\hat{x_i}).$$

Then T has a fixed point.

Proof. It is enough in the above corollary to set $O_x = \text{int}_K S^-(x)$ for all $x \in K$. □

Remark 3.6. *By the same argument as in the above corollary,one can also obtain a proof for Theorem 2.1 in [2].*

4 Equilibrium Problems

We now give some new applications of Theorem 2.1 in obtaining existence results of equilibrium problem.

Theorem 4.1. *Let K be a nonempty convex subset of a Hausdorff (t.v.s.) X. Suppose f is a pseudomomtone real-valued on $K \times K$ such that:*

(A1) $f(x,x) = 0$ for any $x \in X$;

(A2) for each $x, y, z \in X$ if $f(x,y) < 0$ and $f(x,z) \leq 0$, then $f(x,u) < 0$ for all $u \in]y,z[$;

(A3) for each $A \in < K >$, f is transfer l.s.c. in the second variable on conv(A);

(A4) f is 0-segmentary closed;

(A5) there exist a nonempty compact subset $D \subseteq K$ and a nonempty convex compact subset B of K such that for each $x \in K \setminus D$, there exists $y \in$ conv$(B \cup \{x\})$ such that $f(x,y) > 0$.
Then, there exists $\bar{x} \in X$ such that $f(y, \bar{x}) \leq 0$ for all $y \in X$.

Proof. Assume that $\hat{\Gamma} : K \rightrightarrows K, \Gamma : K \rightrightarrows K$ are defined by:

$$\hat{\Gamma}(y) = \{x \in X : f(x,y) \geq 0\},$$
$$\Gamma(y) = \{x \in X : f(y,x) \leq 0\}.$$

Then, as f is pseudomonotone, $\hat{\Gamma}(y) \subseteq \Gamma(y)$ for all $y \in K$. By (A1) and (A2) $\hat{\Gamma}$ is a KKM map, so Γ is a KKM map. Condition (A5) implies condition (H4). Therefore, the conditions (H1) and (H4) are fulfilled by the set-valued map Γ. The condition (A3) implies that the condition (H2) holds for Γ. For condition (H3), suppose

$$z \in \mathrm{cl}_K \left(\bigcap_{u \in [x,y]} \Gamma(u) \right) \cap [x,y].$$

Then, there exists a net (z_α) converging to z such that $f(u, z_\alpha) \leq 0$ for all $u \in [x,y]$. Since $z \in [x,y]$, for each $v \in [u,z]$, we have also $f(v, z_\alpha) \leq 0$, hence by (A4), we obtain $f(u,z) \leq 0$ for each $u \in [x,y]$. Thus, we have

$$z \in \left(\bigcap_{u \in [x,y]} \Gamma(u) \right) \cap [x,y].$$

Hence Γ satisfies also the condition (H3). Therefore, from Theorem 2.1, we have

$$\bigcap_{y \in X} \Gamma(y) \neq \emptyset.$$

Thus, any point $\hat{x}$ in this intersection is a solution for our problem.

□

Corollary 4.2. *In Theorem 4.1, if $f(.,y)$ is upper sign continuous for every $y \in K$, then there exists $\bar{x} \in K$ such that $f(x, \bar{x}) \geq 0$ for all $x \in K$.*

Proof. By Theorem 4.1, suppose that $\bar{x} \in K$ such that $f(y, \bar{x}) \leq 0$ for all $y \in X$. Assume that there exists $\bar{y} \in K$ such that $f(\bar{x}, \bar{y}) < 0$. By our assumption on $\bar{x}$ we have also $f(\bar{y}, \bar{x}) \leq 0$. We will show that $f(u, \bar{y}) \geq 0$ for all $u \in \;]\bar{x}, \bar{y}[$. Indeed, if $f(u, \bar{y}) < 0$ for some $u \in \;]\bar{x}, \bar{y}[$, then as

$f(u,\bar{x}) \leq 0$, we obtain from (A2) that $f(u,u) < 0$, which contradicts (A1). Now by upper sign continuity of f, we have $f(\bar{x},\bar{y}) \geq 0$, which is a contradiction. $\square$

The following example shows that Corollary 4.2 improves the corresponding results in [3] and [13].

Example 4.3. Assume that $K = \mathbb{R}$ and $f : K \times K \to \mathbb{R}$ is defined as follows:

$$f(x,y) := \begin{cases} -y & \text{if } \ x = 0, \\ 1-y & \text{if } \ x = 1, \\ 1 \quad \text{if} & x = \frac{3}{2}, 5 > |y| \geq 3, \\ -2+y & \text{if } \ x = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let $D = [-1,2]$ and $B = [1,2]$. For $y < -1$, we have $f(1,y) > 0$. If $y > 2$, then $f(2,y) > 0$. Therefore, f satisfies all of the condition of Corollary 4.2. But for $x = 3/2$, the set

$$\{y \in \mathbb{R} : f(3/2,y) \leq 0\} =]-3,3[\cup]-\infty,-5[\cup]5,\infty[,$$

which is not convex and K is not compact.

As a consequence of our results, we conclude a new version of Theorem 15 in [12] and its corollary for existence of variational inequalities problem.

Corollary 4.3. *Let K be a nonempty convex subset of a Hausdorff (t.v.s.) X and $T : K \rightrightarrows X^*$. Suppose that*

- *(i) T is upper semicontinuous from conv(A) of any $A \in < K >$ to X^* endowed with w^*-topology and for each $x \in K$, $T(x)$ is convex w^*-compact;*
- *(ii) $f(x,y) := \inf_{y^* \in T(y)} \langle y^*, y-x \rangle$ is 0-segmentary closed;*
- *(iii) there exist a nonempty compact subset D and a nonempty convex compact subset B of K such that, for each $y \in K \backslash D$, there exists $x \in$ conv$(B \cup \{y\})$ such that*

$$\inf_{y^* \in T(y)} \langle y^*, y-x \rangle > 0.$$

Then, there exist $\bar{y} \in K$ and $y_0^* \in T(\bar{y})$ such that $\langle y_0^*, x-\bar{y} \rangle \geq 0, \forall x \in K$.

Proof. Let

$$f(x,y) = \inf_{y^* \in T(y)} \langle y^*, y-x \rangle.$$

We will show that all of the conditions of Theorem 4.1 and its corollary are fulfilled by f. Lemma 2.2 in [7] implies that for each fixed $x \in K$, the function $y \to \inf_{y^* \in T(y)} \langle y^*, y - x\rangle$ is l.s.c. on conv(A) of any $A \in < K >$, hence we trivially have condition(A3) of Theorem 4.1. Since $f(x,x) = 0$, for every $x \in K$and f is affine in the first argument, hence f satisfies conditions (A1) and (A2). Trivially $f(.,y)$ is upper sign continuous, thus from Corollary 4.2, there exists $\bar{y} \in K$ such that

$$\inf_{y^* \in T(\bar{y})} \langle y^*, \bar{y} - x\rangle \leq 0, \forall x \in K.$$

Now, let $g : K \times T(\bar{y}) \to \mathbb{R}$ be defined as follows:

$$g(x, y^*) = \langle y^*, \bar{y} - x\rangle,$$

then since $T(\bar{y})$ is convex, by Kneser's minimax theorem, we have

$$\inf_{y^* \in T(\bar{y})} \sup_{x \in K} \langle y^*, \bar{y} - x\rangle = \sup_{x \in K} \inf_{y^* \in T(\bar{y})} \langle y^*, \bar{y} - x\rangle \leq 0.$$

Therefore, there exists a point $y_0^* \in T(\bar{y})$ such that

$$\sup_{x \in K} \langle y_0^*, \bar{y} - x\rangle \leq 0.$$

□

Acknowledgment. The authors were partially supported by the Center of Excellence for Mathematics (University of Isfahan). The first author was partially supported by a grant from IPM (No. 86470016).

`Received: May 2008. Revised: May 2008.`

References

[1] Ansari, Q.H. and Yao, J.C., *A fixed point theorem and its applications to a system of variational inequalities*, Bull. Austral. Math. Soc., **59** (1999), no. 3, 433–442.

[2] Ansari, Q.H. and Lin, Y.C., *Fixed point and maximal element theorems with application to abstract economy*, Nonlinear Stud., **13** (2006), no. 1, 43–52.

[3] Bianchi, M. and Pini,R., *Coercivity conditions for equilibrium problems*, J. Optim. Theory Appl., **124** (2005), 79–92.

[4] Brézis, H., Nirenberg, L. and Stampacchia, G., *A Remark on Ky Fan's minimax principle*, Bolletino della Unione Matematica Italiana, **6** (1972), 293–300.

[5] Chowdhury, M.S.R. and Tan, K.K., *Generalization of Ky Fan's minimax inequality with applications to generalized variational inequalities for pseudomonotone operators and fixed-point theorem*, J. Math. Anal. Appl., **204** (1996), 910–929.

[6] Djafari Rouhani, B., Tarafdar, E. and Watson, P.J., *Existence of solutions to some equilibrium problems*, J. Optim. Theory Appl., **126** (2005), 97–107.

[7] Ding, X.P. and Tarafdar, E., *Generalized variational-like inequalities with pseudomonotone set-valued mappings*, Archiv. Math., **74** (2000), 302–313.

[8] Fakhar, M. and Zafarani, J., *Generalized equilibrium problems for quasimonotone and pseudomonotone bifunctions*, J. Optim. Theory Appl., **123** (2004), 349–364.

[9] Fakhar, M. and Zafarani, J., *Generalized vector equilibrium problems for pseudomonotone multi-valued bifunctions*, J. Optim. Theory Appl., **126** (2005), 109–124.

[10] Fakhar, M. and Zafarani, J., *On generalized variational inequalities*, to appear in J. Global Optim.

[11] Fan, K., *A Generalization of Tychonoff's fixed-point theorem*, Math. Ann., **142** (1961), 305–310.

[12] Inoan, D. and Kolumban, J., *On pseudomonotone set-valued mappings*, Nonlinear Anal., **68** (2008), 47–53.

[13] Iusem, A.N. and Sosa, W., *New existence results for equilibrium problems*, Nonlinear. Anal., **52** (2003), 621–635.

[14] Kalmoun, E.M., *On Ky Fan's minimax inequalities, mixed equilibrium problems and hemivariational inequalities*, JIPAM. J. Inequal. Pure Appl. Math., **2** (2001), 1–13.

[15] Knaster, B., Kuratowski, C. and Mazurkiewicz, S., *Ein Bewies des Fixpunktsatzes für n-dimensional simplexe*, Fund. Math., **14** (1929), 132–137.

CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$04, (1–13). December 2008

Remarks on KKM Maps and Fixed Point Theorems in Generalized Convex Spaces

Sehie Park
The National Academy of Sciences, Republic of Korea,
Department of Mathematical Sciences, Seoul National University,
Seoul 151–747, KOREA
email: `shpark@math.snu.ac.kr`

ABSTRACT

Various types of ϕ_A-spaces $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ are simply G-convex spaces. Various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on G-convex spaces. Therefore, our G-convex space theory can be applied to various types of ϕ_A-spaces. As such examples, we obtain KKM type theorems and a very general fixed point theorem on ϕ_A-spaces.

RESUMEN

Varios tipos de ϕ_A-espacios $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ son simplemente espacios G-convexos. Varios tipos de aplicaciones KKM generalizadas sobre ϕ_A-espacios son aplicaciones simplemente KKM sobre espacios G-convexos. Por lo tanto, nuestra teoria de espacios G-Convexos puede ser aplicada a varios tipos de ϕ_A-espacios. Como ejemplo obtenemos teoremas do tipo KKM y un teorema general de punto fijo sobre ϕ_A-espacios.

Key words and phrases: *Abstract convex space, generalized (G-) convex space, ϕ_A-space, L-spaces, FC-spaces, property (H); H-condition.*

Math. Subj. Class.: *47H04, 47H10, 49J27, 49J35, 54H25, 91B50.*

1 Introduction

The KKM theory, first called by the author, is the study on applications of equivalent formulations of the KKM principle due to Knaster, Kuratowski, and Mazurkiewicz. The KKM principle provides the foundations for many of the modern essential results in diverse areas of mathematical sciences.

Since 1993, the author has initiated the study of the KKM theory on generalized convex spaces (or G-convex spaces) $(X, D; \Gamma)$ as a common generalization of various general convexities without linear structures due to other authors. We have established within such a frame the foundations of the KKM theory, as well as fixed point theorems and many other equilibrium results for multimaps. This direction of study has been followed by a number of other authors.

In the last decade, some authors who introduced spaces of the form $(X, \{\varphi_A\})$ having a family $\{\varphi_A\}$ of continuous functions defined on simplices claimed that such spaces generalize G-convex spaces without giving any justifications or proper examples. In fact, a number of modifications or imitations of the G-convex spaces have followed; for example, L-spaces due to Ben-El-Mechaiekh et al. [1], spaces having property (H) due to Huang [10], FC-spaces due to Ding [6,7], convexity structures satisfying the H-condition [22], and others. Some authors also tried to generalize the KKM principle for their own settings. They introduced various types of generalized KKM maps; for example, generalized KKM maps on L-spaces [5,20], generalized R-KKM maps [2,8,9], and many others.

In order to destroy such inadequate concepts and to upgrade the KKM theory, recently, we proposed new concepts of abstract convex spaces and the KKM spaces which are proper generalizations of G-convex spaces and adequate to establish the KKM theory; see [15-18]. Moreover, we noticed that all spaces of the form $(X, \{\varphi_A\})$ can be unified to ϕ_A-spaces $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ or spaces having a family $\{\phi_A\}_{A \in \langle D \rangle}$ of singular simplices.

In the present note, we show that various types of ϕ_A-spaces $(X, D;\ \{\phi_A\}_{A \in \langle D \rangle})$ are simply G-convex spaces, and various types of generalized KKM maps on ϕ_A-spaces are simply KKM maps on G-convex spaces. Therefore, our G-convex space theory can be applied to various types of ϕ_A-spaces. As such examples, we obtain KKM type theorems and a very general fixed point theorem on ϕ_A-spaces.

2 Abstract convex spaces

In this section, we follow mainly [15,16]. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

Definition. An *abstract convex space* $(E, D; \Gamma)$ consists of nonempty sets E, D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

Examples. In [15-17], we gave plenty of examples of abstract convex spaces. Here we give only two classes of them as follows:

1. A *convexity space* $(E, \mathcal{C})$ *in the classical sense* consists of a nonempty set E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. For details, see [21], where the bibliography lists 283 papers.

2. A *generalized convex space* or a *G-convex space* $(E, D; \Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is a standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. For G-convex spaces; see [11-14,19] and references therein.

From now on, in an abstract convex space $(E, D; \Gamma)$, E is assumed to be a topological space.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$*-map* [resp., a $\mathfrak{KO}$*-map*] if, for any closed-valued [resp., open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property.

The following is the origin of the KKM theory; see [11,12].

The KKM Principle. *Let D be the set of vertices of an n-simplex Δ_n and $G : D \multimap \Delta_n$ be a KKM map (that is, $\mathrm{co}\, A \subset G(A)$ for each $A \subset D$) with closed [resp., open] values. Then $\bigcap_{z \in D} G(z) \neq \emptyset$.*

3 ϕ_A-spaces

Recently, there have appeared authors in [2,3,6-10,20,22] and others who introduced spaces of the form $(X, \{\varphi_A\})$. Some of them tried to rewrite some results on G-convex spaces by simply replacing $\Gamma(A)$ by $\varphi_A(\Delta_n)$ everywhere and claimed to obtain generalizations without giving any justifications or proper examples.

Motivated by this fact, we are concerned with a reformulation of the class of G-convex spaces as follows [17]:

Definition. A *ϕ_A-space*

$$(X, D; \{\phi_A\}_{A\in\langle D\rangle})$$

consists of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ (that is, singular n-simplices) for $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$.

Any G-convex space is a ϕ_A-space. The converse also holds:

Theorem 1. *A ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ can be made into a G-convex space $(X, D; \Gamma)$.*

Proof. This can be done in two ways.

(1) For each $A \in \langle D\rangle$, by putting $\Gamma_A := X$, we obtain a trivial G-convex space $(X, D; \Gamma)$.

(2) Let $\{\Gamma^\alpha\}_\alpha$ be the family of maps $\Gamma^\alpha : \langle D\rangle \multimap X$ giving a G-convex space $(X, D; \Gamma^\alpha)$. Note that, by (1), this family is not empty. Then, for each α and each $A \in \langle D\rangle$ with $|A| = n + 1$, we have

$$\phi_A(\Delta_n) \subset \Gamma^\alpha_A \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma^\alpha_J \quad \text{for} \quad J \subset A.$$

Let $\Gamma := \bigcap_\alpha \Gamma^\alpha$, that is, $\Gamma_A := \bigcap_\alpha \Gamma^\alpha_A$ for each $A \in \langle D\rangle$. Then

$$\phi_A(\Delta_n) \subset \Gamma_A \quad \text{and} \quad \phi_A(\Delta_J) \subset \Gamma_J \quad \text{for} \quad J \subset A.$$

Therefore, $(X, D; \Gamma)$ is a G-convex space.

Consequently, G-convex spaces and ϕ_A-spaces are essentially the same.

Definition. For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \quad \text{for each} \quad A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is called a *KKM map.*

Theorem 2. (1) *A KKM map $G : D \multimap X$ on a G-convex space $(X, D; \Gamma)$ is a KKM map on the corresponding ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$.*

(2) *A KKM map $T : D \multimap X$ on a ϕ_A-space $(X, D; \{\phi_A\})$ is a KKM map on a new G-convex space $(X, D; \Gamma)$.*

Proof. (1) This is clear from the definition of a KKM map on a G-convex space.

(2) Define $\Gamma : \langle D\rangle \multimap X$ by $\Gamma_A := T(A)$ for each $A \in \langle D\rangle$. Then $(X, D; \Gamma)$ becomes a G-convex space. In fact, for each A with $|A| = n+1$, we have a continuous function $\phi_A : \Delta_n \to T(A) =: \Gamma(A)$

such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset T(J) =: \Gamma(J)$. Moreover, note that $\Gamma_A \subset T(A)$ for each $A \in \langle D \rangle$ and hence $T : D \multimap X$ is a KKM map on a G-convex space $(X, D; \Gamma)$.

The following is a KKM theorem for ϕ_A-spaces. The proof is just a simple modification of the corresponding one in [12,13,19]:

Theorem 3. *For a ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$, let $G : D \multimap X$ be a KKM map with closed [resp., open] values. Then $\{G(z)\}_{z \in D}$ has the finite intersection property. (More precisely, for each $N \in \langle D \rangle$ with $|N| = n + 1$, we have $\phi_N(\Delta_n) \cap \bigcap_{z \in N} G(z) \neq \emptyset$).*

Further, if

(3) $\bigcap_{z \in M} \overline{G(z)}$ is compact for some $M \in \langle D \rangle$,

then we have $\bigcap_{z \in D} \overline{G(z)} \neq \emptyset$.

Proof. Let $N = \{z_0, z_1, \ldots, z_n\}$. Since G is a KKM map, for each vertex e_i of Δ_n, we have $\phi_N(e_i) \in G(z_i)$ for $0 \leq i \leq n$. Then $e_i \mapsto \phi_N^{-1} G(z_i)$ is a closed [resp., open] valued map such that $\Delta_k = \text{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi_N^{-1} G(z_{i_j})$ for each face Δ_k of Δ_n. Therefore, by the original KKM principle, $\Delta_n \supset \bigcap_{i=0}^{n} \phi_N^{-1} G(z_i) \neq \emptyset$ and hence $\phi_N(\Delta_n) \cap \left(\bigcap_{z \in N} G(z)\right) \neq \emptyset$.

The second conclusion is clear.

Remarks. (1) We may assume that, for each $a \in D$ and $N \in \langle D \rangle$, $G(a) \cap \phi_N(\Delta_n)$ is closed [resp., open] in $\phi_N(\Delta_n)$. This is said by some authors that G has finitely closed [resp., open] values. However, by replacing the topology of X by its finitely generated extension, we can eliminate "finitely"; see [13].

(2) For $X = \Delta_n$, if D is the set of vertices of Δ_n and $\Gamma = \text{co}$, the convex hull, Theorem 3 reduces to the original KKM principle and its open version; see [11,12].

(3) If D is a nonempty subset of a topological vector space X (not necessarily Hausdorff), Theorem 3 extends Fan's KKM lemma; see [11,12].

(4) Note that any KKM theorem on spaces of the form $(X, \{\varphi_A\})$ can not generalize the original KKM principle or Fan's KKM lemma.

4 Examples of ϕ_A-spaces

In this section, we give some examples of spaces of the form $(X, \{\varphi_A\})$ given by other authors:

(I) In 1998, Ben-El-Mechaiekh et al. [1] defined an L-space (E, Γ), which is a particular form of our G-convex space $(X, D; \Gamma)$ for the case $E = X = D$. Some authors incorrectly claimed that the class of L-spaces contains our class of G-convex spaces; for example, [4,5], which contain a number of particular results (with certain defects) of known ones.

(II) In 2003, the authors of [20] considered the L-space.

(III) [10] A topological space Y is said to have property (H) if, for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$.

(IV) [6,7] $(Y, \{\varphi_N\})$ is said to be a FC-space if Y is a topological space and for each $N = \{y_0, \dots, y_n\} \in \langle Y \rangle$ where some elements in N may be same, there exists a continuous mapping $\varphi_N : \Delta_n \to Y$. This definition appears in a large number of papers of the same author and his followers. Note that for each N, there should be infinitely many φ_N's.

The author of [6,7] wrote in more than one dozen papers that: "It is easy to see that the class of FC-spaces includes the classes of convex sets in topological vector spaces, C-spaces (or H-spaces) [20], G-convex spaces, L-convex spaces [1], and many topological spaces with abstract convexity structure as true subclasses. Hence, it is quite reasonable and valuable to study various nonlinear problems in FC-spaces." There he failed to give any justification or any proper example of his space which is not G-convex. One wonders how could a pair $(Y, \{\varphi_N\})$ generalize a triple $(X, D; \Gamma)$.

(V) In [22], a pair $(Y, \mathcal{C})$ is introduced, where Y is a topological space and $\mathcal{C}$ is a family of subsets of Y such that $(Y, \mathcal{C})$ is similar to the convexity space in the classical sense.

A pair $(X, \mathcal{C})$ is said to have the selection property with respect to a topological space S if every multimap $F : S \multimap X$ admits a single-valued continuous selection whenever F is lower semicontinuous and nonempty closed convex valued.

A pair $(Y, \mathcal{C})$ is said to satisfy H-condition if $\mathcal{C}$ has the following property:

(H) For each finite subset $\{y_0, \dots, y_n\} \subset Y$, there exists a continuous mapping $f : \Delta_N \to \overline{\text{conv}}\{y_0, \dots, y_n\}$, where Δ_N is the standard n-simplex, such that $f(\Delta_J) \subset \overline{\text{conv}}\{y_j : j \in J\}$ for each nonempty subset $J \subset N = \{0, 1, \dots, n\}$, where $\overline{\text{conv}}$ denotes the closed convex hull.

For these definitions, we note the following remarks:

(i) A pair $(Y, \mathcal{C})$ is a particular form of our abstract convex space $(E, D; \Gamma)$ with $Y = E = D$ and $\Gamma_A {:=} \text{conv}(A) = \bigcap\{B \in \mathcal{C} \mid A \subset B\}$ for $A \in \langle Y \rangle$. Then $(Y, \mathcal{C})$ becomes our abstract convex space $(Y; \Gamma)$.

(ii) The selection property would be better to call the Michael selection property.

(iii) A pair $(Y, \mathcal{C})$ satisfying the H-condition is a particular form of our G-convex space $(X, D; \Gamma)$ with $Y = X = D$ such that Γ is closed-valued.

The following new result gives an example of ϕ_A-spaces:

Theorem 4. *If an abstract convex space $(E, D; \Gamma)$ has the Michael selection property with respect to a simplex and if Γ is closed-valued, then we have a ϕ_A-space*

$$(E, D; \{\phi_A\}_{A\in\langle D\rangle}).$$

Corollary 4.1. [22, Theorem 1] *If a pair $(Y, \mathcal{C})$ has the selection property with respect to any simplex, then the pair satisfies the H-condition.*

In fact, just following the proof of [22, Theorem 1], we can easily deduce the more general Theorem 4.

Moreover, in [22], several results on the pairs $(Y, \mathcal{C})$ satisfying the H-condition are obtained. Some of such results are particular forms of known results on G-convex spaces.

5 Various KKM maps

A number of authors tried to generalize the concept of KKM maps on particular forms of ϕ_A-spaces. In this section, we show that all of them are particular forms of our KKM maps.

(I) In 2003 [20, Definition 2], for an L-space (X, Γ) and a topological space Y, a correspondence $G : Y \multimap X$ is called a *generalized KKM-correspondence*, if for all $A = \{y_0, y_1, \ldots, y_n\} \in \langle Y\rangle$, there exists a subset $B = \{x_0, x_1, \ldots, x_n\} \in \langle X\rangle$, such that for all $J \subseteq \{0, 1, \ldots, n\}$, it is satisfied that $\phi_B(\Delta_J) \subseteq \bigcup_{j\in J} G(y_j)$.

Note that a generalized KKM-correspondence becomes simply our KKM map on a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ by putting $D := Y$ and, for any $A \in \langle D\rangle$, by defining $\phi_A(\Delta_{|A|-1}) := \phi_B(\Delta_{|B|-1})$ for $B \in \langle X\rangle$ corresponding to A.

(II) In 2003 [2, Definition 2.1], for a nonempty set X and a topological space Y, $T : X \to 2^Y$ is said to be generalized relatively KKM (R-KKM) mapping if for any $N = \{x_0, x_1, \ldots, x_n\} \in \langle X\rangle$, there exists a continuous mapping $\phi_N : \Delta_n \to Y$ such that, for each $e_{i_0}, e_{i_1}, \cdots, e_{i_k}$,

$$\phi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j},$$

where Δ_k is a standard k-simplex of Δ_n with vertices $e_{i_0}, e_{i_1}, \cdots, e_{i_k}$.

For a ϕ_A-space $(Y, X; \{\phi_N\}_{N\in\langle X\rangle})$, $T : X \to 2^Y$ is simply a KKM map.

(III) Let X be a nonempty set and Y be a topological space with property (H). In 2005 [10], $T : X \to 2^Y$ is said to be a generalized R-KKM mapping if for each $\{x_0, \ldots, x_n\} \in \langle X\rangle$, there exists $N = \{y_0, \ldots, y_n\} \in \langle Y \rangle$ such that

$$\varphi_N(\Delta_k) \subset \bigcup_{j=0}^{k} Tx_{i_j},$$

for all $\{i_0, \ldots, i_k\} \subset \{0, \ldots, n\}$.

Similarly to (II), a generalized R-KKM map $T : X \to 2^Y$ is simply a KKM map for the ϕ_A-space $(Y, X; \{\phi_A\}_{A \in \langle X \rangle})$.

The author of [8] claimed as follows: "The above class of generalized *R-KKM* mappings includes those classes of *KKM* mappings, *H-KKM* mappings, *G-KKM* mappings, generalized *G-KKM* mappings, generalized *S-KKM* mappings, *GLKKM* mappings and *GMKKM* mappings defined in topological vector spaces, H-spaces, G-convex spaces, G-H-spaces, L-convex spaces and hyperconvex metric spaces, respectively, as true subclasses." This is partially incorrect.

In view of this claim and Theorem 2, so many variants of KKM type theorems in [2-10,20,22] and a large number of other papers can be reduced to the ones in our G-convex space theory. We should recognize that, in the KKM theory on G-convex spaces, every argument is related to the finite intersection property of functional values of KKM maps having closed [resp., open] values, in other words, related to some $N \in \langle D \rangle$ in $(X, D; \Gamma)$.

(IV) Motivated by a large number of recent works on generalized KKM maps, we introduced the following definition in [19]: Let (X, D, Γ) be a G-convex space and I a nonempty set. A map $F : I \multimap X$ is called a *generalized KKM map* provided that for each $N \in \langle I \rangle$, there exists a function $\sigma : N \to D$ such that $\Gamma_{\sigma(M)} \subset F(M)$ for each $M \in \langle N \rangle$.

In [19], a unified account on results for such maps was given; for example, the KKM type theorem, characterizations of such maps, an equilibrium theorem implying minimax inequalities, variational inequalities, and so on.

A little later than [19], similar results appeared in [4,5], which has trivial defects in certain aspects.

6 Various KKM type theorems

For particular forms of G-convex spaces, some authors obtained KKM type theorems or equivalents which can not be applicable even to the KKM principle for $(\Delta_n, V; \mathrm{co})$ or to the Ky Fan lemma for $(X \supset D; \mathrm{co})$, where X is a topological vector space.

In this section, we give two KKM type theorems which improve corresponding ones in [2,20]:

Theorem 5. *Let X be a topological space, D a nonempty set, and $G : D \multimap X$ a map such that*

(1) *G is transfer closed-valued [that is, $\bigcap_{z\in D}\overline{G(z)} = \bigcap_{z\in D} G(z)$];*

(2) *there exists $a^* \in Y$ with $\overline{G(a^*)}$ compact.*

Then, there exists a G-convex space $(X, D; \Gamma)$ such that G is a KKM map if and only if $\bigcap_{z\in D} G(z) \neq \emptyset$.

Proof. (Necessity) Follows from Theorem 3.

(Sufficiency) Choose an $x^* \in \bigcap_{z\in D} G(z) \neq \emptyset$. Define a map $\Gamma : \langle D\rangle \multimap X$ given by the constant function $\Gamma(A) = \{x^*\}$ for all $A \in \langle D\rangle$ with $|A| = n+1$, and a function $\phi_A : \Delta_n \to \Gamma(A)$ by $\phi_A(\lambda) = x^*$ for all $\lambda \in \Delta_n$. Then it is easy to verify that, with this G-convex space $(X, D; \Gamma)$, G is a KKM map.

Corollary 5.1. [20, Theorem 1] *Let X and Y be topological spaces and $\Gamma : Y \to X$ a transfer closed-valued correspondence on Y such that there exists $y^* \in Y$ with $cl[\Gamma(y^*)]$ compact. Then, there exists an L-structure on X such that Γ is a generalized KKM-correspondence if and only if $\bigcap_{y\in Y}\Gamma(y) \neq \emptyset$.*

Recall that several generalizations of [20, Theorem 1] already appeared in [19].

Theorem 6. *For a ϕ_A-space $(Y, D; \{\phi_N\}_{N\in\langle D\rangle})$, let $T : D \to 2^Y$ be a map such that $T(z)$ is nonempty and closed for each $z \in D$.*

(i) *If T is a KKM map, then for each $N \in \langle D\rangle$ with $|N| = n+1$,*

$$\phi_N(\Delta_n) \cap \bigcap_{x\in N} T(x) \neq \emptyset.$$

(ii) *If the family $\{T(z) \mid z \in Z\}$ has finite intersection property, then T is a KKM map.*

Proof. (i) Apply Theorem 3.

(ii) Just follow the sufficiency part of Theorem 5.

The following is the key result in [2] with almost a page proof:

Corollary 6.1. [2, Theorem 3.1] *Let X be a nonempty set and Y be a topological space. Let $T : X \to 2^Y$ be a set-valued mapping such that $T(x)$ is nonempty and compactly closed in Y for each $x \in X$.*

(i) *If T is a generalized R-KKM mapping, then for each $N = \{x_0, x_1, \ldots, x_n\} \in \langle X\rangle$,*

$$\phi_N(\Delta_n) \cap \left(\bigcap_{x\in N} Tx\right) \neq \emptyset,$$

where ϕ_N is the continuous mapping in touch with N in definition of a generalized R-KKM map-

ping.

(ii) *If the family $\{T(x) \mid x \in X\}$ has finite intersection property, then T is a generalized R-KKM mapping.*

Proof. Switch the topology of Y to its compactly generated extension [13]. Then we can eliminate 'compactly' and apply Theorem 6.

Remark. In [2], its authors used the partition of unity subordinated to a cover of $\phi_{N_0}(\Delta_n)$ which should be assumed Hausdorff. They claim that, applying their Theorem 3.1, they obtained new theorems which unify and extend many known results in recent literature. However, theirs are all disguised forms of known results and their practical applicability is doubtful.

7 Fixed points of $\mathfrak{B}$-maps

In this section, the well-known better admissible class $\mathfrak{B}$ on G-convex spaces [14] can be introduced on ϕ_A-spaces.

A ϕ_A-space $(E, D; \{\phi_A\}_{A\in\langle D\rangle})$ is an abstract convex space $(E, D; \Gamma)$, where $\Gamma_N := \phi_N(\Delta_n)$ for each $N \in \langle D\rangle$ with $|N| = n+1$, and hence there is a continuous function $\phi_N : \Delta_n \to \Gamma_N$. This $(E, D; \Gamma)$ is not necessarily a G-convex space.

Definition. Let $(E, D; \Gamma)$ be an abstract convex space, X a nonempty subset of E and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of maps from X into Y as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D\rangle$ with the cardinality $|N| = n+1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, there exists a continuous function $\phi_N : \Delta_n \to \Gamma_N$ such that the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that $\phi_N(\Delta_n)$ is a compact subset of X.

Recall that for a ϕ_A-space $(E, D; \{\phi_A\}_{A\in\langle D\rangle})$, by letting $\Gamma_N := \phi_N(\Delta_n)$, the above definition works. There are a large number of examples of $\mathfrak{B}$-maps; see [14] and references therein.

We introduce particular types of subsets of abstract convex uniform spaces adequate to establish our fixed point theory. In fact, as in [14], we introduce the Klee approximability of ranges of maps:

Definition. Let $(E, D; \{\phi_A\}_{A\in\langle D\rangle}; \mathcal{U})$ be a uniform ϕ_A-space. A subset K of E is said to be *Klee approximable* if, for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to E$ satisfying

(1) $(x, h(x)) \in U$ for all $x \in K$;

(2) $h(K) \subset \phi_N(\Delta_n)$ for some $N \in \langle D \rangle$ with $|N| = n+1$; and

(3) there exist a continuous function $p : K \to \Delta_n$ such that $h = \phi_N \circ p$.

Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K) \subset \phi_N(\Delta_n) \subset X$ for some $N \in \langle D \rangle$ in condition (2).

We have given a lot of examples of Klee approximable subsets in [14]. Now we have the following generalizations of the main result of [14]:

Theorem 7. *Let $(E, D; \{\phi_A\}_{A \in \langle D \rangle}; \mathcal{U})$ be a uniform ϕ_A-space, $X \subset Y$ subsets of E, and $F : Y \multimap Y$ a map such that $F|_X \in \mathfrak{B}(X, Y)$ and $F(X)$ is Klee approximable into X. Then F has the almost fixed point property (that is, for any $U \in \mathcal{U}$, there exist $x_U \in X$ such that $F(x_U) \cap U[x_U] \neq \emptyset$).*

Further if $(E,\mathcal{U})$ is Hausdorff, F is closed, and $\overline{F(X)}$ is compact in Y, then F has a fixed point $x_0 \in Y$ (that is, $x_0 \in F(x_0)$).

Proof. Since $K := F(X)$ is a Klee approximable into X, for each symmetric entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to X$ satisfying conditions (1) - (3) of the definition of Klee approximable subsets, and we have

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \overset{F|_{\Gamma_N}}{\multimap} K \xrightarrow{p} \Delta_n$$

for some $N \in \langle D \rangle$ with $|N| = n+1$ and $\Gamma_N := \phi_N(\Delta_n) \subset X$. Let $p' := p|_{F(\Gamma_N)}$. Since $F|_X \in \mathfrak{B}(X, Y)$, the composition $p' \circ (F|_{\Gamma_N}) \circ \phi_N : \Delta_n \multimap \Delta_n$ has a fixed point $a_U \in \Delta_n$. Let $x_U := \phi_N(a_U)$. Then

$$a_U \in (p' \circ F \circ \phi_N)(a_U) = (p' \circ F)(x_U)$$

and hence

$$x_U = \phi_N(a_U) \in (\phi_N \circ p' \circ F)(x_U).$$

Since $h = \phi_N \circ p$ by definition, we have

$$x_U = h(y_U) \quad \text{for some} \quad y_U \in (F|_{\Gamma_N})(x_U).$$

Therefore, for each entourage $U \in \mathcal{U}$, there exist points $x_U \in X$ and $y_U \in F(x_U)$ such that $(x_U, y_U) = (h(y_U), y_U) \in U$. So, for each U, there exist $x_U, y_U \in X$ such that $y_U \in F(x_U)$ and $y_U \in U[x_U]$.

Now suppose that F is closed and $\overline{F(X)}$ is compact. Since $F(X)$ is relatively compact, we may assume that the net y_U in $F(X)$ converges to some $x_0 \in \overline{F(X)}$. Since $(x_U, y_U) \in U$ for each $U \in \mathcal{U}$, by the Hausdorffness of E, the net x_U also converges to x_0. Since the graph of F is closed in $Y \times Y$ and $(x_U, y_U) \in \mathrm{Gr}(F)$, we have $(x_0, y_0) \in \mathrm{Gr}(F)$ and hence we have $x_0 \in F(x_0)$. This completes our proof.

Note that, by choosing particular subclass of multimaps or particular types of ϕ_A-spaces, we can deduce a large number of known or new fixed point theorems from Theorem 7.

`Received: December 2007. Revised: December 2007.`

References

[1] H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano and J.-V. Llinares, *Abstract convexity and fixed points*, J. Math. Anal. Appl. **222** (1998), 138–150.

[2] L. Deng and X. Xia, *Generalized R-KKM theorems in topological space and their applications*, J. Math. Anal. Appl. **285** (2003), 679–690.

[3] L. Deng and M.G. Yang, *Coincidence theorems with applications to minimax inequalities, section theorem, best approximation and multiobjective games in topological spaces*, Acta Math. Sinica, English Ser. **22** (2006), 1809–1818.

[4] X.P. Ding, *Abstract convexity and generalizations of Himmelberg type fixed-point theorems*, Comp. Math. Appl. **41** (2001), 497–504.

[5] X.P. Ding, *Generalized L-KKM type theorems in L-convex spaces with applications*, Comp. Math. Appl. **43** (2002), 1240–1256.

[6] X.P. Ding, *Maximal element theorems in product FC-spaces and generalized games*, J. Math. Anal. Appl. **305** (2005), 29–42.

[7] X.P. Ding, *Generalized KKM type theorems in FC-spaces with applications (I)*, J. Glob. Optim. **36** (2006), 581–596.

[8] X.P. Ding, *New generalized R-KKM type theorems in general topological spaces and applications*, Acta Math. Sinica, English Ser. (2006) DOI: 10.1007/s10114-005-0876-y.

[9] X.P. Ding, Y.C. Liou and J.C. Yao, *Generalized R-KKM type theorems in topological spaces with applications*, Appl. Math. Letters **18** (2005), 1345–1350.

[10] J. Huang, *The matching theorems and coincidence theorems for generalized R-KKM mapping in topological spaces*, J. Math. Anal. Appl. **312** (2005), 374–382.

[11] S. Park, *Ninety years of the Brouwer fixed point theorem*, Vietnam J. Math. **27** (1999), 193–232.

[12] S. Park, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comput. & Appl. Math. **7** (2000), 1–28.

[13] S. Park, *Remarks on topologies of generalized convex spaces*, Nonlinear Func. Anal. Appl. **5** (2000), 67–79.

[14] S. Park, *A unified fixed point theory in generalized convex spaces*, Acta Math. Sinica, English Ser. 23(8) (2007), 1509–1526.

[15] S. Park, *On generalizations of the KKM principle on abstract convex spaces*, Nonlinear Anal. Forum 11 (2006), 67–77.

[16] S. Park, *Elements of the KKM theory on abstract convex spaces*, J. Korean Math. Soc. 45(1) (2007), 1–27.

[17] S. Park, *Various subclasses of abstract convex spaces for the KKM theory*, Proc. Nat. Inst. Math. Sci. 2(4) (2007), 35–47.

[18] S. Park, *Equilibrium existence theorems in the KKM spaces*, Nonlinear Analysis (2007), doi:10.1016/j.na.2007.10.058.

[19] S. Park and W. Lee, *A unified approach to generalized KKM maps in generalized convex spaces*, J. Nonlinear Convex Anal. **2** (2001), 157–166.

[20] M.C. Sánchez, J.-V. Llinares and B. Subiza, *A KKM-result and an application for binary and non-binary choice function*, Econom. Th. **21** (2003), 185–193.

[21] B.P. Sortan, *Introduction to Axiomatic Theory of Convexity*, Kishyneff, 1984. [Russian with English summary]

[22] S.-w. Xiang and H. Yang, *Some properties of abstract convexity structures on topological spaces*, Nonlinear Analysis **67** (2007), 803–808.

CUBO A Mathematical Journal
Vol.10, $N^{\underline{o}}$04, (15–26). December 2008

A Strong Convergence Theorem by a New Hybrid Method for an Equilibrium Problem with Nonlinear Mappings in a Hilbert Space

Rinko Shinzato and Wataru Takahashi
Department of Mathematical and Computing Sciences,
Tokyo Institute of Technology,
Ohokayama, Meguro-ku, Tokyo 152-8552, Japan
emails: `shinzato.l.aa@m.is.titech.ac.jp, wataru@is.titech.ac.jp`

ABSTRACT

In this paper, we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem, the set of solutions of the variational inequality for a monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space by using a new hybrid method. Using this theorem, we obtain three new results for finding a solution of an equilibrium problem, a solution of the variational inequality for a monotone mapping and a fixed point of a nonexpansive mapping in a Hilbert space.

RESUMEN

En este artículo, probamos un teorema de convergencia fuerte para encontrar un elemento común del conjunto de soluciones de un problema de equilibrio; del conjunto de soluciones de una desigualdad variacional para una aplicación monótona y del conjunto de punto fijos de una aplicación no expansiva en un espacio de Hilbert mediante el uso de un nuevo método híbrido. Usando nuestro teorema obtenemos tres nuevos resultados para encontrar una solución de un problema de equilíbrio; una solución de la desigualdad variacional para una aplicación monótona y un punto fijo para una aplicación no expansiva en un espacio de Hilbert.

Key words and phrases: *Hilbert space, equilibrium problem, nonexpansive mapping, inverse-strongly monotone mapping, iteration, strong convergence theorem.*

Math. Subj. Class.: *47H05, 47H09, 47J25.*

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a nonempty closed convex subset of H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $f : C \times C \to \mathbb{R}$ is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \geq 0 \tag{1.1}$$

for all $y \in C$. The set of such solutions $\hat{x}$ is denoted by $EP(f)$. The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncoopetative games and others; see, for instance, [1] and [6]. A mapping S of C into H is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. A mapping $A : C \to H$ is called inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$. The variational inequality problem is to find a $u \in C$ such that

$$\langle v - u, Au \rangle \geq 0 \tag{1.2}$$

for all $v \in C$. The set of such solutions u is denoted by $VI(C, A)$. Setting $A = I - S$, where $S : C \to H$ is nonexpansive, we have from [14] that $A : C \to H$ is a $\frac{1}{2}$-inverse-strongly monotone mapping. Recently, Tada and Takahashi [9, 10] and Takahashi and Takahashi [11] obtained weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. In particular, Tada and Takahashi [10] established a strong convergence theorem for finding a common element of such two sets by using the hybrid method introduced in Nakajo and Takahashi [7]. On the other hand, Takahashi and Toyoda [16] introduced an iterative method for finding a common element of the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping. Very recently, Takahashi, Takeuchi and Kubota [15] proved the following theorem by a new hybrid method which is different from Nakajo and Takahashi's hybrid method. We call such a method the shrinking projection method.

Theorem 1.1 (Takahashi, Takeuchi and Kubota [15])**.** *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n \leq a < 1$. Then, $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$, where $P_{F(T)}$ is the metric projection of H onto $F(T)$.

In this paper, motivated by Tada and Takahashi [10], Takahashi and Toyoda [16], and Takahashi, Takeuchi and Kubota [15], we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem, the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space by using the shrinking projection method. Using this theorem, we obtain three new results for finding a solution of an equilibrium problem, a solution of the variational inequality for an inverse-strongly monotone mapping and a fixed point of a nonexpansive mapping in a Hilbert space.

2 Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote by "$\to$" strong convergence and by "$\rightharpoonup$" weak convergence. We know from [14] that, for all $x, y \in H$ and $\lambda \in [0, 1]$, there holds

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2.$$

Let C be a nonempty closed convex subset of H. For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|$$

for all $y \in C$. P_C is called the metric projection of H onto C. We know that P_C satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \tag{2.1}$$

for all $x, y \in H$. Further, we have that

$$\langle x - P_C x, P_C x - y \rangle \geq 0 \tag{2.2}$$

for all $x \in H$ and $y \in C$. A mapping $A : C \to H$ is called inverse-strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2$$

for all $x, y \in C$. The set of solutions of the variational inequality for A is denoted by $VI(C, A)$. We know that, for all $\lambda > 0$,

$$u \in VI(C, A) \iff u = P_C(u - \lambda Au).$$

We also know that, for any λ with $0 < \lambda \leq 2\alpha$, a mapping $I - \lambda A : C \to H$ is nonexpansive; see [16, 14] for more details. It is also known that H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. A Hilbert space H also has the Kadec-Klee property, i.e., if $\{x_n\}$ is a sequence of H with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then there holds $x_n \to x$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be an inverse-strongly monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w \rangle \geq 0, \ \forall u \in C\}$, and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$; see [8].

For solving an equilibrium problem for a bifunction $f : C \times C \to \mathbb{R}$, let us assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e. $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,
$$\limsup_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$

(A4) for all $x \in C$, $f(x, \cdot)$ is convex and lower semicontinuous.

The following lemma appears implicitly in Blum and Oettlli [1].

Lemma 2.1 (Blum and Oettli). *Let C be a nonempty closed convex subset of H and let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying $(A1) - (A4)$. Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0 \ \textit{for all} \ y \in C.$$

The following lemma was also given in [2].

Lemma 2.2. *Assume that* $f : C \times C \to \mathbb{R}$ *satisfies* $(A1) - (A4)$. *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C \right\}$$

for all $x \in H$. *Then, the following hold:*

(1) T_r *is single-valued;*

(2) T_r *is a firmly nonexpansive mapping, i.e., for all* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(f)$;

(4) $EP(f)$ *is closed and convex.*

3 Strong convergence theorem

In this section, using the shrinking projection method, we prove a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem, the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* f *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *satisfying* $(A1) - (A4)$ *and let* S *be a nonexpansive mapping from* C *into* H *and let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H *such that* $F(S) \cap VI(C, A) \cap EP(f) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence in* C *generated by* $x_0 = x \in C$, $C_0 = C$ *and*

$$\begin{cases} u_n = T_{r_n}(x_n), \\ y_n = \alpha_n x_n + (1 - \alpha_n) S P_C(u_n - \lambda_n A u_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $0 \leq \alpha_n \leq c < 1, \quad 0 < d \leq r_n < \infty$ *and* $0 < a \leq \lambda_n \leq b < 2\alpha$. *Then,* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap VI(C,A)\cap EP(f)} x$.

Proof. From [7], we know that

$$\begin{aligned} &\|y_n - z\| \leq \|x_n - z\| \\ \Longleftrightarrow &\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z\rangle \leq 0. \end{aligned}$$

So, C_n is a closed convex subset of H for all $n \in \mathbb{N} \cup \{0\}$. Next we show by mathematical induction that $F(S) \cap VI(C, A) \cap EP(f) \subset C_n$ for all $n \in \mathbb{N} \cup \{0\}$. Put $z_n = P_C(u_n - \lambda_n A u_n)$ for all $n \in \mathbb{N} \cup \{0\}$. From $C_0 = C$, we have

$$F(S) \cap VI(C, A) \cap EP(f) \subset C_0.$$

Suppose that $F(S) \cap VI(C, A) \cap EP(f) \subset C_k$ for some $k \in \mathbb{N} \cup \{0\}$. Let $u \in F(S) \cap VI(C, A) \cap EP(f)$. Since $I - \lambda_k A$ and T_{r_k} are nonexpansive and $u = P_C(u - \lambda_k A u)$, we have

$$\begin{aligned}
\|z_k - u\| &= \|P_C(u_k - \lambda_k A u_k) - P_C(u - \lambda_k A u)\| \\
&\le \|(I - \lambda_k A)u_k - (I - \lambda_k A)u\| \\
&\le \|u_k - u\| \\
&= \|T_{r_k} x_k - T_{r_k} u\| \\
&\le \|x_k - u\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|y_k - u\| &= \|\alpha_k x_k + (1 - \alpha_k) S z_k - u\| \\
&\le \alpha_k \|x_k - u\| + (1 - \alpha_k)\|S z_k - u\| \\
&\le \alpha_k \|x_k - u\| + (1 - \alpha_k)\|z_k - u\| \\
&\le \alpha_k \|x_k - u\| + (1 - \alpha_k)\|x_k - u\| \\
&= \|x_k - u\|.
\end{aligned}$$

Since $u \in C_k$, we have $u \in C_{k+1}$. This implies that

$$F(S) \cap VI(C, A) \cap EP(f) \subset C_n$$

for all $n \in \mathbb{N} \cup \{0\}$. So, $\{x_n\}$ is well-defined.

From the definition of x_{n+1}, we have

$$\|x_{n+1} - x\| \le \|u - x\|$$

for all $u \in F(S) \cap VI(C, A) \cap EP(f) \subset C_{n+1}$. Then, $\{x_n\}$ is bounded. Therefore, $\{y_n\}$, $\{z_n\}$, $\{u_n\}$ and $\{S z_n\}$ are also bounded.

Let us show that $\|x_{n+1} - x_n\| \to 0$. From $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = P_{C_n} x$, we have

$$\|x_n - x\| \le \|x_{n+1} - x\|$$

for all $n \in \mathbb{N} \cup \{0\}$. Thus $\{\|x_n - x\|\}$ is nondecreasing. Thus $\lim_{n\to\infty} \|x_n - x\|$ exists. Since

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x\|^2 + \|x_n - x\|^2 + 2\langle x_{n+1} - x, x - x_n\rangle \\
&= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2\langle x_n - x_{n+1}, x - x_n\rangle \\
&\le \|x_{n+1} - x\|^2 - \|x_n - x\|^2
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$, we have $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Since $x_{n+1} \in C_{n+1}$, we have

$$\|x_n - y_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \le 2\|x_n - x_{n+1}\|.$$

This together with $\|x_{n+1} - x_n\| \to 0$ implies that

$$\|x_n - y_n\| \to 0.$$

We also show that $\|Au_n - Au\| \to 0$. For all $u \in F(S) \cap VI(C, A) \cap EP(f)$, we have

$$\begin{aligned}
\|z_n - u\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(u - \lambda_n Au)\|^2 \\
&\le \|(u_n - \lambda_n Au_n) - (u - \lambda_n Au)\|^2 \\
&= \|u_n - u - \lambda_n(Au_n - Au)\|^2 \\
&= \|u_n - u\|^2 - 2\lambda_n\langle u_n - u, Au_n - Au\rangle + \lambda_n^2\|Au_n - Au\|^2 \\
&\le \|u_n - u\|^2 - 2\lambda_n\alpha\|Au_n - Au\|^2 + \lambda_n^2\|Au_n - Au\|^2 \\
&= \|u_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Au_n - Au\|^2 \\
&\le \|u_n - u\|^2 + a(b - 2\alpha)\|Au_n - Au\|^2.
\end{aligned}$$

Since $\|\cdot\|^2$ is convex and $\|u_n - u\| \le \|x_n - u\|$, we have

$$\begin{aligned}
\|y_n - u\|^2 &\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\|Sz_n - u\|^2 \\
&\le \alpha_n\|x_n - u\|^2 + (1 - \alpha_n)\{\|u_n - u\|^2 + a(b - 2\alpha)\|Au_n - Au\|^2\} \\
&\le \|x_n - u\|^2 + a(b - 2\alpha)\|Au_n - Au\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-a(b - 2\alpha)\|Au_n - Au\|^2 &\le \|x_n - u\|^2 - \|y_n - u\|^2 \\
&= (\|x_n - u\| + \|y_n - u\|)(\|x_n - u\| - \|y_n - u\|) \\
&\le (\|x_n - u\| + \|y_n - u\|)\|x_n - y_n\|.
\end{aligned}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded and $\|x_n - y_n\| \to 0$, we obtain $\|Au_n - Au\| \to 0$. Further we show that $\|z_n - u_n\| \to 0$. For all $u \in F(S) \cap VI(C, A) \cap EP(f)$, we have from (2.1) that

$$\begin{aligned}
\|z_n - u\|^2 &= \|P_C(u_n - \lambda_n Au_n) - P_C(u - \lambda_n Au)\|^2 \\
&\le \langle (u_n - \lambda_n Au_n) - (u - \lambda_n Au), z_n - u\rangle \\
&= \frac{1}{2}\{\|(u_n - \lambda_n Au_n) - (u - \lambda_n Au)\|^2 + \|z_n - u\|^2 \\
&\qquad - \|(u_n - \lambda_n Au_n) - (u - \lambda_n Au) - (z_n - u)\|^2\} \\
&\le \frac{1}{2}\{\|u_n - u\|^2 + \|z_n - u\|^2 - \|(u_n - z_n) - \lambda_n(Au_n - Au)\|^2\} \\
&= \frac{1}{2}\{\|u_n - u\|^2 + \|z_n - u\|^2 - \|u_n - z_n\|^2 \\
&\qquad + 2\lambda_n\langle u_n - z_n, Au_n - Au\rangle - \lambda_n^2\|Au_n - Au\|^2\},
\end{aligned}$$

and hence

$$\|z_n - u\|^2 \leq \|u_n - u\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\langle u_n - z_n, Au_n - Au\rangle.$$

From this inequality and $\|u_n - u\| \leq \|x_n - u\|$, we have

$$\begin{aligned}\|y_n - u\|^2 &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|z_n - u\|^2\\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\{\|u_n - u\|^2 - \|u_n - z_n\|^2\\ &\qquad + 2\lambda_n\langle u_n - z_n, Au_n - Au\rangle\}\\ &\leq \|x_n - u\|^2 - (1-\alpha_n)\|u_n - z_n\|^2\\ &\qquad + 2\lambda_n(1-\alpha_n)\langle u_n - z_n, Au_n - Au\rangle,\end{aligned}$$

and hence

$$\begin{aligned}(1-\alpha_n)\|u_n - z_n\|^2 &\leq \|x_n - u\|^2 - \|y_n - u\|^2\\ &\qquad + 2\lambda_n(1-\alpha_n)\langle u_n - z_n, Au_n - Au\rangle\\ &\leq (\|x_n - u\| + \|y_n - u\|)\|x_n - y_n\|\\ &\qquad + 2\lambda_n(1-\alpha_n)\langle u_n - z_n, Au_n - Au\rangle.\end{aligned}$$

Since $0 \leq \alpha_n \leq c < 1$, $\|x_n - y_n\| \to 0$ and $\|Au_n - Au\| \to 0$, we have that

$$\|u_n - z_n\| \to 0.$$

Let us show $\|x_n - u_n\| \to 0$. For all $u \in F(S) \cap VI(C,A) \cap EP(f)$, we have from Lemma 2.2 and $F(T_{r_n}) = EP(f)$ that

$$\begin{aligned}\|u_n - u\|^2 = \|T_{r_n}x_n - T_{r_n}u\|^2 &\leq \langle T_{r_n}x_n - T_{r_n}u, x_n - u\rangle\\ &= \langle u_n - u, x_n - u\rangle\\ &= \frac{1}{2}\{\|u_n - u\|^2 + \|x_n - u\|^2 - \|u_n - x_n\|^2\},\end{aligned}$$

and hence

$$\|u_n - u\|^2 \leq \|x_n - u\|^2 - \|u_n - x_n\|^2.$$

From this inequality and $\|z_n - u\| \leq \|u_n - u\|$, we have

$$\begin{aligned}\|y_n - u\|^2 &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|z_n - u\|^2\\ &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\{\|x_n - u\|^2 - \|u_n - x_n\|^2\},\end{aligned}$$

and hence

$$(1-\alpha_n)\|u_n - x_n\|^2 \leq \|x_n - u\|^2 - \|y_n - u\|^2 \leq (\|x_n - u\| + \|y_n - u\|)\|x_n - y_n\|.$$

Therefore, we obtain

$$\|u_n - x_n\| \to 0.$$

Since $(1-\alpha_n)(Sz_n - z_n) = \alpha_n(z_n - x_n) + (y_n - z_n)$, we have

$$\begin{aligned}(1-\alpha_n)\|Sz_n - z_n\| &\le \|z_n - x_n\| + \|y_n - z_n\| \\ &\le \|z_n - x_n\| + \|y_n - x_n\| + \|x_n - z_n\| = 2\|z_n - x_n\| + \|y_n - x_n\| \\ &\le 2(\|z_n - u_n\| + \|u_n - x_n\|) + \|y_n - x_n\|.\end{aligned}$$

Therefore, we also obtain $\|Sz_n - z_n\| \to 0$.

Since $\{z_n\}$ is bounded, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that $z_{n_i} \rightharpoonup z_0$. Then, we can obtain that $z_0 \in F(S) \cap VI(C,A) \cap EP(f)$. In fact, let us first show $z_0 \in F(S)$. Assume that $z_0 \notin F(S)$. By Opial's condition,

$$\begin{aligned}\liminf_{i\to\infty} \|z_{n_i} - z_0\| < \liminf_{i\to\infty} \|z_{n_i} - Sz_0\| &= \liminf_{i\to\infty} \|z_{n_i} - Sz_{n_i} + Sz_{n_i} - Sz_0\| \\ &= \liminf_{i\to\infty} \|Sz_{n_i} - Sz_0\| \\ &\le \liminf_{i\to\infty} \|z_{n_i} - z_0\|.\end{aligned}$$

This is a contradiction. Therefore, we have $z_0 \in F(S)$. Let us show $z_0 \in VI(C,A)$. Define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = VI(C,A)$; see [8]. Let $(v,u) \in G(T)$. Since $u - Av \in N_C v$ and $z_n = P_C(u_n - \lambda_n Au_n) \in C$, we have $\langle v - z_n, u - Av\rangle \ge 0$. By the definition of z_n, we also have

$$\langle v - z_n, z_n - (u_n - \lambda_n Au_n)\rangle \ge 0,$$

and hence

$$\langle v - z_n, \frac{z_n - u_n}{\lambda_n} + Au_n\rangle \ge 0.$$

Therefore,

$$\begin{aligned}\langle v - z_{n_i}, u\rangle &\ge \langle v - z_{n_i}, Av\rangle \\ &\ge \langle v - z_{n_i}, Av - \{\frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}} + Au_{n_i}\}\rangle \\ &= \langle v - z_{n_i}, Av - Az_{n_i}\rangle + \langle v - z_{n_i}, Az_{n_i} - Au_{n_i}\rangle - \langle v - z_{n_i}, \frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\rangle \\ &\ge -\|v - z_{n_i}\|\|Az_{n_i} - Au_{n_i}\| - \|v - z_{n_i}\|\|\frac{z_{n_i} - u_{n_i}}{\lambda_{n_i}}\|.\end{aligned}$$

Since $\|z_n - u_n\| \to 0$ and A is Lipschits continuous, we have $\langle v - z_0, u\rangle \ge 0$. Since T is maximal monotone, we have $z_0 \in T^{-1}0$ and hence $z_0 \in VI(C,A)$.

Finally, we show that $z_0 \in EP(f)$. By $u_n = T_{r_n} x_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0$$

for all $y \in C$. From $(A2)$ we also have

$$\frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq f(y,u_n)$$

and hence

$$\langle y-u_{n_i}, \frac{u_{n_i}-x_{n_i}}{r_{n_i}}\rangle \geq f(y,u_{n_i}).$$

Since $\|u_n-z_n\| \to 0$ and $z_{n_i} \rightharpoonup z_0$, we have $u_{n_i} \rightharpoonup z_0$. Since $0<d\leq r_n<\infty$ and $\|u_n-x_n\|\to 0$, we have from $(A4)$ that $0 \geq f(y,z_0)$ for all $y\in C$. For $t\in(0,1]$ and $y\in C$, let $y_t = ty+(1-t)z_0$. Since $y\in C$ and $z_0\in C$, we have $y_t\in C$ and hence $f(y_t,z_0)\leq 0$. So, from $(A1)$ and $(A4)$ we have

$$0=f(y_t,y_t)\leq tf(y_t,y)+(1-t)f(y_t,z_0)\leq tf(y_t,y)$$

and hence $0\leq f(y_t,y)$. From $(A3)$, we have $0\leq f(z_0,y)$ for all $y\in C$ and hence $z_0\in EP(f)$. Therefore $z_0\in F(S)\cap VI(C,A)\cap EP(f)$.

From $z' = P_{F(S)\cap VI(C,A)\cap EP(f)}x$, $z_0\in F(S)\cap VI(C,A)\cap EP(f)$ and $\|x_n-x\|\leq\|z'-x\|$, we have

$$\begin{aligned}\|z'-x\| \leq \|z_0-x\| &\leq \liminf_{i\to\infty}\|z_{n_i}-x\| \\ &\leq \limsup_{i\to\infty}\|z_{n_i}-x\| \\ &\leq \limsup_{i\to\infty}\{\|z_{n_i}-u_{n_i}\|+\|u_{n_i}-x_{n_i}\|+\|x_{n_i}-x\|\} \\ &\leq \|z'-x\|.\end{aligned}$$

Thus, we have

$$\lim_{i\to\infty}\|z_{n_i}-x\| = \|z_0-x\| = \|z'-x\|.$$

This implies $z_0 = z'$. Further, since a Hilbert space has the Kadec-Klee property, we have that $z_{n_i}\to z'$. From $\|z_n-x_n\|\to 0$, we also have $x_{n_i}\to z'$. Therefore, $x_n\to z'$. This completes the proof. □

4 Applications

In this section, using Theorem 3.1, we prove three new results for finding a solution of an equilibrium problem, a solution of the variational inequality for an inverse-strongly monotone mapping and a fixed point of a nonexpansive mapping in a Hilbert space. First, we obtain a result for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ and let S be a nonexpansive mapping from C*

into H such that $F(S) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_0 = x \in C$, $C_0 = C$ and

$$\begin{cases} u_n = T_{r_n}(x_n), \\ y_n = \alpha_n x_n + (1-\alpha_n)S(u_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $0 \leq \alpha_n \leq c < 1$ and $0 < d \leq r_n < \infty$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap EP(f)}x$.

Proof. Putting $A = 0$ in Theorem 3.1, we obtain the desired result. □

Next, we obtain a result for finding a common element of the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for an inverse-strongly monotone mapping in a Hilbert space.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and let A be an α-inverse-strongly monotone mapping of C into H such that $VI(C, A) \cap EP(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{cases} u_n = T_{r_n}(x_n), \\ y_n = \alpha_n x_n + (1-\alpha_n)P_C(u_n - \lambda_n A u_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $0 \leq \alpha_n \leq c < 1$, $\quad 0 < d \leq r_n < \infty$ and $0 < a \leq \lambda_n \leq b < 2\alpha$. Then, $\{x_n\}$ converges strongly to $P_{VI(C,A)\cap EP(f)}x$.

Proof. Putting $S = I$ in Theorem 3.1, we obtain the desired result. □

Finally, we obtain a result for finding a common element of the set of solutions of the variational inequality for an inverse-strongly monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let S be a nonexpansive mapping from C into H and let A be an α-inverse-strongly monotone mapping of C into H such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C generated by $x_0 = x \in C$, $C_0 = C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n A x_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x, \quad n \in \mathbb{N} \cup \{0\}, \end{cases}$$

where $0 \leq \alpha_n \leq c < 1$ and $0 < a \leq \lambda_n \leq b < 2\alpha$. Then, $\{x_n\}$ converges strongly to $P_{F(S)\cap VI(C,A)}x$.

Proof. Putting $f = 0$ in Theorem 3.1, we obtain the desired result. □

Received: January 2008. Revised: February 2008.

References

[1] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student, **63** (1994), 123–145.

[2] P.L. Combettes and S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal., **6** (2005), 117–136.

[3] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc., **73** (1967), 957–961.

[4] W.R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., **4** (1953), 506–510.

[5] A. Moudafi, *Second-order differential proximal methods for equilibrium problems*, J. Inequal. Pure Appl. Math., **4** (2003), art. 18.

[6] A. Moudafi and M. Thera, *Proximal and dynamical approaches to equilibrium problems*, Lecture Notes in Economics and Mathematical Systems, Springer, **477** (1999), pp. 187–201.

[7] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl., **279** (2003), 372–379.

[8] R.T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc., **149** (1970), 75–88.

[9] A. Tada and W. Takahashi, *Strong convergence theorem for an equilibrium problem and a nonexpansive mapping*, in: W. Takahashi and T. Tanaka (Eds.), Nonlinear Analysis and Convex Analysis, Yokohama Publishers, Yokohama, 2007, pp. 609–617.

[10] A. Tada and W. Takahashi, *Weak and Strong convergence theorems for a nonexpansive mapping and an equilibrium problem*, J. Optim. Theory Appl., **133** (2007), 359–370.

[11] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl., **331** (2007), 506–515.

[12] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[13] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000 (Japanese).

[14] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2005 (Japanese).

[15] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl., **341** (2008), 276–286.

[16] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl., **118** (2003), 417–428.

[17] W. Takahashi and K. Zembayashi, *Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces*, Nonlinear Anal., to appear.

[18] R. Wittmann, *Approsimation of fixed points of nonexpansive mappings*, Arch. Math., **58** (1992), 486–461.

CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$04, (27–43). December 2008

Simple Fixed Point Theorems on Linear Continua

Jan Andres*, Karel Pastor* and Pavla Šnyrychová*
Dept. of Math. Analysis, Faculty of Science,
Palacký University, Tomkova 40,
779 00 Olomouc-Hejčín, Czech Republic
emails: `andres@inf.upol.cz, pastor@inf.upol.cz, snyrychp@inf.upol.cz`

ABSTRACT

A simple fixed point theorem is formulated for multivalued maps with a connected graph on closed intervals of linear continua. These intervals either cover themselves or are concerned with self-maps. We discuss a question when the original map can possess a fixed point, provided the same assumptions are satisfied only for some of its iterate. We are particularly interested in a situation on noncompact connected linearly ordered spaces. Many illustrating examples are supplied.

RESUMEN

Un teorema simple de punto fijo es formulado para aplicaciones multivaluadas con gráfico conexo sobre intervalos cerrados de un linear contínuo. Estos intervalos cubrem ellos mismos o son relacionados con auto-aplicaciones. Discutimos cuando la aplicación original puede poseer un punto fijo, con tal que las mismas condiciones sean satisfechas solamente para algunos de sus iterados. Nosostros estamos particurlamente interesados en una situación sobre espacios linealmente ordenados conexos no compactos. Ejemplos son exhibidos.

---

*Supported by the Council of Czech Goverment (MSM 6198959214).

Key words and phrases: *Fixed point, periodic orbit, linear continuum, multivalued map, iterates, connected graph, period implications.*

Math. Subj. Class.: *37E05, 47H04, 47H10, 58C30.*

1. Introduction (Fixed point theorems in low dimensions)

The celebrated Sharkovskii cycle coexistence theorem is, roughly speaking, based on many times repeated *Bolzano's intermediate value theorem* (cf. [Sh]) which in turn *is equivalent with a one-dimensional version of the Brouwer fixed point theorem* (cf. [B1], p. 273). Since Bernard Bolzano proved his statement already in 1817, the intermediate value theorem can be regarded in a certain sense as probably the oldest fixed point theorem at all.

More precisely, if a closed interval covers continuously itself, then there exists a subinterval which is mapped onto the image of the given closed interval (endpoints onto the endpoints), and the application of Bolzano's theorem to the restriction on this subinterval leads to the existence of a fixed point. Moreover, a related periodic point theorem can be regarded as such a fixed point theorem applied to the iterate, but the minimal period must be still guaranteed which follows from the fact that the interiors of the subintervals are mapped onto the interiors.

Although there exists an n-dimensional version of the intermediate value theorem due to H. Poincaré which was shown after more than fifty years to be equivalent with the Brouwer fixed point theorem by C. Miranda (for more details, see e.g. [B1], p. 273), a *closed square covering continuously itself need not contain a fixed point* (see Example 1 and cf. [Ka], [Kl]).

Example 1. Letting $f(x,y) := (1-x-y, 2-2x)$ and $A := [0,1]^2$, f is obviously continuous on A and $A \subset f(A) = [-1,1] \times [0,2]$. Fixed points of f must satisfy the system $1-x-y=x$, $2-2x=y$ which is equivalent with finding the intersection points of lines $y_1 = 1-2x$ and $y_2 = 2-2x$ inside the square A.

Since there are evidently no intersections inside A (see Fig. 1), f is fixed point free in A.

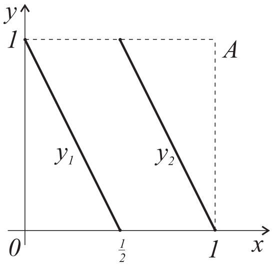

Figure 1: *Lines y_1 and y_2 from Example 1.*

The additional conditions imposed on given maps which assert fixed points are rather drastic (cf. [A2], [Sn]). That is also why *Sharkovskii's theorem holds in principle only in one dimension.*

For multivalued maps, the situation is much more delicate. B. O'Neil gave in 1947 an example (see e.g. [Mi], p. 6) of a continuous fixed point free mapping whose values are homeomorphic to S^1 which sends a closed ball in the plane into itself. J. Jezierski constructed in 1987 (see e.g. [G2], pp. 249-250) a continuous fixed point free map whose values are 1, 2 or 3 points, again on a closed ball in the plane. One can easily find only upper semicontinuous maps whose values are 1 or a fixed number $n \in \mathbb{N}$ of points which are fixed point free on closed intervals (for $n = 2$, see Example 2).

Example 2. The upper semicontinuous map (observe that its graph is closed)

$$\varphi(x) := \begin{cases} 1, & \text{for } x \in \left[0, \frac{1}{2}\right), \\ \{0,1\}, & \text{for } x = \frac{1}{2}, \\ 0, & \text{for } x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

is evidently fixed point free on the unit iterval (see Fig. 2).

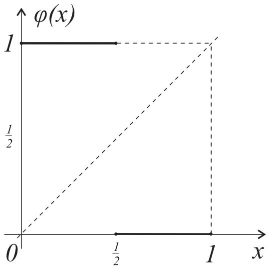

Figure 2: *Function φ from Example 2.*

In spite of these counter-examples, there exists a fixed point theory for multivalued maps with discontinuous values, of course under suitable regularity assumptions (see [Dz], [G1], [G2], [Sk], and the references therein).

Although the conjecture posed in [G1] by L. Górniewicz that the Brouwer fixed point theorem holds for Borsuk-continuous maps (for the definition, see e.g. [G2]) with compact connected values was recently answered in a negative way on B^4 by D. Miklaszewski [Mi], the same author proved that (even a weaker) *notion of Hausdorff-continuity implies for maps with ANR-values a fixed point, on any finite-dimensional ball.* In particular, for (Hausdorff-) continuous n-valued (n fixed) maps, a similar result was already achieved in 1984 by H. Schirmer on finite polyhedra (see [S1]). For n-valued maps, R. F. Brown obtained the Anosov theorem on the circle, namely that the well-defined Nielsen and Lefschetz numbers are absolutely equal (for more details, see [B2], [B3]). Let us note that E. Kudryavtseva (Moscow State University) disproved the Anosov property for 2-maps on the two-dimensional torus after its conjecturing by R. F. Brown during his talk at the conference TTFPP 2007 in Polish Będlewo.

Hence, *in one dimension* (which is sufficient for our needs here), *the Brouwer theorem holds at least for continuous maps whose values are finite unions of closed* (possibly degenerate) *intervals.* For upper semicontinuous maps with closed connected values, a special case of the well-known Kakutani theorem applies (cf. [AG], [G2]), while fixed point theorems for those with disconnected values must be formulated in a more sophisticated way (cf. [Dz], [G2], [Sk]).

If a closed interval only covers itself, then one can easily check that the same assumptions are insufficient for the existence of a fixed point. For instance, continuous 2-point maps are often fixed point free (see Example 3).

Example 3. Since $\varphi(x) = f_1(x) \cup f_2(x)$, where $f_1(x) := x - \frac{1}{2}$ and $f_2(x) := x + \frac{1}{2}$, φ is a continuous 2-point map such that $[-\frac{1}{2}, \frac{1}{2}] \subset \varphi([-\frac{1}{2}, \frac{1}{2}]) = [-1, 1]$, but there are evidently (see Fig. 3) no fixed points of φ on $[-\frac{1}{2}, \frac{1}{2}]$.

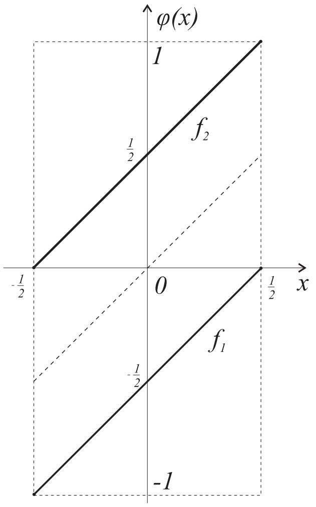

Figure 3: *Function φ from Example 3.*

Thus, some additional connectivity restrictions should be imposed (see Theorem 1 below).

The main purpose of our paper is to discuss possible improvements and consequences of Theorem 1, especially in terms of the iterates of given maps. This will be done, including the formulation of Theorem 1, in Section 3. Before we recall auxiliary definitions. Concluding remarks concern some further possibilities, higher-dimensional analogies and open problems.

2. Auxiliary definitions

In the entire text, all topological spaces will be Hausdorff and all multivalued maps will have nonempty values, i.e. by $\varphi : X \multimap Y$, we mean $\varphi : X \to 2^Y \setminus \{\emptyset\}$. By a *fixed point* of φ, we mean a point $x_0 \in X \cap Y$ such that $x_0 \in \varphi(x_0)$. By a *k-orbit* of φ, we mean a sequence $\{x_1, \dots, x_k\}$ such that

(i) $x_{i+1} \in \varphi(x_i)$, for all $i = 1, \dots, k-1$, $x_1 \in \varphi(x_k)$, and

(ii) the orbit is not a product orbit formed by going p-times around a shorter m-orbit, where $mp = k$.

We say that a linearly ordered set $\mathbb{L}$ with more than one point is a *linear continuum* (cf. [Mu], [S2]) if

(i) $\mathbb{L}$ has the least upper bound property,

(ii) $\mathbb{L}$ is ordered densely, i.e. if $x < y$, then there exists z so that $x < z < y$,

(iii) $\mathbb{L}$ is endowed with the order topology by which $\mathbb{L}$ becomes a topological (Hausdorff) space.

It is well-known (see e.g. [AS]) that *connected linearly ordered topological spaces* can be fully characterized by conditions (i)-(iii). The typical examples of linear continua are the real line, its intervals of all types, the long line, the unit square in the dictionary order, etc.

By *intervals* (a, b) or $[a, b]$ of $\mathbb{L}$, where $a, b \in \mathbb{L}$, we understand the sets $\{x \in \mathbb{L} : a < x < b\}$ or $\{x \in \mathbb{L} : a \leq x \leq b\}$, respectively.

Let us also recall that a multivalued map $\varphi : X \multimap Y$ is *upper* (*lower*) *semicontinuous* if $\varphi^{-1}(U) := \{x \in X : \varphi(x) \subset U\}$ is open (closed) in X, for every open (closed) subset U of Y, or equivalently, if $\varphi_+^{-1}(U) := \{x \in X : \varphi(x) \cap U \neq \emptyset\}$ is closed (open) in X, for every closed (open) subset U of Y. The map φ is *continuous* if it is both upper and lower semicontinuous.

We call multivalued maps with compact connected values which are upper semicontinuous or lower semicontinuous or continuous as *M-maps* or *N-maps* or *S-maps*, respectively (cf. [AFP]).

It is well-known (see e.g. [AG], [G2]) that, for compact-valued maps, the notions of continuity and Hausdorff continuity (i.e. the continuity w.r.t. the Hausdorff metric) coincide and that compact maps are upper semicontinuous if and only if their graph is a closed set.

We say that a multivalued mapping $\varphi : \mathbb{L} \multimap \mathbb{L}$ is determined by a *connectivity relation* in $\mathbb{L}^2$ if its graph Γ_φ, restricted to every interval $I \subset \mathbb{L}$, i.e. $\Gamma_{\varphi\restriction_I}$, is connected, for every $I \subset \mathbb{L}$ (including intervals degenerated to one point).

Lemma 1. *The mapping $\varphi : \mathbb{L} \multimap \mathbb{L}$ is determined by a connectivity relation in $\mathbb{L}^2$ if and only if φ has a connected graph and connected values.*

Proof. It suffices to show that if $\varphi : \mathbb{L} \multimap \mathbb{L}$ has a connected graph and connected values, then it it is determined by a connectivity relation in $\mathbb{L}^2$, because the reverse implication follows directly from the definition.

Consider an arbitrary closed interval $I = [a, b] \subset \mathbb{L}$.

At first, we show that the set $M_b^{\bullet} := \{(x, y) \in \Gamma_{\varphi} : x \leq b\}$ is connected. We suppose that there exist nonempty sets A and B such that $M_b^{\bullet} = A \cup B$ and

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset. \tag{1}$$

Considering the set $M^b := \{(x, y) \in \Gamma_{\varphi} : x > b\}$, we have

$$\Gamma_{\varphi} = M_b^{\bullet} \cup M^b = (A \cup M^b) \cup B.$$

Since Γ_{φ} is connected, either $\overline{A \cup M^b} \cap B \neq \emptyset$ or $(A \cup M^b) \cap \overline{B} \neq \emptyset$. Because of formula (1) and the form of M^b, we can find $y_0 \in \varphi(b)$ such that $y_0 \in (\overline{M^b} \cap B)$. Therefore, since $\varphi(b)$ is connected, formula (1) implies that

$$\overline{M^b} \cap A = M^b \cap \overline{A} = \emptyset.$$

Thus, using formula (1) again, we obtain

$$\overline{(M^b \cup B)} \cap A = (M^b \cup B) \cap \overline{A} = \emptyset,$$

but it is a contradiction, because $\Gamma_{\varphi} = M^b \cup B \cup A$ is connected.

We can show in an analogous way that the set $\{(x, y) \in M_b^{\bullet} : x \geq a\}$, i.e. $\Gamma_{\varphi\restriction_I} := \{(x, y) \in \Gamma_{\varphi} : a \leq x \leq b\}$, is connected, too.

Since any interval $J \subset \mathbb{L}$ can be expressed as the union of closed intervals of $\mathbb{L}$ that have a point in common, the graph $\Gamma_{\varphi\restriction_J}$ can be expressed, in view of the above conclusions, as the union of connected sets of $\mathbb{L}^2$ that have a point in common. This is sufficient (see e.g. [Mu, p. 150]) in order the graph $\Gamma_{\varphi\restriction_J}$ to be connected which completes the proof. □

The map φ is determined by a *G_δ-relation* in $\mathbb{L}^2$ if its graph Γ_{φ} is a G_δ-subset of $\mathbb{L}^2$, i.e. if $\Gamma_{\varphi} = \bigcap_{m \in \mathbb{N}} G_m$, where all $G_m \subset \mathbb{L}^2$ are open. Map φ is determined by a *connectivity G_δ-relation* if it has both the above properties.

In [ASS] resp. [AFP], we have shown that M-maps resp. N-maps in $\mathbb{R}$ are determined by connectivity G_δ-relations in $\mathbb{R}^2$.

3. Statements on linear continua

The following fixed point theorem is intuitively obvious (for $\mathbb{L} = \mathbb{R}$, cf. Lemma 2.4 in [ASS]).

Theorem 1. *Let $I = [a, b] \subset \mathbb{L}$ be a closed interval of a linear continuum $\mathbb{L}$ and $\varphi : I \multimap \mathbb{L}$ be a multivalued mapping with a connected graph. Assume that either $I \subset \varphi(I)$ or $\varphi(I) \subset I$. Then φ has a fixed point in I.*

Proof. Denote by $\Gamma_\varphi \subset I \times \mathbb{L} \subset \mathbb{L}^2$ the graph of φ and define the sets P, P_1 and P_2 as

$$P := \{(x,x) \in \mathbb{L}^2\},\ P_1 := \{(x,y) \in \mathbb{L}^2 : x < y\},\ P_2 := \{(x,y) \in \mathbb{L}^2 : y < x\}.$$

Obviously, P_1 and P_2 are nonempty disjoint open sets in $\mathbb{L}^2$ and $\mathbb{L}^2 = P \cup P_1 \cup P_2$.

Assume that Fix $\varphi := \{x \in I : x \in \varphi(x)\} = \emptyset$, i.e. $P \cap \Gamma_\varphi = \emptyset$.

- If $I \subset \varphi(I)$, then there exist points $c, d \in [a,b]$ such that $a \in \varphi(c)$ and $b \in \varphi(d)$. Moreover, $a < c$ (otherwise, $a \in \varphi(a)$ and a is a fixed point) and $d < b$ (otherwise, $b \in \varphi(b)$ and b is a fixed point).

  Then
  $$d < b \Rightarrow (d,b) \in P_1 \cap \Gamma_\varphi \Rightarrow P_1 \cap \Gamma_\varphi \neq \emptyset$$
  and
  $$a < c \Rightarrow (c,a) \in P_2 \cap \Gamma_\varphi \Rightarrow P_2 \cap \Gamma_\varphi \neq \emptyset.$$

  From the above arguments, we have $\Gamma_\varphi \subset P_1 \cup P_2$, where Γ_φ is connected and $P_1 \cup P_2$ is disconnected which is a contradiction.

- If $\varphi(I) \subset I$, then $a < p$, for all $p \in \varphi(a)$ (otherwise, $a \in \varphi(a)$ and a is a fixed point) and $q < b$, for all $q \in \varphi(b)$ (otherwise, $b \in \varphi(b)$ and b is a fixed point).

  Then
  $$a < p \Rightarrow (a,p) \in P_1 \cap \Gamma_\varphi \Rightarrow P_1 \cap \Gamma_\varphi \neq \emptyset$$
  and
  $$q < b \Rightarrow (b,q) \in P_2 \cap \Gamma_\varphi \Rightarrow P_2 \cap \Gamma_\varphi \neq \emptyset.$$

  From the above arguments, we have $\Gamma_\varphi \subset P_1 \cup P_2$, where Γ_φ is connected and $P_1 \cup P_2$ is disconnected which is again a contradiction.

□

The following slight generalization of a one-dimensional version of the Brouwer theorem is well-known, because it can be easily deduced from the evident intermediate value property.

Corollary 1. *If a single-valued map $f : I \to I$, where $I \subset \mathbb{L}$ is a closed interval of a linear continuum $\mathbb{L}$, has a connected graph, then f has a fixed point in I.*

Example 4. The function $f : [-1,1] \to [-1,1]$ defined by

$$f(x) := \begin{cases} \sin \frac{1}{x}, & \text{for } x \in [-1,1] \setminus \{0\}, \\ 1, & \text{for } x = 0. \end{cases}$$

is not continuous, but has a connected graph. It admits, in fact, infinitely many fixed points in $[-1,1]$ (see Fig. 4).

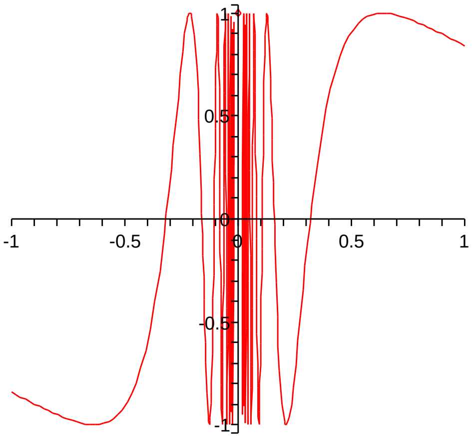

Figure 4: *Function f from Example 4.*

Lemma 2. *Let $\varphi : \mathbb{L} \multimap \mathbb{L}$ have a connected graph Γ_φ. If φ has an n-orbit, for some $n \in \mathbb{N}$, then it also has a fixed point.*

Proof. Let $\{x_1, x_2, \dots, x_n\}$ be an n-orbit of φ. Assume that Fix $\varphi := \{x \in \mathbb{L} : x \in \varphi(x)\} = \emptyset$. Denote $a := \min\{x_i : i = 1, \dots n\}$ and $b := \max\{x_i : i = 1, \dots n\}$. There exist $k, l \in \{1, 2, \dots n\}$ such that $x_k \in \varphi(a)$ and $x_l \in \varphi(b)$. Then $x_k > a$ and $x_l < b$ (otherwise, φ has a fixed point x_k or x_l). Hence,

$$(a, x_k) \in P_1 := \{(x, y) \in \Gamma_\varphi : x < y\} \quad \text{and} \quad (b, x_l) \in P_2 := \{(x, y) \in \Gamma_\varphi : y < x\}.$$

Since the sets P_1 and P_2 are nonempty disjoint open sets in Γ_φ and $\Gamma_\varphi = P_1 \cup P_2$ (we suppose Fix $\varphi = \emptyset$), we obtain a contradiction with the connectedness of Γ_φ. □

The class of maps with a connected graph is rather large. In particular, it trivially contains maps determined by connectivity relations, and since in $\mathbb{R}$ upper and lower semicontinuous maps with closed connected values are determined by connectivity G_δ-relations (see [ASS] or [AFP]), they also have connected graph.

Since the maps satisfying the assumptions of Theorem 1 possess a fixed point, so obviously do their iterates whose graph is not necessarily connected like e.g. the a map $\varphi : [0, 1] \multimap [\frac{1}{4}, \frac{3}{4}]$, where

$$\varphi(x) := \begin{cases} \{\frac{1}{4}, \frac{3}{4}\}, & \text{for } x \in [0, 1)\ , \\ \\ \left[\frac{1}{4}, \frac{3}{4}\right], & \text{for } x = 1, \end{cases}$$

because $\varphi^2 : [0,1] \multimap [\frac{1}{4}, \frac{3}{4}]$, where $\varphi^2(x) = \{\frac{1}{4}, \frac{3}{4}\}$, for $x \in [0,1]$.

On the other hand, if φ has still connected values (i.e. if it is determined by a connectivity relation; cf. Lemma 1), then all the iterates $\varphi^n, n \in \mathbb{N}$, of φ have the same property. Indeed. It directly follows from the definition of a connectivity relation that, for any (possibly degenerate) interval $I \subset \mathbb{L}$, $\varphi(I)$ is connected, i.e. an interval. Thus, $\varphi^2(I) = \varphi(\varphi(I))$ must be also connected, i.e. φ^2 is determined by a connectivity relation and, in particular, it has a connected graph. By induction, we get that it holds for all the iterates, as claimed.

Moreover, there exist maps with a disconnected graph or disconnected values whose some iterate determines a connectivity relation like the mapping $\varphi : [0,1] \multimap [\frac{1}{4}, \frac{3}{4}]$, where

$$\varphi(x) := \begin{cases} \frac{3}{4}, & \text{for } x \in \left[0, \frac{1}{2}\right) \setminus \{\frac{1}{4}\}, \\ \left[\frac{1}{4}, \frac{3}{4}\right], & \text{for } x = \frac{1}{4}, \\ \{\frac{1}{4}, \frac{3}{4}\}, & \text{for } x = \frac{1}{2}, \\ \left[\frac{1}{4}, \frac{3}{4}\right], & \text{for } x = \frac{3}{4}, \\ \frac{1}{4}, & \text{for } x \in \left(\frac{1}{2}, 1\right] \setminus \{\frac{3}{4}\}, \end{cases}$$

because $\varphi^2 : [0,1] \multimap [\frac{1}{4}, \frac{3}{4}]$, $\varphi^2(x) = [\frac{1}{4}, \frac{3}{4}]$ (see Fig. 5).

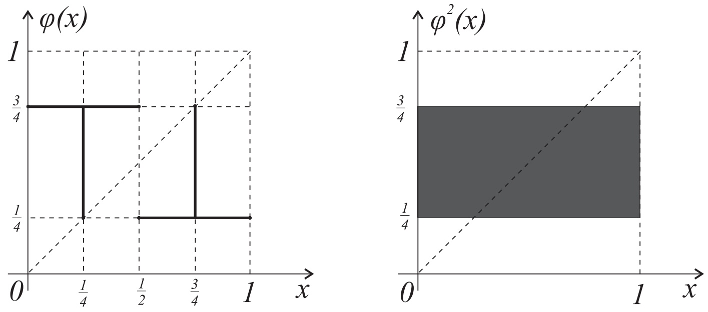

Figure 5: *Maps φ and φ^2.*

If, in Theorem 1, $\varphi = \xi^n$, for some $n \in \mathbb{N}$, then a natural question therefore arises whether or not mapping ξ itself admits a fixed point. As a partial answer, we can give the two following corollaries.

Corollary 2. *Let $\varphi : \mathbb{L} \multimap \mathbb{L}$ be a multivalued mapping with a connected graph. Assume that, for some $n \in \mathbb{N}$, the n-th iterate φ^n of φ has also a connected graph and that there exists a closed* *interval $I \subset \mathbb{L}$ of a linear continuum $\mathbb{L}$ such that either $I \subset \varphi^n(I)$ or $\varphi^n(I) \subset I$. Then φ has a fixed point.*

Proof. According to Theorem 1, φ^n has a fixed point in I. If it is not at the same time a fixed point of φ, then a nontrivial k-orbit of φ occurs, for some $k|n$. By means of Lemma 2, φ must have a fixed point. □

Corollary 3. *Let $\varphi : \mathbb{L} \multimap \mathbb{L}$ be a multivalued mapping with a connected graph and connected values (i.e. let φ determine a connectivity relation; cf. Lemma 1). Assume that, for the n-th iterate φ^n, $n \in \mathbb{N}$, of φ there exists a closed interval $I \subset \mathbb{L}$ of a linear continuum $\mathbb{L}$ such that either $I \subset \varphi^n(I)$ or $\varphi^n(I) \subset I$. Then φ has a fixed point.*

Proof. Since φ^n has, by the above arguments, a connected graph, an application of Corollary 2 completes the proof. □

The following example demonstrates that the graph connectedness in Corollaries 2 and 3 cannot be avoided.

Example 5. The mapping $\varphi : [0,1] \multimap [0,1]$ with closed connected values (observe that $\varphi([0,1]) = [0,1]$, see Fig. 6), where

$$\varphi(x) := \begin{cases} \left[\frac{1}{2}, 1\right], & \text{for } x = 0, \\ -x + 1, & \text{for } x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\ 0, & \text{for } x = \frac{1}{2}, \\ \left[0, \frac{1}{2}\right], & \text{for } x = 1, \end{cases}$$

has the second iterate $\varphi^2 : [0,1] \multimap [0,1]$, where

$$\varphi^2(x) = \begin{cases} \left[0, \frac{1}{2}\right], & \text{for } x = 0, \\ x, & \text{for } x \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1), \\ \left[\frac{1}{2}, 1\right], & \text{for } x \in \{\frac{1}{2}, 1\}, \end{cases}$$

which is an M-mapping (see Fig. 6), but despite the fact that the set of fixed points of φ^2 is the whole interval $[0,1]$, φ itself is fixed point free.

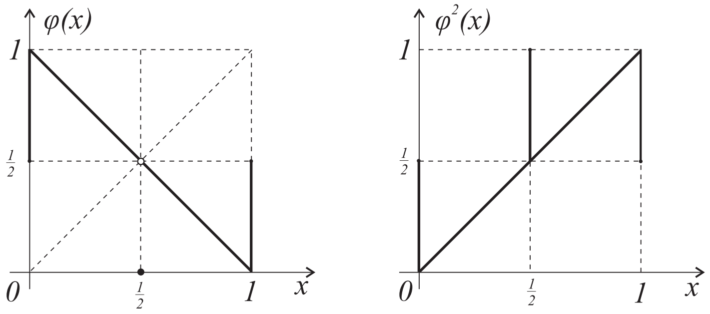

Figure 6: *Maps φ and φ^2 from Example 5.*

As a simple example of an application of Corollary 3, let us consider a continuous (single-valued) function $f : (0, \infty) \to (0, \infty)$, where $f(x) := \frac{1}{x}$, whose second iterate $f^2 : (0, \infty) \to (0, \infty)$ is $f^2(x) = x$. One can readily check that, for $I = [\frac{1}{4}, \frac{1}{2}]$, we have $f([\frac{1}{4}, \frac{1}{2}]) = [2, 4]$, i.e. $f(I) \not\subset I$ and $I \not\subset f(I)$, but $f^2([\frac{1}{4}, \frac{1}{2}]) = [\frac{1}{4}, \frac{1}{2}]$. Thus, according to Corollary 3, f has a fixed point. Observe that the only fixed point of f, $x = 1 \notin [\frac{1}{4}, \frac{1}{2}]$. On the other hand, e.g. for the interval $[\frac{1}{2}, 2]$, we already have $f([\frac{1}{2}, 2]) = [\frac{1}{2}, 2]$, and it is sufficient to apply Theorem 1, according to which f has a fixed point in $[\frac{1}{2}, 2]$.

For M-maps, Theorem 1 can be improved in the form of the following lemma.

Lemma 3 (cf. [AP], Lemma 2.2)**.** *Let $\varphi : \mathbb{L} \multimap \mathbb{L}$ be an M-map. Assume that $I_k \subset \mathbb{L}, k = 0, 1, \dots, n-1$, are closed intervals such that $I_{k+1} \subset \varphi(I_k)$, for $k = 0, 1, \dots, n-1$, and $I_n = I_0$, which we write as $I_0 \to I_1 \to \cdots \to I_n = I_0$. Then the n-th iterate φ^n of φ (i.e. the n-fold composition of φ with itself) has a fixed point x_0 (i.e. $x_0 \in \varphi^n(x_0)$) with $x_{k+1} \in \varphi(x_k)$, $x_n = x_0$, where $x_k \in I_k$, for $k = 0, 1, \dots, n-1$.*

We will finally show how Lemma 3 can be employed for restricting the problem of coexistence of periodic orbits from noncompact linearly ordered spaces to closed intervals.

Theorem 2. *Let an M-mapping $\varphi : \mathbb{L} \multimap \mathbb{L}$ have an n-orbit $\{x_1, \dots, x_n\}$, and let*

$$a := \min\{x_1, \dots, x_n\}, \qquad b := \max\{x_1, \dots, x_n\}.$$

Then there exist a closed inteval I, $[a, b] \subset I \subset \mathbb{L}$, and an M-mapping $\hat{\varphi} : I \multimap I$ such that $\hat{\varphi}(x) = \varphi(x)$ for every $x \in (a, b)$ and, for every $k \in \mathbb{N}$, the existence of a k-orbit of $\hat{\varphi}$ implies the existence of a k-orbit of φ.

Proof. If $\varphi([a, b]) = [a, b]$, it suffices to put $\hat{\varphi} = \varphi$ and $I = [a, b]$. On the contrary, let $[d, c] := \varphi([a, b])$. Now, the proof splits into the following cases.

I. $\varphi((b, c]) \not\subset [d, c]$. Setting $s := \inf\{x \in (b, c] : \varphi(x) \not\subset [d, c]\}$, then due to the upper semicontinuity of φ just one of the following possibilities occurs:

1. $c \in \varphi(s)$

   If $\varphi([d,a]) \subset [d,c]$, we put $I = [d,c]$ and define

$$\hat{\varphi}(x) = \begin{cases} \varphi(x), & \text{for every } x \in [d,s), \\ \varphi(x) \cap [d,c], & \text{for } x = s, \\ c, & \text{for every } x \in (s,c]. \end{cases}$$

   The points forming a k-orbit, $k \in \mathbb{N} \setminus \{1\}$, of $\hat{\varphi}$ form the same orbit of φ, because the point c can only form a 1-orbit of $\hat{\varphi}$ in addition to φ. On the other hand, the existence of 1-orbit of φ on $[a,b]$ follows from Theorem 1.

   If $\varphi([d,a]) \not\subset [d,c]$, then we consider $t := \sup\{x \in [d,a) : \varphi(x) \not\subset [d,c]\}$.
   If $c \in \varphi(t)$, we put $I = [d,c]$ and define

$$\hat{\varphi}(x) = \begin{cases} c, & \text{for every } x \in [d,t) \cup (s,c] \\ \varphi(x) \cap [d,c], & \text{for } x = t, s, \\ \varphi(x), & \text{for every } x \in (t,s). \end{cases}$$

   If $c \notin \varphi(t)$, then $d \in \varphi(t)$. Indeed, supposing $d \notin \varphi(t)$, either (if $t < a$) the upper semicontinuity of φ leads to a contradiction with the definition of t or (if $t = a$) we obtain a contradiction with the fact that $\varphi(a)$ is a connected interval and $\varphi(a) \cap [a,b] \neq \emptyset$. We put $I = [d,c]$ and define

$$\hat{\varphi}(x) = \begin{cases} d, & \text{for every } x \in [d,t), \\ \varphi(x) \cap [d,c], & \text{for } x = t, s, \\ \varphi(x), & \text{for every } x \in (t,s), \\ c, & \text{for every } x \in (s,c]. \end{cases}$$

   The points forming a k-orbit, $k \in \mathbb{N} \setminus \{1\}$, of $\hat{\varphi}$ form the same orbit of φ, because the points c or d can only form a 1-orbit of $\hat{\varphi}$ in addition to φ. Again, Theorem 1 implies the existence of a fixed point on $[a,b]$.

2. $c \notin \varphi(s)$ and $d \in \varphi(s)$.

   Setting $e := \min\{y \in \varphi(x) : x \in [b,c]\}$ and $r := \min\{x \in [b,c] : e \in \varphi(x)\}$, there are the following possibilities depending on function values of φ on $[e,a]$:

   A) $\varphi([e,a]) \subset [e,c]$.

      If $\varphi((s,c]) \leq c$ (i.e., $y \leq c$, for every $y \in \varphi(x)$, where $x \in (s,c]$), it suffices to put $I = [e,c]$ and $\hat{\varphi} = \varphi$. Otherwise, setting $q := \inf\{x \in (s,c] : \varphi(x) > c\}$, we put $I = [e,c]$ and define

$$\hat{\varphi}(x) = \begin{cases} \varphi(x), & \text{for every } x \in [e,q), \\ \varphi(x) \cap [e,c], & \text{for } x = q, \\ c, & \text{for every } x \in (q,c]. \end{cases}$$

B) $\varphi([e,a]) \not\subset [e,c]$.
We consider $u := \sup\{x \in [e,a) : \varphi(x) \not\subset [e,c]\}$, and put $I = [e,c]$. The definition of $\hat{\varphi}$ depends on the relation of e and $\varphi(u)$, and on the relation of $\varphi((s,r))$ and c. If $e \in \varphi(u)$ and $\varphi((s,r)) \leq c$, then we define

$$\hat{\varphi}(x) = \begin{cases} e, & \text{for every } x \in [e,u) \cup (r,c], \\ \varphi(x) \cap [e,c], & \text{for } x = u, r, \\ \varphi(x), & \text{for every } x \in (u,r). \end{cases}$$

If $e \in \varphi(u)$ and $m := \inf\{x \in (s,r) : \exists y \in \varphi(x), y > c\} \in (s,r)$, then we define

$$\hat{\varphi}(x) = \begin{cases} e, & \text{for every } x \in [e,u), \\ \varphi(x) \cap [e,c], & \text{for } x = u, m, \\ \varphi(x), & \text{for every } x \in (u,m), \\ c, & \text{for every } x \in (m,c], \end{cases}$$

and the same arguments as those at the end of part I., 1. conclude this case.
If $e \notin \varphi(u)$, then $c \in \varphi(u)$. If, moreover, $\varphi((s,r]) \leq c$, we define

$$\hat{\varphi}(x) = \begin{cases} c, & \text{for every } x \in [e,u), \\ \varphi(x) \cap [e,c], & \text{for } x = u, r, \\ \varphi(x), & \text{for every } x \in (u,r), \\ e, & \text{for every } x \in (r,c]. \end{cases}$$

The points forming a k-orbit of mapping $\hat{\varphi}$, for $k \in \mathbb{N} \setminus \{1,2\}$, form the same k-orbit of φ, because the points c, e can only form a 2-orbit $\{e,c\}$ of $\hat{\varphi}$ in addition to φ. The existence of a 2-orbit of φ is also guaranteed, because it holds

$$[e,a] \to [b,c] \to [e,a].$$

Finally, if $e \notin \varphi(u)$, $c \in \varphi(u)$ and $m \in (s,r)$, we define

$$\hat{\varphi}(x) = \begin{cases} c, & \text{for every } x \in [e,u) \cup (m,c], \\ \varphi(x) \cap [e,c], & \text{for } x = u, m, \\ \varphi(x), & \text{for every } x \in (u,m). \end{cases}$$

II. $\varphi((b,c]) \subset [d,c]$.
We will discuss functional values of φ on $[d,a]$.

1. $\varphi([d,a]) \subset [d,c]$.
   It suffices to put $I = [d,c]$ and $\hat{\varphi} = \varphi$.
2. $\varphi([d,a]) \not\subset [d,c]$.
   We consider $v := \sup\{x \in [d,a) : \varphi(x) \not\subset [d,c]\}$.

If $d \in \varphi(v)$, we put $I = [d, c]$ and define

$$\hat{\varphi}(x) = \begin{cases} d, & \text{for every } x \in [d, v), \\ \varphi(x) \cap [d, c], & \text{for } x = v, \\ \varphi(x), & \text{for every } x \in (v, c]. \end{cases}$$

If $d \notin \varphi(v)$, we set $f := \max\{y \in \varphi(x) : x \in [d, a]\}$ and $p := \max\{x \in [d, a] : f \in \varphi(x)\}$.
There are two possibilities w.r.t. functional values of φ on $(c, f]$:

A) $\varphi((c, f]) \subset [d, f]$.
If $\varphi([d, v)) \geq d$, it suffices to put $I = [d, f]$ and $\hat{\varphi} = \varphi$. Otherwise, setting $n := \sup\{x \in [d, v) : \varphi(x) < d\}$, we put $I = [d, f]$ and define

$$\hat{\varphi}(x) = \begin{cases} d, & \text{for every } x \in [d, n), \\ \varphi(x) \cap [d, f], & \text{for } x = n, \\ \varphi(x), & \text{for every } x \in (n, f]. \end{cases}$$

B) $\varphi((c, f]) \not\subset [d, f]$.
We consider $w := \inf\{x \in (c, f] : \varphi(x) \not\subset [d, f]\}$, and put $I = [d, f]$. The definition of $\hat{\varphi}$ depends on the relation of f and $\varphi(w)$ and on the relation of $\varphi((p, v))$ and d.
If $f \in \varphi(w)$ and $\varphi((p, v)) \geq d$, then we define

$$\hat{\varphi}(x) = \begin{cases} f, & \text{for every } x \in [d, p) \cup (w, f], \\ \varphi(x) \cap [d, f], & \text{for } x = p, w, \\ \varphi(x), & \text{for every } x \in (p, w). \end{cases}$$

If $f \in \varphi(w)$ and $n := \sup\{x \in (p, v) : \exists y \in \varphi(x), y < d\} \in (p, v)$, then we define

$$\hat{\varphi}(x) = \begin{cases} d, & \text{for every } x \in [d, n), \\ \varphi(x) \cap [d, f], & \text{for } x = n, w, \\ \varphi(x), & \text{for every } x \in (n, w), \\ f, & \text{for every } x \in (w, f]. \end{cases}$$

We can use the same ideas as in the previous cases to conclude this situation.
If $f \notin \varphi(w)$, then $d \in \varphi(w)$. If, moreover, $\varphi((p, v)) \geq d$, we define

$$\hat{\varphi}(x) = \begin{cases} f, & \text{for every } x \in [d, p), \\ \varphi(x) \cap [d, f], & \text{for } x = p, w, \\ \varphi(x), & \text{for every } x \in (p, w), \\ d, & \text{for every } x \in (w, f]. \end{cases}$$

and the analogous arguments as before conclude this case, jointly with the fact that by Lemma 3 φ has a 2-orbit, because

$$[d,a] \longrightarrow [b,f] \longrightarrow [d,a].$$

Finally, if $f \notin \varphi(w)$, $d \in \varphi(w)$ and $n \in (p,v)$, we define

$$\hat{\varphi}(x) = \begin{cases} d, & \text{for every } x \in [d,n) \cup (w,f], \\ \varphi(x) \cap [d,f], & \text{for } x = n, w, \\ \varphi(x), & \text{for every } x \in (n,w). \end{cases}$$

□

4. Concluding remarks

If Lemma 3 could be generalized for maps determined by connectivity G_δ-relations on linear continua, then a Sharkovskii-type theorem might be formulated for these maps by means of the appropriately modified statements like Theorem 2. On the real line, this was already done in [ASS].

The Sharkovskii-type theorems establish an order relationship among the periods that the mapping can possess by means of a new (Sharkovskii's) ordering of positive integers. We already pointed out that this Sharkovskii phenomenon is in principle one-dimensional. Nevertheless, there exists a two-dimensional analogy in the sense that an order relationship can be replaced by forcing relations on braid types (see [Ha], [M1], [M2]). More precisely, one braid type is larger than the second if whenever a homeomorphism has a periodic orbit of the first type, then it also has a periodic orbit of the second type. The theory of braid types on surface dynamics was developed by several authors, but the standard reference for us is here the paper [Bo] by P. Boyland.

Higher than two-dimensional analogies of Sharkovskii's theorem require special structure of maps. For triangular maps, it was achieved (in a single-valued case) by P. Kloeden [Kl] and further extended (in a multivalued case) in [AFP], [AP], [APS]. Since we were able to do it in [APS] on a Cartesian product of linear continua $\mathbb{L}_1 \times \cdots \times \mathbb{L}_N$, where $\mathbb{L}_N$ was only a closed interval, a natural question arises whether Theorem 2 can be extended to triangular M-maps on $\mathbb{L}_1 \times \cdots \times \mathbb{L}_N$, where $\mathbb{L}_N$ is not necessarily a closed interval.

A combination of Theorem 1 and Corollary 1 in [A3] leads directly to the following random fixed point theorem (for definitions and more details, see [A3]).

Theorem 3. *Let $\Phi : \Omega \times \mathbb{L} \multimap \mathbb{L}$ be a random operator, where Ω is a complete measurable space and $\mathbb{L}$ is a complete separable metric linear continuum. Assume that, for each $\omega \in \Omega$, there exists a closed interval $I_\omega \subset \mathbb{L}$ such that $\Phi(\omega,\cdot) : I_\omega \multimap \mathbb{L}$ has a connected graph and either $I_\omega \subset \Phi(\omega, I_\omega)$ or $I_\omega \supset \Phi(\omega, I_\omega)$. Then Φ has a random fixed point, i.e. a measurable function $x : \omega \to \mathbb{L}$ such that $x(\omega) \in \Phi(\omega, x(\omega))$, for a.a. $\omega \in \Omega$.*

For more sophisticated fixed point theorems, where closed subsets are covered by their images or just intersect their images, see e.g. [A1] and the references therein.

`Received: January 2008. Revised: February 2008.`

References

[A1] J. Andres, *Some standard fixed–point theorems revisited*, Atti Sem. Mat. Fis. Univ. Modena **49** (2001), 455–471.

[A2] J. Andres, *Period three implications for expansive maps in $\mathbb{R}^N$*, J. Difference Eqns Appl. **10**, 1 (2004), 17–28.

[A3] J. Andres, *Randomization of Sharkovskii–type theorems*, Proc. Amer. Math. Soc. **136** (2008), 1385–1395.

[AP] J. Andres and K. Pastor, *On a multivalued version of the Sharkovskii theorem and its application to differential inclusions, III*, Topol. Meth. Nonlin. Anal., **22** (2003), 369–386.

[AFP] J. Andres, T. Fürst and K. Pastor, *Full analogy of Sharkovsky's theorem for lower semicontinuous maps*, J. Math. Anal. Appl. **340** (2008), 1132–1144.

[AG] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Kluwer, Dordrecht, 2003.

[APS] J. Andres, K. Pastor and P. Šnyrychová, *A multivalued version of Sharkovskii's theorem holds with at most two exceptions*, J. Fixed Point Theory Appl. **2** (2007), 153–170.

[AS] D. Alcaraz and M. Sanchiz, *A note on Šarkovskii's theorem in connected linearly ordered spaces*, Int. J. Bifurc. Chaos **13**, 7 (2003), 1665–1671.

[ASS] J. Andres, P. Šnyrychová and P. Szuca, *Sharkovskii's theorem for connectivity G_δ–relations*, Int. J. Bifurc. Chaos, **16**, 8 (2006), 2377–2393.

[Bo] P. Boyland, *Topological methods in surface dynamics*, Topol. Appl. **58** (1994), 224–298.

[B1] R.F. Brown, *Fixed point theory*, In “History of Topology (Chapter 10)” (ed. by I.M. James), Elsevier, Amsterdam, 1999, pp. 271–299.

[B2] R.F. Brown, *Fixed points of n–valued multimaps of the circle*, Bull. Polish Acad. Sci. Math. **54** (2006), 153–162.

[B3] R.F. Brown, *The Lefschetz number of an n–valued multimaps*, JP Jour. Fixed Point Theory Appl. **2** (2007), 53–60.

[Dz] Z. Dzedzej, *Fixed point index theory for a class of nonacyclic multivalued maps*, Dissertationes Math. **235** (1985), 1–58.

[G1] L. Górniewicz, *Present state of the Brouwer fixed point theorem for multivalued mappings*, Ann. Sci. Math. Québec **22**, 2 (1998), 169–179.

[G2] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings* (2nd edition). Springer, Berlin, 2006.

[Ha] M. Handel, *The forcing partial order on three times punctured disk*, Ergod. Th. Dynam. Sys. **17**, (1997), 593–610.

[Ka] J. Kampen, *On fixed points of maps and iterated maps and applications*, Nonlin. Anal. **42**, (2000), 509–532.

[Kl] P.E. Kloeden, *On Sharkovsky's cycle coexisting ordering*, Bull. Austral. Math. Soc. **20** (1979), 171–177.

[Mi] D. Miklaszewski, *The Role of Various Kinds of Continuity in the Fixed Point Theory of Set–Valued Mappings*, Lecture Notes in Nonlin. Anal., Vol. 7, J. Schauder Center for Nonlinear Studies, N. Copernicus Univ., Toruń, 2005.

[M1] T. Matsuoka, *Braids of periodic points and a 2–dimensional analogue of Sharkovskii's ordering*, In: "Dynamical Systems and Nonlinear Oscillations" (G. Ikegami, ed.) World Sci. Press, Singapore, 1986, pp. 58–72.

[M2] T. Matsuoka, *Periodic points and braid theory*, In: "Handbook of Topological Fixed Point Theory" (ed. by R.F. Brown, M. Furi, L. Górniewicz and B. Jiang), Springer, Berlin, 2005, pp. 171–216.

[Mu] J. R. Munkres, *Topology. A First Course*, Prentice–Hall, New Jersey, 1975.

[S1] H. Schirmer, *An index and a Nielsen number for n–valued multifunctions*, Fund. Math. **124** (1984), 207–219.

[S2] H. Schirmer, *A topologist's view of Sharkovsky's theorem*, Houston J. Math. **11**, 3 (1985), 385–395.

[Sh] A.N. Sharkovskii, *Coexistence of cycles of a continuous map of a line into itself*, Ukrain. Math. J. **16** (1964), 61–71 (in Russian); Int. J. Bifurc. Chaos **5** (1995), 1263–1273 (English translation).

[Sk] R. Skiba, *Fixed Points of Multivalued Weighted Maps*, Lecture Notes in Nonlin. Anal., Vol. 9, J. Schauder Center for Nonlinear Studies, N. Copernicus Univ., Toruń, 2007.

[Sn] P. Šnyrychová, *Periodic points for maps in $\mathbb{R}^n$*, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica **42** (2003), 87–104.

CUBO A Mathematical Journal
Vol.10, $N^{\underline{o}}$ 04, (45–66). December 2008

Fixed Points for Operators on Generalized Metric Spaces

Adrian Petruşel, Ioan A. Rus and Marcel Adrian Şerban
Department of Applied Mathematics,
Babeş-Bolyai University Cluj-Napoca,
Kogălniceanu 1, 400084, Cluj-Napoca, Romania
emails: `petrusel`{*iarus*, *mserban*}`@math.ubbcluj.ro`

ABSTRACT

The purpose of this paper is to present the fixed point theory for operators (singlevalued and multivalued) on generalized metric spaces in the sense of Luxemburg.

RESUMEN

El proposito de este artículo es presentar la teoria de punto fijo para operadores (univariados y multivaluados) sobre espacios métricos generalizados en el sentido de Luxemburg.

Key words and phrases: *Generalized metric in the sense of Luxemburg, Pompeiu-Hausdorff generalized functional, weakly Picard operator, fixed point, strict fixed point, generalized contraction, fibre generalized contraction, data dependence, pseudo-contractive multivalued operator.*

Math. Subj. Class.: *47H10, 54H25.*

1. Introduction

Let X be a nonempty set. A functional $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ is said to be a generalized metric in the sense of Luxemburg on X ([9], [13]) if:

i) $d(x, y) = 0 \;\Leftrightarrow\; x = y$;

ii) $d(x, y) = d(y, x)$;

iii) $x, y, z \in X$ with $d(x, z), d(z, y) < +\infty \;\Rightarrow d(x, y) \le d(x, z) + d(z, y)$.

The pair (X, d) is called a generalized metric space. In a generalized metric space, the concepts of open and closed ball, Cauchy sequence, convergent sequence, etc. are defined in a similar way to the case of a metric space.

There are some contributions to fixed point theory for singlevalued operators (W.A.J. Luxemburg [13], J.B. Diaz and B. Margolis [7], C.F.G. Jung [9], S. Kasahara [10], G. Dezso [6],...) and multivalued operators (H. Covitz and S.B. Nadler [5], P.Q. Khanh [11],...) on a generalized metric space in the sense of Luxemburg.

The aim of this paper is to establish some new fixed point theorems for operators on a generalized metric space and, in this framework, to study the basic problems of the metrical fixed point theory.

2. Generalized metric spaces in the sense of Luxemburg

We start our considerations by presenting some examples of generalized metric spaces.

Example 2.1 *Let X be a nonempty set and $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$, given by*

$$d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ +\infty, & \text{otherwise.} \end{cases}$$

Example 2.2 *Let $X := C(\mathbb{R})$ and $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ given by $d(x, y) := \sup\limits_{t\in\mathbb{R}} |x(t) - y(t)|$.*

Example 2.3 *Let $X := C(\mathbb{R})$ (the space of all continuous functions on $\mathbb{R}$) and $d : X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ given by $d(x, y) := \sup\limits_{t\in\mathbb{R}}(|x(t) - y(t)| \cdot e^{-\tau|t|})$, where $\tau > 0$.*

Example 2.4 (Generic example) *Let (X_i, d_i), $i \in I$ be a family of metric spaces such that each two elements of the family are disjoint. Denote $X := \bigcup\limits_{i\in I} X_i$. If we define*

$$d(x, y) := \begin{cases} d_i(x, y), & \text{if } x, y \in X_i \\ +\infty, & \text{if } x \in X_i, y \in X_j, i \neq j \end{cases},$$

then the pair (X, d) is a generalized metric space.

The following characterization theorem of a generalized metric space was given by Jung.

Theorem 2.5 (Jung [9]) *Let* (X, d) *be a generalized metric space. Then there exists a partition* $X := \bigcup_{i \in I} X_i$ *of* X *such that* $d_i := d_{|X_i \times X_i}$ *is a metric, for each* $i \in I$. *Moreover,* (X, d) *is complete if and only if* (X_i, d_i) *is complete, for each* $i \in I$.

Notice that the above partition is induced by the following equivalence relation: $x \sim y \;\Leftrightarrow\; d(x, y) < +\infty$.

Let (X, d) be a generalized metric space. Then, the partition $X := \bigcup_{i \in I} X_i$ given by Jung's theorem is called the canonical decomposition of X into metric spaces. Moreover, if $x \in X$, then there exists $i(x) \in I$ such that $x \in X_{i(x)}$.

We will denote $B_d(x_0; r) := \{x \in X | d(x_0, x) < r\}$ and $\widetilde{B}_d(x_0; r) := \{x \in X | d(x_0, x) \leq r\}$. If $x \in X_i$, then $\widetilde{B}_d(x_0; r) = \widetilde{B}_{d_i}(x_0; r)$ and $B_d(x_0; r) = B_{d_i}(x_0; r)$.

If (X, d) is a generalized metric space, then the metric topology induced on X is given by:

$$\tau_d := \{Y \subseteq X | y \in Y \Rightarrow \exists r > 0 \;:\; B_d(y, r) \subset Y\}.$$

By this definition, it follows that:

$$(x_n)_{n \in \mathbb{N}} \subset X, x^* \in X, x_n \overset{\tau_d}{\to} x^* \;\Leftrightarrow\; d(x_n, x^*) \to 0.$$

A subset Y of X is said to be d-closed (closed with respect to the topology induced by d) if and only if $(y_n)_{n \in \mathbb{N}} \subset Y$ with $d(y_n, y) \to 0$, as $n \to +\infty$ implies $y \in Y$. Also, Y is d-open if for each $y \in Y$ there exists a ball $B(x_0, r) := \{x \in Y | d(x_0, x) < r\} \subset Y$.
Let us remark that if $X := \bigcup_{i \in I} X_i$ is the canonical decomposition of X, then X_i is d-closed and d-open, for each $i \in I$.

Definition 2.6 Two generalized metrics d_1 and d_2 on X are said to be:

(a) topological equivalent if $\tau_{d_1} = \tau_{d_2}$;

(b) metric equivalent if there exist $c_1, c_2 > 0$ such that:

i) $d_1(x, y) < +\infty$ implies $d_2(x, y) \leq c_1 d_1(x, y)$;

ii) $d_2(x, y) < +\infty$ implies $d_1(x, y) \leq c_2 d_2(x, y)$.

Remark 2.7 *If* d_1 *is a generalized metric on* X, *then there exists a bounded metric* d_2 *on* X, *topological equivalent to* d_1 *(for example take* $d_2(x, y) := \min\{d_1(x, y), 1\}$*).*

3. Functionals on generalized metric spaces

Throughout this section (X, d) will be a generalized metric space in the sense of Luxemburg.

Let us consider now the following families of subsets of the space (X, d):

$$P(X) := \{Y \subseteq X \mid Y \neq \emptyset\} \ ; \ P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded }\};$$

$$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed }\}; \ P_{b,cl}(X) := \{Y \in P(X) \mid Y \text{ is bounded and closed }\}.$$

Consider now some functionals on $P(X) \times P(X)$ (see also [3], [16]).

(i) the gap functional D_d defined by:

$$D_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$D_d(A, B) := \inf\{d(a, b) \mid a \in A, \ b \in B\}.$$

(ii) the excess generalized functional ρ_d defined by:

$$\rho_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$\rho_d(A, B) := \sup\{D(a, B) \mid a \in A\}.$$

(iii) the Pompeiu-Hausdorff generalized functional H_d defined by:

$$H_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},$$

$$H_d(A, B) := \max\{\rho(A, B), \rho(B, A)\}.$$

(iv) the delta functional δ_d defined by:

$$\delta_d : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

$$\delta_d(A, B) := \sup\{d(a, b) \mid a \in A, \ b \in B\}.$$

Let $A, B \in P(X)$. For the rest of the paper, we denote

$$A_i := A \cap X_i \text{ and } B_i := B \cap X_i,$$

where X_i are the sets from the characterization Theorem 2.1.

From (i), Theorem 2.5 and Example 2.4 we have:

Lemma 3.1 *Let (X, d) be a generalized metric space and $A, B \in P(X)$. Then:*

(i) $D(A, B) = \inf\limits_{i \in I} D(A_i, B_i)$;

(ii) $D(A, B) < +\infty$ if and only if there exists $i \in I$ such that $A_i \neq \emptyset$ and $B_i \neq \emptyset$.

A useful result is:

Lemma 3.2 *Let (X, d) be a generalized metric space $x \in X$ and $A \in P(X)$. Then $D(x, A) = 0$ if and only if $X_{i(x)} \cap A \neq \emptyset$ and $x \in \overline{A}$ (where $X_{i(x)}$ denotes the unique element of the canonical decomposition of X where x belongs).*

From (iv) and Theorem 2.5 we obtain:

Lemma 3.3 *Let* (X, d) *be a generalized metric space and* $A, B \in P(X)$. *Then* $\delta(A, B) < +\infty$ *if and only if there exists* $i \in I$ *such that* $A, B \in P_b(X_i)$. *In particular,* $A \in P_b(X)$ *if and only if there exists* $i \in I$ *such that* $A \in P_b(X_i)$.

From (ii), Theorem 2.5 and Example 2.4 we have:

Lemma 3.4 *Let* (X, d) *be a generalized metric space and* $Y, Z \in P(X)$. *Then* $\rho(Y, Z) < +\infty$ *if and only if there exists* $\eta > 0$ *such that for each* $y \in Y$ *there is* $z \in Z$ *such that* $d(y, z) < \eta$.

Proof. If $\rho(Y, Z) < +\infty$, then there is $\eta > 0$ such that $\rho(Y, Z) < \eta$. Thus $D(y, Z) < \eta$ for each $y \in Y$. Hence there exists $z \in Z$ such that $d(y, z) < \eta$.

Suppose now there is $\eta > 0$ such that for each $y \in Y$ there exists $z \in Z$ with $d(y, z) < \eta$. Then, $y, z \in X_i$, where X_i is an element of the partition of the generalized metric space X. Hence $D(y, Z) \leq \eta$, for each $y \in Y$. Thus, $\rho(Y, Z) \leq \eta$. □

Let (X, d) be a generalized metric space, $Y \in P(X)$ and $\varepsilon > 0$. An open neighborhood of radius ε for the set Y is the set denoted $V_\varepsilon(Y)$ and defined by:

$$V_\varepsilon(Y) := \{x \in X |\ D(x, Y) < \varepsilon\}.$$

Let us remark that $V_\varepsilon(Y) = \bigcup_{i \in I, Y_i \neq \emptyset} V_\varepsilon(Y_i)$.

In the usual case of a metric space (X, d) the following equivalent definitions of the Pompeiu-Hausdorff functional are well-known.

$$(iii)'\ H_d(A, B) := \inf\{\varepsilon > 0 | A \subset V_\varepsilon(B), B \subset V_\varepsilon(A)\},$$

and

$$(iii)''\ H_d(A, B) := \sup_{x \in X} |D(x, A) - D(x, B)|.$$

We have:

Lemma 3.5 *Let* (X, d) *be a generalized metric space. Then, the definitions* (iii), $(iii)'$ *and* $(iii)''$ *are equivalent.*

We can also prove the following result.

Lemma 3.6 *Let* (X, d) *be a generalized metric space and* $A, B \in P(X)$. *Then the following assertions are equivalent:*

(a) $H(A, B) < +\infty$;

(c) there exists $\eta > 0$ *such that [for each* $a \in A$ *there exists* $b \in B$ *such that* $d(a, b) < \eta$*] and [for each* $b \in B$ *there exists* $a \in A$ *such that* $d(a, b) < \eta$*].*

Lemma 3.7 *Let* (X, d) *be a generalized metric space. Then the following assertions hold:*

i) Let $\varepsilon > 0$ and $Y, Z \in P(X)$ such that $H(Y,Z) < +\infty$. Then for each $y \in Y$ there exists $z \in Z$ such that $d(y,z) \leq H(Y,Z) + \varepsilon$.

ii) Let $q > 1$ and $Y, Z \in P(X)$ such that $H(Y,Z) < +\infty$. Then, for each $y \in Y$ there exists $z \in Z$ such that $d(y,z) \leq qH(Y,Z)$.

Proof. i) Let $Y, Z \in P(X)$ and $\varepsilon > 0$. Suppose that $H(Y,Z) < +\infty$. Then, supposing, by contradiction, there is $y \in Y$ such that for every $z \in Z$ we have $d(y,z) > H(Y,Z) + \varepsilon$. If $d(y,z) < +\infty$ then since $H(Y,Z) \geq D(y,Z) \geq H(Y,Z) + \varepsilon$ we get a contradiction. If $d(y,z) = +\infty$ then, we get a contradiction to the supposition $H(Y,Z) < +\infty$, since, by Lemma 3.6, there is $\eta > 0$ such that for each $y \in Y$ there is $z \in Z$ with $d(y,z) < \eta$. □

Lemma 3.8 *Let (X,d) be a generalized metric space and $A, B \in P(X)$. Then:*

a) $H(A,B) = \sup\limits_{i \in I} H(A \cap X_i, B \cap X_i)$;

b) $A \in P_{cp}(X) \Leftrightarrow card\{i \in I | A \cap X_i \neq \emptyset\} < +\infty$ and $A_i \in P_{cp}(X_i)$.

Remark 3.9 *Let (X,d) be a generalized metric space. Then $P_{cp}(X) \nsubseteq P_b(X)$. Consider, for example, $x, y \in X$ with $d(x,y) = +\infty$, then $\{x,y\}$ is compact but it is not bounded.*

4. Singlevalued operators on generalized metric spaces

4.1 General considerations

Let X be a nonempty set, $s(X) := \{(x_n)_{n\in\mathbb{N}} | x_n \in X,\ n \in \mathbb{N}\}$, $c(X) \subset s(X)$ and $Lim : c(X) \to X$ an operator. By definition the triple $(X, c(X), Lim)$ is called an L-space if the following conditions are satisfied:

(i) If $x_n = x$, for all $n \in \mathbb{N}$, then $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$.

(ii) If $(x_n)_{n\in\mathbb{N}} \in c(X)$ and $Lim(x_n)_{n\in\mathbb{N}} = x$, then for all subsequences, $(x_{n_i})_{i\in\mathbb{N}}$, of $(x_n)_{n\in\mathbb{N}}$ we have that $(x_{n_i})_{i\in\mathbb{N}} \in c(X)$ and $Lim(x_{n_i})_{i\in\mathbb{N}} = x$.

By definition an element of $c(X)$ is convergent sequence and $x := Lim(x_n)_{n\in\mathbb{N}}$ is the limit of this sequence and we write $x_n \to x$ as $n \to \infty$.

In what follows we will denote an L-space by $(X, \to)$.

Actually, an L-space is any set endowed with a structure implying a notion of convergence for sequences. For example, Hausdorff topological spaces, metric spaces, generalized metric spaces in Perov' sense (i.e., $d(x,y) \in \mathbb{R}^m_+$), generalized metric spaces in Luxemburg' sense (i.e., $d(x,y) \in \mathbb{R}_+ \cup \{+\infty\}$), K-metric spaces (i.e., $d(x,y) \in K$, where K is a cone in an ordered Banach space), gauge spaces, 2-metric spaces, D-R-spaces, probabilistic metric spaces, syntopogenous spaces, are such L-spaces. For more details see Fréchet [8], Blumenthal [4] and I.A. Rus [22].

Let (X, d) and (Y, ρ) be two generalized metric spaces and $f : X \to Y$.

Definition 4.1 The operator $f : (X, d) \to (Y, \rho)$ is said to be:

a) continuous, if $x_n \to x^*$ implies $f(x_n) \to f(x^*)$;

b) closed, if $x_n \to x^*$ and $f(x_n) \to y^*$ imply $f(x^*) = y^*$;

c) α-Lipschitz if $\alpha > 0$ and

$$d(x, y) < +\infty \Longrightarrow \rho(f(x), f(y)) \leq \alpha \cdot d(x, y).$$

d) α-contraction if f is α-Lipschitz with $\alpha < 1$.

4.2 Weakly Picard operators on L-spaces

Let $(X, \to)$ be an L-space and $f : X \to X$. We denote by $f^0 := 1_X$, $f^1 := f$, $f^{n+1} := f \circ f^n$, $n \in \mathbb{N}$ the iterate operators of f. Also:

$$F_f := \{x \in X \mid f(x) = x\},$$
$$I(f) := \{Y \in P(X) \mid f(Y) \subseteq Y\}.$$

Definition 4.2 (I.A. Rus [22]) Let $(X, \to)$ be an L-space. Then $f : X \to X$ is said to be

1) a Picard operator if:

i) $F_f = \{x^*\}$;

ii) $(f^n(x))_{n \in \mathbb{N}} \to x^*$ as $n \to +\infty$, for all $x \in X$.

2) a weakly Picard (briefly WP) operator if the sequence $(f^n(x))_{n \in \mathbb{N}}$ converges for all $x \in X$ and the limit (which may depend on x) is a fixed point of f.

If $f : X \to X$ is a weakly Picard operator, then we define the operator $f^\infty : X \to X$ by:

$$f^\infty(x) := \lim_{n \to \infty} f^n(x).$$

Notice that $f^\infty(X) = F_f$. Moreover, if f is a Picard operator and we denote by x^* its unique fixed point, then $f^\infty(x) = x^*$, for each $x \in X$.

Definition 4.3 Let $(X, \to)$ be an L-space, $c > 0$ and $d : X \times X \to \mathbb{R}_+$. By definition, the operator f is called c-weakly Picard with respect to d, if f is a weakly Picard operator and

$$d(x, f^\infty(x)) \leq c \cdot d(x, f(x)), \quad \text{for all } x \in X.$$

If f is Picard operator and the above condition holds, then f is said to be c-Picard.

Theorem 4.4 *(Characterization Theorem) (I.A. Rus [25], [22]) Let $(X, \to)$ be an L-space and $f : X \to X$ be an operator. Then, f is a weakly Picard operator if and only if there exists a partition of X, $X = \bigcup_{\lambda \in \Lambda} X_\lambda$, such that:*

a) $X_\lambda \in I(f)$, *for all* $\lambda \in \Lambda$*;*

b) $f|_{X_\lambda} : X_\lambda \to X_\lambda$ *is a Picard operator, for all* $\lambda \in \Lambda$*.*

4.3 Contractions on generalized metric spaces

We present first some important auxiliary results.

Lemma 4.5 *Let* (X, d) *be a complete generalized metric space and* $f : X \to X$ *be an* α*-contraction. The following statements are equivalent:*

i) $F_f \neq \emptyset$*;*

ii) there exists $x \in X$ *such that* $d(x, f(x)) < +\infty$*;*

iii) there exist $x \in X$ *and* $n(x) \in \mathbb{N}$ *such that* $d\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$*;*

iv) there exists $i \in I$ *such that* $X_i \in I(f)$*.*

Proof. $i) \Longrightarrow ii)$ Let $x^* \in F_f$. We have

$$d(x^*, f(x^*)) = d(x^*, x^*) = 0 < +\infty.$$

$ii) \Longrightarrow iii)$ We choose $n(x) = 0$;

$iii) \Longrightarrow i)$ Since f is an α-contraction we have that $(f^n(x))$ is a Cauchy sequence. This implies $f^n(x) \to x^*$, as $n \to +\infty$. From the continuity of f it follows that $x^* \in F_f$.

$ii) \Longrightarrow iv)$ Since $d(x, f(x)) < +\infty$, there exists $i \in I$ such that $x \in X_i$. Let $y \in X_i$ then $d(x, y) < +\infty$. We have:

$$d(x, f(y)) \leq d(x, f(x)) + d(f(x), f(y)) \leq d(x, f(x)) + \alpha \cdot d(x, y) < +\infty$$

which implies $f(y) \in X_i$.

$iv) \Longrightarrow ii)$ Let $x \in X_i$. Since $X_i \in I(f)$, we get that $f(x) \in X_i$. Therefore $d(x, f(x)) < +\infty$.□

Lemma 4.6 *Let* (X, d) *be a complete generalized metric space and* $f : X \to X$ *be an* α*-contraction. We suppose that:*

i) there exists $x \in X$ *such that* $d(x, f(x)) < +\infty$*;*

ii) if $u, v \in F_f$ *then* $d(u, v) < +\infty$*;*

Then:

a) $F_f = \{x^*\}$*;*

b) $f\big|_{X_{i(x)}} : X_{i(x)} \to X_{i(x)}$ *is a Picard operator.*

Proof. From $i)$ and Lemma 4.5 we have that there exists $i \in I$ such that $X_i \in I(f)$, $f^n(x) \in X_i$ for every $n \in \mathbb{N}$, $F_f \neq \emptyset$, $f^n(x) \to x^* \in F_f \cap X_i$. Let $u, v \in F_f$. Then $d(u, v) < +\infty$ and

$$d(u, v) = d(f(u), f(v)) \leq \alpha \cdot d(u, v).$$

Therefore $d(u,v)=0$, which implies $u=v$. Hence $F_f=\{x^*\}$.

Since $X_i \in I(f)$ then $d(y,f(y))<+\infty$ for every $y\in X_i$ and applying again Lemma 4.5 we get that $f\big|_{X_{i(x)}}:X_{i(x)}\to X_{i(x)}$ is a Picard operator. □

Theorem 4.7 *Let (X,d) be a complete generalized metric space and $f:X\to X$. We suppose that:*

i) f is an α-contraction;

ii) for every $x\in X$ there exists $n(x)\in\mathbb{N}$ such that $d\left(f^{n(x)}(x),f^{n(x)+1}(x)\right)<+\infty$.

Then:

a) f is a weakly Picard operator. If in addition, for every $x\in X$ we have $d(x,f(x))<+\infty$, then f is $\frac{1}{1-\alpha}$-weakly Picard;

b) If, in addition:

b_1) for every $x\in X$ we have $d(x,f(x))<+\infty$;

b_2) $u,v\in F_f$ implies $d(u,v)<+\infty$,

then f is $\frac{1}{1-\alpha}$-Picard.

Proof. *a*) The first part follows from Lemma 4.5 and Lemma 4.6. For the second conclusion, notice that for every $x\in X$ such that $d(x,f(x))<+\infty$ and each $n\in\mathbb{N}$ we have:

$$d\left(f^n(x),f^\infty(x)\right)\le\frac{\alpha^n}{1-\alpha}\cdot d(x,f(x))$$

which implies

$$d\left(x,f^\infty(x)\right)\le\frac{1}{1-\alpha}\cdot d(x,f(x)).$$

b) From b_2) we obtain $F_f=\{x^*\}$ and from *a*) we obtain that f is $\frac{1}{1-\alpha}$-Picard operator. □

Theorem 4.8 *Let (X,d) be a complete generalized metric space and $f,g:X\to X$ two operators. We suppose that:*

i) f and g are α-contractions;

ii) $d(x,f(x))<+\infty$ and $d(x,g(x))<+\infty$, for every $x\in X$;

iii) there exists $\eta>0$ such that

$$d(f(x),g(x))\le\eta,\quad \textit{for all } x\in X.$$

Then:

$$H(F_f,F_g)\le\frac{\eta}{1-\alpha}.$$

Proof. Let $x\in F_f$ and $y\in F_g$. From *ii*) and Theorem 4.7 we have:

$$d\left(x,g^\infty(x)\right)\le\frac{1}{1-\alpha}\cdot d(x,g(x))=\frac{1}{1-\alpha}\cdot d(f(x),g(x))\le\frac{\eta}{1-\alpha}.$$

Since $g^{\infty}(x) \in F_g$ then

$$D(x, F_g) \leq d(x, g^{\infty}(x)) \leq \frac{\eta}{1-\alpha}.$$

By taking the supremum over $x \in F_f$ we get

$$\rho(F_f, F_g) \leq \frac{\eta}{1-\alpha}.$$

Using the same technique we have:

$$\rho(F_g, F_f) \leq \frac{\eta}{1-\alpha}$$

which implies the conclusion. □

Theorem 4.9 *(Fibre contraction principle) Let $(X_0, \to)$ be an L-space and (X_k, d_k), $k \in \{0, 1, \cdots, p\}$ (where $p \geq 1$) be complete generalized metric spaces. We consider the operators:*

$$f_k : X_0 \times ... \times X_k \to X_k, \quad k \in \{0, 1, \cdots, p\}.$$

We suppose that:

i) $f_0 : X_0 \to X_0$ is a weakly Picard operator;

ii) $f_k(x_0, ..., x_{k-1}, \cdot)$ is an α_k-contraction, $k \in \{1, 2, \cdots, p\}$;

iii) f_k is continuous, $k \in \{1, 2, \cdots, p\}$;

iv) for every $(x_0, x_1, ..., x_k) \in X_0 \times ... \times X_k$ we have

$$d_k(x_k, f_k(x_0, x_1, ..., x_k)) < +\infty, \quad k \in \{1, 2, \cdots, p\}.$$

Then the operator

$$g_p : X_0 \times ... \times X_p \to X_0 \times ... \times X_p$$
$$g_p(x_0, x_1, ..., x_p) = (f_0(x_0), f_1(x_0, x_1), ..., f_p(x_0, x_1, ..., x_p))$$

is weakly Picard.

Proof. We will prove by induction. For $p = 1$ the conclusion follows by Theorem 3.1 in M.A. Şerban [31]. We suppose that conclusion holds for $k \leq p$ and we prove the conclusion for $k + 1$. We know that $g_{k+1} = (g_k, f_{k+1})$, g_k are weakly Picard and from *ii)* $f_{k+1}(x_0, ..., x_k, \cdot)$ is an α_{k+1}-contraction, so we apply again Theorem 3.1 from M.A. Şerban [31] and we get that g_{k+1} is weakly Picard. □

Theorem 4.10 *Let X be a nonempty set, $\alpha \in]0; 1[$ and $f : X \to X$ an operator. The following statements are equivalent:*

i) $F_f = F_{f^n} \neq \emptyset$ for every $n \in \mathbb{N}$;

ii) there exists a complete generalized metric d on X such that:

a) $f : (X, d) \to (X, d)$ is an α-contraction;

b) $d(x, f(x)) < +\infty$ *for every* $x \in X$.

Proof. $i) \Longrightarrow ii)$ $F_f = F_{f^n} \neq \emptyset$ for every $n \in \mathbb{N}$ implies that there exists a partition of X, $X = \bigcup\limits_{i \in I} X_i$ such that $X_i \in I(f)$, $card\,(F_f \cap X_i) = 1$ and $f|_{X_i}$ is a Bessaga operator (see I.A. Rus [24]). From Bessaga's theorem [2] there exists a complete metric d_i on X_i such that $f|_{X_i} : X_i \to X_i$ is an α-contraction for all $i \in I$. So, $d : X \times X \to R_+ \cup \{+\infty\}$

$$d(x, y) = \begin{cases} d_i(x, y) & if \;\; x, y \in X_i \\ +\infty & if \;\; x \in X_i, \; y \in X_i, \; i \neq j \end{cases}$$

is the complete generalized metric on X that we are looking for.

$ii) \Longrightarrow i)$ is Theorem 4.7. □

4.4 Graphic contractions

Let (X, d) be a generalized metric space and $f : X \to X$.

Definition 4.11 $f : X \to X$ is a graphic contraction if there exists $\alpha \in [0; 1[$ such that:

$$d\left(f^2(x), f(x)\right) \leq \alpha \cdot d(x, f(x)) \text{ for all } x \in X \text{ with } d(x, f(x)) < +\infty.$$

Theorem 4.12 *Let* (X, d) *be a complete generalized metric space and* $f : X \to X$. *We suppose that:*

i) f *is a closed graphic contraction;*

ii) for every $x \in X$ *there exists* $n(x) \in \mathbb{N}$ *such that* $d\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$.

Then:

a) f *is a weakly Picard operator. If, in addition, for every* $x \in X$ *we have that* $d(x, f(x)) < +\infty$, *then* f *is* $\frac{1}{1-\alpha}$*-weakly Picard;*

b) If, in addition:

$b_1)$ *for every* $x \in X$ *we have* $d(x, f(x)) < +\infty$;

$b_2)$ *if* $u, v \in F_f$ *implies* $d(u, v) < +\infty$,

then f *is* $\frac{1}{1-\alpha}$*-Picard.*

Proof. $a)$ From $i)$ and $ii)$ we have that for each $x \in X$, the sequence $(f^n(x))$ is Cauchy. Therefore there exists $x^* \in X$ such that $f^n(x) \to x^*$, as $n \to +\infty$ and

$$d(f^n(x), x^*) \leq \frac{\alpha^{n-n(x)}}{1-\alpha} \cdot d\left(f^{n(x)}(x), f^{n(x)+1}(x)\right), \quad n \geq n(x).$$

Since f is closed we get that $x^* \in F_f$ and $f^\infty(x) = x^*$. This means that f is a weakly Picard operator.

If for every $x \in X$ we have $d(x, f(x)) < +\infty$, then $n(x) = 0$ and letting $n = 0$ in the above relation, we conclude that f is $\frac{1}{1-\alpha}$-weakly Picard operator.

b) If for $u, v \in F_f$ we have $d(u, v) < +\infty$ then $F_f = \{x^*\}$, which means that f is a $\frac{1}{1-\alpha}$-Picard operator. □

4.5 Meir-Keeler operators

Let us consider now the case of Meir-Keeler operators on generalized metric spaces.

Definition 4.13 *Let (X, d) be a generalized metric space. Then, $f : X \to X$ is called a Meir-Keeler type operator if for each $\epsilon > 0$ there exists $\eta = \eta(\epsilon) > 0$ such that for $x, y \in X$ with $\epsilon \leq d(x, y) < \epsilon + \eta$ we have $d(f(x), f(y)) < \epsilon$.*

By using an argument similar to the one in the Meir-Keeler fixed point theorem [14] we have:

Theorem 4.14 *Let (X, d) be a generalized complete metric space and $f : X \to X$ be a Meir-Keeler type operator. Suppose there exists $x_0 \in X$ such that $d(x_0, f(x_0)) < +\infty$.*
Then $F_f \neq \emptyset$. Moreover, if additionally $x, y \in F_f$ implies $d(x, y) < +\infty$, then $F_f = \{x^*\}$.

Proof. Denote $x_n := f^n(x_0)$, $n \in \mathbb{N}$.

The proof of the theorem can be organized in five steps.
Step 1. We prove that

$$d(f(x), f(y)) < d(x, y), \text{ for each } x, y \in X \text{ with } x \neq y \text{ and } d(x, y) < +\infty.$$

Let $x, y \in X$ be such that $x \neq y$ and $d(x, y) < +\infty$. Then by letting $\epsilon := d(x, y)$ in the definition of Meir-Keeler operators we get $d(f(x), f(y)) < d(x, y)$.
Step 2. We can prove, by induction, that $d(x_n, x_{n+1}) < +\infty$, for all $n \in \mathbb{N}$.
Step 3. We prove that the sequence $a_n := d(x_n, x_{n+1}) \searrow 0$ as $n \to +\infty$.

If there is $n_0 \in \mathbb{N}$ such that $a_{n_0} = 0$ then $x_{n_0} \in F_f$.

If $a_n \neq 0$, for each $n \in \mathbb{N}$, then $a_n = d(f(x_{n-1}), f(x_n)) < d(x_{n-1}, x_n) = a_{n-1}$. Hence the sequence $(a_n)_{n \in \mathbb{N}}$ converges to a certain $a \geq 0$. Suppose that $a > 0$. Then, for each $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $\epsilon \leq a_n < \epsilon + \eta$, for all $n \geq n_\epsilon$. Then, by the Meir-Keeler condition we obtain $a_{n+1} < \epsilon$, which is a contradiction with the above relation.
Step 4. We will prove that the sequence (x_n) is Cauchy.

Suppose, by contradiction, that (x_n) is not a Cauchy sequence. Then, there exists $\epsilon > 0$ such that $\limsup d(x_m, x_n) > 2\epsilon$. For this ϵ there exists $\eta := \eta(\epsilon) > 0$ such that for $x, y \in X$ with $\epsilon \leq d(x, y) < \epsilon + \eta$ we have $d(f(x), f(y)) < \epsilon$. Choose $\delta := \min\{\epsilon, \eta\}$. Since $a_n \searrow 0$ as $n \to +\infty$ it follows that there is $p \in \mathbb{N}$ such that $a_p < \frac{\delta}{3}$. Let $m, n \in \mathbb{N}^*$ with $n > m > p$ such that $d(x_n, x_m) > 2\epsilon$. For $j \in [m, n]$ we have $|d(x_m, x_j) - d(x_m, x_{j+1}| \leq a_j < \frac{\delta}{3}$. Also, $d(x_m, x_{m+1} < \epsilon$ and $d(x_m, x_n) > \epsilon + \delta$ we obtain that there exists $k \in [m, n]$ such that $\epsilon < \epsilon + \frac{2\delta}{3} < d(x_m, x_k) < \epsilon + \delta$. On the other hand, for any $m, l \in \mathbb{N}$ we have: $d(x_m, x_l) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{l+1}) +$

$d(x_{l+1}, x_l) = a_m + d(f(x_m), f(x_l)) + a_l < \frac{\delta}{3} + \epsilon + \frac{\delta}{3}$. The contradiction proves that (x_n) is Cauchy.

Step 5. We prove that $x^* := \lim\limits_{n\to+\infty} x_n$ is a fixed point of f.

Since f is continuous and $x_{n+1} = f(x_n)$, we get by passing to the limit that $x^* = f(x^*)$.

If $x^*, y \in F_f$ are two distinct fixed points of f then, by the contractive condition, we get the following contradiction: $d(x^*, y) = d(f(x^*), f(y)) < d(x^*, y)$. This completes the proof. □

4.6 Caristi operators

Let (X, d) be a generalized metric space.

Definition 4.15 *A space X is said to be sequentially complete in Weierstrass' sense (see [33]) if each sequence $(x_n)_{n\in\mathbb{N}}$ in X such that $\sum\limits_{n=0}^{+\infty} d(x_n, x_{n+1}) < +\infty$ is convergent in X.*

Definition 4.16 *Let (X, d) be a generalized metric space. Then, $f : X \to X$ is called a Caristi operator if there exists a functional $\varphi : X \to \mathbb{R}_+$ such that*

$$d(x, f(x)) \le \varphi(x) - \varphi(f(x)), \text{ for every } x \in X .$$

Theorem 4.17 *Let (X, d) be a sequentially complete (in Weierstrass' sense) generalized metric space and $f : X \to X$ be a closed Caristi operator. Then f is a weakly Picard operator.*

Proof. We remark that if f is a Caristi operator, then $d(x, f(x)) < +\infty$ for every $x \in X$. Denote by $x_n := f^n(x)$, for $n \in \mathbb{N}$. Then:

$$\sum_{n=0}^{+\infty} d(x_n, x_{n+1}) = \sum_{n=0}^{+\infty} d(f^n(x), f^{n+1}(x)).$$

We will prove that the series $\sum\limits_{n=0}^{+\infty} d(f^n(x), f^{n+1}(x))$ is convergent. For this purpose we need to show that the sequence of its partial sums is convergent in $\mathbb{R}_+$. Denote by $s_n := \sum\limits_{k=0}^{n} d(f^k(x), f^{k+1}(x))$. Then $s_{n+1} - s_n = d(f^{n+1}(x), f^{n+2}(x)) \ge 0$, for each $n \in \mathbb{N}$. Moreover $s_n = \sum\limits_{k=0}^{n} d(f^k(x), f^{k+1}(x)) \le \varphi(x)$. Hence $(s_n)_{n\in\mathbb{N}}$ is upper bounded and increasing in $\mathbb{R}_+$. Then the sequence $(s_n)_{n\in\mathbb{N}}$ is convergent.

It follows that the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy and, from the sequentially completeness of the space, convergent to a certain element $x^* \in X$. The conclusion follows from the fact that f is closed. □

4.7 Fixed point theorems in a set with two generalized metrics

Let X be a nonempty set and $d, \rho : X \times X \to R_+ \cup \{+\infty\}$ be two generalized metrics on X. In this subsection we will present Maia's fixed point theorem for the case of a set with two generalized metrics.

Theorem 4.18 *Let X be a nonempty set, $d, \rho : X \times X \to R_+ \cup \{+\infty\}$ two generalized metrics on X and $f : X \to X$. We suppose that:*

i) (X, d) is a complete generalized metric space;

ii) there exists $c > 0$ such that $d(x, y) \leq c \cdot \rho(x, y)$ for all $x, y \in X$ with $\rho(x, y) < +\infty$;

iii) for every $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $\rho\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$;

iv) $f : (X, \rho) \to (X, \rho)$ is an α-contraction.

Then f is weakly Picard.

Proof. For each $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $\rho\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$. Also, there exists $i \in I$ such that $X_i \in I(f)$ and $f^n(x) \in X_i$ for all $n \geq n(x)$. Since $f : (X, \rho) \to (X, \rho)$ is an α-contraction, the sequence $(f^n(x))_{n\in\mathbb{N}}$ is Cauchy in (X, ρ). Using conditions *ii*), *iii*) and *iv*) we get

$$d\left(f^n(x), f^{n+p}(x)\right) \leq c \cdot \rho\left(f^n(x), f^{n+p}(x)\right) \leq c \cdot \frac{\alpha^{n-n(x)}}{1-\alpha} \rho\left(f^{n(x)}(x), f^{n(x)+1}(x)\right), \quad n \geq n(x),$$

so $d(f^n(x), f^{n+p}(x)) \to 0$ as $n \to +\infty$. Thus $(f^n(x))_{n\in\mathbb{N}}$ is Cauchy sequence in (X, d), which implies that $f^n(x) \to x^* \in X_i$. By condition *iv*) we have that $x^* \in F_f$. Hence f is weakly Picard.□

An improved version of Maia's theorem can be obtained by replacing the assumption *ii*) with a more useful condition (from an application point of view), see I.A. Rus [20].

Theorem 4.19 *Let X be a nonempty set, $d, \rho : X \times X \to R_+ \cup \{+\infty\}$ two generalized metrics on X and $f : X \to X$. We suppose that:*

i) (X, d) is a complete generalized metric space;

ii) there exists $c > 0$ such that $d(f(x), f(y)) \leq c \cdot \rho(x, y)$, for all $x, y \in X$ with $\rho(x, y) < +\infty$;

iii) for every $x \in X$ there exists $n(x) \in \mathbb{N}$ such that $\rho\left(f^{n(x)}(x), f^{n(x)+1}(x)\right) < +\infty$;

iv) $f : (X, \rho) \to (X, \rho)$ is an α-contraction.

Then f is a weakly Picard operator.

Proof. The proof follows the method in Theorem 4.18. □

5. Multivalued operators in generalized metric spaces

5.1 General considerations

Let (X, d) be a generalized metric space. Let Y, Z be two nonempty subsets of X and $T : Y \to P(Z)$ be a multivalued operator. By definition, $t : Y \to Z$ is a selection of T if $t(x) \in T(x)$, for each $x \in Y$. If $T : X \to P(X)$ is a multivalued operator, then $x^* \in X$ is a fixed point for T if and only if $x^* \in T(x^*)$. Denote by F_T the set of all fixed points for T. Also, $x^* \in X$ is called a strict fixed point for T if and only if $\{x^*\} = T(x^*)$. We will denote by $(SF)_T$ the set of all strict fixed points of T. By $Graph(T) := \{(x, y) \in X \times X | y \in T(x)\}$ we denote the graph of the multivalued operator T and by $T(Y) := \bigcup_{x \in Y} T(x)$ the image through T of the set $Y \in P(X)$.

Recall that if $Y \subseteq X$, then $T(Y) := \bigcup_{x \in Y} T(x)$. We also denote by $T^n := T \circ T \cdots \circ T$ (the n times composition).

Recall that, if (X, d) is a metric space, then $T : X \to P_{cl}(X)$ is said to be a multivalued a-contraction if

$$a \in [0, 1[\text{ and } H_d(T(x), T(y)) \leq a d(x, y), \text{ for each } x, y \in X.$$

The following result is known as Covitz-Nadler fixed point principle.

Theorem 5.1 (Covitz-Nadler [5]) *Let (X, d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Then, for each $x_0 \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X with $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N}$, which converges to a fixed point of T.*

Remark 5.2 *From the proof of the above result it follows that for each $x \in X$ and each $y \in T(x)$ there exists in X a sequence $(x_n)_{n\in\mathbb{N}}$ with the properties:*
a) $x_0 = x, x_1 = y$;
b) $x_{n+1} \in T(x_n)$ for all $n \in \mathbb{N}^*$;
c) $(x_n)_{n\in\mathbb{N}}$ converges to a fixed point of T.

This principle gave rise to the following concept.

Definition 5.3 (Rus-Petruşel-Sîntămărian [28], [29]) Let $(X, \to)$ be an L-space. Then $T : X \to P(X)$ is a multivalued weakly Picard operator (briefly MWP operator) if for each $x \in X$ and each $y \in T(x)$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in X such that:

i) $x_0 = x,\ x_1 = y$

ii) $x_{n+1} \in T(x_n)$, for all $n \in \mathbb{N}$

iii) the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent and its limit is a fixed point of T.

A sequence $(x_n)_{n\in\mathbb{N}}$ in X satisfying the conditions (i) and (ii) in Definition 5.3 is called a sequence of successive approximations for T starting from (x, y).

The aim of this section is to establish some fixed point results for multivalued operators of contractive type on generalized metric space.

5.2 Multivalued contractions on generalized metric spaces

Let us recall first some contractive-type conditions for multivalued operators.

Definition 5.4 Let (X,d) be a generalized metric space. Then $T: X \to P_{cl}(X)$ is called a multivalued a-contraction if $a \in [0,1[$ and

$$H_d(T(x),T(y)) \leq ad(x,y), \text{ for each } x,y \in X, \text{ with } d(x,y) < +\infty.$$

Let (X,d) be a generalized metric space. We denote by $\mathcal{P}(X)$ the set of all subsets of a nonempty set X.

Definition 5.5 Let (X,d) be a generalized metric space. If $T: X \to P(X)$ is a multivalued operator, then we consider the following multivalued operators generated by T:

$$\widehat{T}: X \to \mathcal{P}(X),\ \widehat{T}(x) := T(x) \cap X_{i(x)}$$

(where $X_{i(x)}$ denotes the unique element of the canonical decomposition of X where x belongs),

$$\tilde{T}^i: X \to \mathcal{P}(X),\ \tilde{T}^i(x) := T(x) \cap X_i$$

(where X_i denotes an arbitrary element of the canonical decomposition of X).

Then we have:

Lemma 5.6 $F_T = F_{\widehat{T}}$.

Lemma 5.7 $F_T \neq \emptyset \ \Leftrightarrow$ *if there exists* $i \in I$ *such that* $F_{\tilde{T}^i} \neq \emptyset$.

The following result is a straightforward version of Covitz and Nadler alternative theorem in [5].

Theorem 5.8 *Let (X,d) be a generalized complete metric space and $T: X \to P_{cl}(X)$ be a multivalued a-contraction. Suppose that for each $x \in X$ there is $y \in T(x)$ such that $d(x,y) < +\infty$. Then there exists a sequence of successive approximations of T starting from any arbitrary $x \in X$ which converges to a fixed point of T.*

The previous result gives rise to the following open question.

Open question. Let $T: X \to P_{cl}(X)$ be a multivalued a-contraction as in the above Covitz-Nadler fixed point result. Is T a MWP operator ?

Theorem 5.9 *Let (X,d) be a generalized complete metric space and $T: X \to P_{cl}(X)$ be a multivalued a-contraction. Suppose there exists $x_0 \in X$ and $x_1 \in T(x_0)$ such that $d(x_0,x_1) < +\infty$.*

Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ *of successive approximations for* T *starting from* x_0 *which converges to a fixed point of* T*.*

Proof. Let $X := \bigcup_{i\in I} X_i$ be the canonical decomposition of X into metric spaces. Recall that X is complete if and only if X_i is complete for each $i \in I$. Let $j \in I$ such that $x_0 \in X_j$.

For $x \in X$ we successively have:

$$D(x,T(x)) < +\infty \Leftrightarrow \text{ there exists } y \in T(x) \text{ such that } d(x,y) < +\infty \Leftrightarrow y \in T(x) \cap X_{i(x)}.$$

Hence

$$D(x,T(x)) < +\infty \Leftrightarrow T(x) \cap X_{i(x)} \neq \emptyset.$$

Consider now the multivalued operator

$$\tilde{T}^j : X \to \mathcal{P}(X),\ \tilde{T}^j(x) := T(x) \cap X_j.$$

We will prove that $\tilde{T}^j_{|X_j} : X_j \to P_{cl}(X_j)$. For this purpose, it is enough to show that

$$D(x,T(x)) < +\infty, \text{ for each } x \in X_j.$$

For $x \in X_j$ we have:
$D(x,T(x)) \le D(x,T(x_0)) + H(T(x_0),T(x)) \le d(x,x_0) + D(x_0,T(x_0)) + ad(x_0,x) < +\infty.$

Hence $\tilde{T}^j_{|X_j} : X_j \to P_{cl}(X_j)$ is a multivalued a-contraction on the complete metric space $(X_j, d_{|X_j\times X_j})$. The conclusion follows from Lemma 5.7 and Theorem 5.1. □

An answer to the above problem is the following result.

Theorem 5.10 *Let* (X,d) *be a generalized complete metric space and* $T : X \to P_{cl}(X)$ *be a multivalued* a*-contraction. Suppose that for each* $x \in X$ *and* $y \in T(x)$ *we have* $d(x,y) < +\infty$ *(or equivalently, for each* $x \in X$ *we have* $T(x) \subset X_{i(x)}$*). Then* T *is a MWP operator.*

Proof. From the hypothesis we have that $D(x,T(x)) < +\infty$, for each $x \in X$. Hence, for each $x \in X$ we have that $T : X_{i(x)} \to P_{cl}(X_{i(x)})$. Since $(X_{i(x)}, d_{|X_{i(x)}\times X_{i(x)}})$ is a complete metric space, by Theorem 5.1 and Remark 5.2, we conclude that T is a MWP operator. □

We introduce now the following concepts.

Definition 5.11 (Rus-Petruşel-Sîntămărian [29]) Let $(X, \to)$ be an L-space and $T : X \to P(X)$ be a MWP operator. Define the multivalued operator $T^\infty : Graph(T) \to P(F_T)$ by the formula $T^\infty(x,y) = \{\ z \in F_T \mid$ there exists a sequence of successive approximations of T starting from (x,y) that converges to $z\ \}$.

Definition 5.12 (see also Rus-Petruşel-Sîntămărian [29]) Let (X,d) be a generalized metric space and $T : X \to P(X)$ be a MWP operator such that for each $x \in X$ and $y \in T(x)$ we have that $d(x,y) < +\infty$. Then, T is called a c-multivalued weakly Picard operator (briefly c-MWP operator) if there exists a selection t^∞ of T^∞ such that $d(x, t^\infty(x,y)) \le c\ d(x,y)$, for all $(x,y) \in Graph(T)$.

As an example, we have:

Theorem 5.13 *Let (X, d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction, such that for each $x \in X$ and $y \in T(x)$ we have $d(x, y) < +\infty$.*

Then T is a $\frac{1}{1-a}$-MWP operator.

We present now an abstract data dependence theorem for the fixed point set of c-MWP operators on generalized metric spaces.

Theorem 5.14 *Let (X, d) be a generalized metric space and $T_1, T_2 : X \to P(X)$ be two multivalued operators. We suppose that:*

i) T_i is a c_i-MWP operator, for $i \in \{1, 2\}$

ii) there exists $\eta > 0$ such that $H(T_1(x), T_2(x)) \leq \eta$, for all $x \in X$.

Then $H(F_{T_1}, F_{T_2}) \leq \eta \ \max \{ \ c_1, \ c_2 \ \}$.

Proof. The proof follows in a similar way to Rus-Petruşel-Sîntămărian [29]. For the sake of completeness we present it here.
Let $t_i : X \to X$ be a selection of T_i for $i \in \{1, 2\}$. Let us remark that

$$H(F_{T_1}, F_{T_2}) \leq \max \left\{ \sup_{x \in F_{T_2}} d(x, t_1^{\infty}(x, t_1(x))), \sup_{x \in F_{T_1}} d(x, t_2^{\infty}(x, t_2(x))) \right\}.$$

Let $q > 1$. Then we can choose t_i $(i \in \{1, 2\})$ such that

$$d(x, t_1^{\infty}(x, t_1(x))) \leq c_1 q H(T_2(x), T_1(x)), \text{ for all } x \in F_{T_2}$$

and

$$d(x, t_2^{\infty}(x, t_2(x)) \leq c_2 q H(T_1(x), T_2(x)), \text{ for all } x \in F_{T_1}.$$

Thus we have $H(F_{T_1}, F_{T_2}) \leq q\eta \max\{c_1, c_2\}$. Letting $q \searrow 1$, the proof is complete. □

Notice that the above conclusions means that the data dependence phenomenon of the fixed point set for c-MWP operators holds.

We also have:

Theorem 5.15 *Let (X, d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a multivalued a-contraction. Suppose:*

(i) $(SF)_T \neq \emptyset$;

(ii) If $x, y \in F_T$ then $d(x, y) < +\infty$.

Then $F_T = (SF)_T = \{x^*\}$.

Proof. We will prove first that $(SF)_T = \{x^*\}$. Indeed, if $z \in (SF)_T$ with $z \neq x^*$, then $d(z, x^*) < +\infty$ and $d(z, x^*) = H(T(z), T(x^*)) \leq ad(z, x^*)$, a contradiction. Next we will prove that $F_T \subseteq$

$(SF)_T$. Let $y \in F_T$. Then $d(y, x^*) < +\infty$. Thus $d(y, x^*) = D(y, T(x^*)) \leq H(T(y), T(x^*)) \leq ad(y, x^*)$, which implies $y = x^*$. This completes the proof. □

5.3 Pseudo-contractive multivalued operators on generalized metric spaces

In D. Azé and J.-P. Penot [1] the following concept is introduced.

Definition 5.16 (Azé-Penot [1]) Let (X, d) be a metric space. A multivalued operator $T : X \to P(X)$ is said to be pseudo-a-Lipschitzian with respect to the subset $U \subset X$ whenever, for all $x, y \in U$, we have

$$\rho_d(T(x) \cap U, T(y)) \leq ad(x, y).$$

Also, the multivalued opeator T is called pseudo-a-contractive with respect to U if it is pseudo-a-Lipschitzian with respect to U for some $a \in [0, 1[$.

In Azé-Penot [1], the fixed point theory for multivalued pseudo-a-contractive operators with respect to the open ball $B_d(x_0, r)$ of a complete metric space (X, d) is studied. The aim of this section is to give some fixed point results for multivalued pseudo-a-contractive operators in the setting of a generalized metric space.

Theorem 5.17 *Let (X, d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a multivalued operator. Let $X := \bigcup_{i \in I} X_i$ be the canonical decomposition of X. Suppose that there exists $x_0 \in X$ such that $D(x_0, T(x_0)) < +\infty$ and T is pseudo a-contractive with respect to $X_{i(x_0)}$. Then $F_T \neq \emptyset$.*

Proof. Since $D(x_0, T(x_0)) < +\infty$ there exists $b > 0$ and $x_1 \in T(x_0)$ such that $d(x_0, x_1) < b < +\infty$. Then $x_1 \in X_{i(x_0)}$ and thus $x_1 \in T(x_0) \cap X_{i(x_0)}$. Hence we have $D(x_1, T(x_1)) \leq \rho(T(x_0) \cap X_{i(x_0)}, T(x_1)) \leq ad(x_0, x_1) < ab$. Thus there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) < ab < +\infty$. Thus $x_2 \in T(x_1) \cap X_{i(x_0)}$. In a similar way, we have $D(x_2, T(x_2)) \leq \rho(T(x_1) \cap X_{i(x_0)}, T(x_2)) \leq ad(x_1, x_2) < a^2 b < +\infty$.

By induction, we obtain a sequence $(x_n)_{n \in \mathbb{N}}$ with the following properties:

(a) $x_{n+1} \in T(x_n) \cap X_{i(x_0)}$, for all $n \in \mathbb{N}$;

(b) $d(x_n, x_{n+1}) < a^n b$, for all $n \in \mathbb{N}$.

From (b) we get that $(x_n)_{n \in \mathbb{N}}$ is Cauchy and hence convergent in $X_{i(x_0)}$. Thus there exists $x^* \in X_{i(x_0)}$ (since $X_{i(x_0)}$ is d-closed), such that $x_n \to x^*$ as $n \to +\infty$. Let us show now that $x^* \in F_T$. We have $D(x^*, T(x^*)) \leq d(x^*, x_{n+1}) + D(x_{n+1}, T(x^*)) \leq d(x^*, x_{n+1}) + \rho(T(x_n) \cap X_{i(x_0)}, T(x^*)) \leq d(x^*, x_{n+1}) + ad(x^*, x_n)) \to 0$ as $n \to +\infty$. Hence $x^* \in T(x^*)$. □

A second answer to the open problem mentioned in Section 3 is the following:

Theorem 5.18 *Let (X, d) be a generalized complete metric space and $T : X \to P_{cl}(X)$ be a* *multivalued operator such that for each* $x \in X$ *and* $y \in T(x)$ *we have* $d(x,y) < +\infty$*. Let* $X := \bigcup_{i \in I} X_i$ *be the canonical decomposition of* X*. Suppose that* T *is pseudo a-contractive with respect to* $X_{i(x)}$*, for each* $x \in X$*. Then* T *is a MWP operator.*

Proof. Let $x_0 \in X$ and $x_1 \in T(x)$ such that $d(x_0, x_1) < b < +\infty$, for some $b > 0$. Thus $x_1 \in T(x_0) \cap X_{i(x_0)}$. Hence we have $D(x_1, T(x_1)) \leq \rho(T(x_0) \cap X_{i(x_0)}, T(x_1)) \leq ad(x_0, x_1) < ab$. We obtain that there exists $x_2 \in T(x_1)$ such that $d(x_1, x_2) < ab < +\infty$. Thus $x_2 \in T(x_1) \cap X_{i(x_0)}$. In a similar way, we have $D(x_2, T(x_2)) \leq \rho(T(x_1) \cap X_{i(x_0)}, T(x_2)) \leq ad(x_1, x_2) < a^2 b < +\infty$.

By induction, we obtain a sequence $(x_n)_{n \in \mathbb{N}}$ with the following properties:

(a) $x_{n+1} \in T(x_n) \cap X_{i(x_0)}$, for all $n \in \mathbb{N}$;

(b) $d(x_n, x_{n+1}) < a^n b$, for all $n \in \mathbb{N}$.

From (b) we get that $(x_n)_{n \in \mathbb{N}}$ is Cauchy and hence convergent in $X_{i(x_0)}$ to a certain x^*. As before, we obtain $x^* \in T(x^*)$. Since $x_0 \in X$ and $x_1 \in T(x_0)$ were arbitrarily chosen, we get that T is a MWP operator. □

Received: February 2008. Revised: February 2008.

References

[1] D. Azé and J.-P. Penot, *On the dependence of fixed points sets of pseudo-contractive multifunctions. Application to differential inclusions*, Nonlinear Dyn. Syst. Theory, **6** (2006), 31–47.

[2] C. Bessaga, *On the converse of the Banach fixed point principle*, Colloq. Math., **7** (1959), 41–43.

[3] G. Beer, *Topologies on Closed and Closed Convex Sets*, Kluwer Acad. Publ., Dordrecht, 1994.

[4] L.M. Blumenthal, *Theory and Applications of Distance Geometry*, Oxford University Press, 1953.

[5] H. Covitz and S.B. Nadler, *Multi-valued contraction mapping in generalized metric spaces*, Israel J. Math., **8** (1970), 5–11.

[6] G. Dezso, *Fixed point theorems in generalized metric spaces*, Pure Math. Appl., **11** (2000), 183–186.

[7] J.B. Diaz and B. Margolis, *A fixed point theorem for the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc., **74** (1968), 305–309.

[8] M. Fréchet, *Les espaces abstraits*, Gauthier-Villars, Paris, 1928.

[9] C.F.K. Jung, *On generalized complete metric spaces*, Bull. A.M.S., **75** (1969), 113–116.

[10] S. Kasahara, *On some generalizations of the Banach contraction theorems*, Mathematics Seinar Notes, **3** (1975), 161–169.

[11] P.Q. Khanh, *Remarks on fixed point theorems based on iterative approximations*, Polish Acad. Sciences, Inst. of Mathematics, Preprint 361, 1986.

[12] R. Kopperman, *All topologies come from generalized metrics*, Amer. Math. Monthly, **95** (1988), 89–97.

[13] W.A.J. Luxemburg, *On the convergences of successive approximations in the theory of ordinary differential equations*, Indag. Math., **20** (1958), 540–546.

[14] A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., **28** (1969) 326–329.

[15] S.B. Nadler jr., *Multivalued contraction mappings*, Pacific J. Math., **30** (1969), 475–488.

[16] A. Petruşel, *Multivalued weakly Picard operators and applications*, Scientiae Mathematicae Japonicae, **59** (2004), 167–202.

[17] A. Petruşel and I.A. Rus, *Multivalued Picard and weakly Picard operators*, Fixed Point Theory and Applications (E. Llorens Fuster, J. Garcia Falset, B. Sims-Eds.), Yokohama Publishers, 2004, 207–226.

[18] S. Reich, *Some remarks concerning contraction mappings*, Canad. Math. Bull., **14** (1971), 121–124.

[19] S. Reich, *Fixed point of contractive functions*, Boll. U.M.I., **5** (1972), 26–42.

[20] I.A. Rus, *Metrical Fixed Point Theorems*, Cluj-Napoca, 1979.

[21] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[22] I.A. Rus, *Picard operators and applications*, Scientiae Mathematicae Japonicae, **58** (2003), 191–219.

[23] I.A. Rus, *Metric sapces with fixed point property with respect to contractions*, Studia Univ. Babeş-Bolyai Math., **51** (2006), 115–121.

[24] I.A. Rus, *Weakly Picard mappings*, Comment. Math. Univ. Carolinae, **34** (1993), 769–773.

[25] I.A. Rus, *Weakly Picard operators and applications*, Seminar on Fixed Point Theory, Cluj-Napoca, **2** (2001), 41–58.

[26] I.A. Rus, *The theory of a metrical fixed point theorem: theoretical and applicative relevances*, Fixed Point Theory, **9** (2008), to appear.

[27] I.A. Rus, A. Petruşel and G. Petruşel, *Fixed Point Theory 1950–2000: Romanian Contributions*, House of the Book of Science, Cluj-Napoca, 2002.

[28] I.A. Rus, A. Petruşel and A. Sîntămărian, *Data dependence of the fixed point set of multivalued weakly Picard operators*, Studia Univ. Babeş-Bolyai Mathematica, **46** (2001), 111–121.

[29] I.A. Rus, A. Petruşel and A. Sîntămărian, *Data dependence of the fixed point set of some multivalued weakly Picard operators*, Nonlinear Analysis, **52** (2003), 1947–1959.

[30] I.A. Rus, A. Petruşel and M.A. Şerban, *Weakly Picard operators: equivalent definitions, applications and open problems*, Fixed Point Theory, **7** (2006), 3–22.

[31] M.A. Şerban, *Fibre contraction theorem in generalized metric spaces*, Automation Computers Applied Mathematics, **16** (2007), No. 1–2, 9–14.

[32] S.-W. Xiang, *Equivalence of completeness and contraction property*, Proc. Amer. Math. Soc., **135** (2007), 1051–1058.

[33] P.P. Zabreiko, *K-metric and K-normed linear spaces: survey*, *Collect. Math.,* **48** (1997), 825–859.

CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$04, (67–72). December 2008

A Fixed Point Theorem for Certain Operators

B.E. Rhoades
Department of Mathematics, Indiana University,
Bloomington, IN 47405-7106
email: `rhoades@indiana.edu`

ABSTRACT

We obtain a fixed point theorem for a class of operators. This result is an extension of a similar theorem of Constantin (1994).

RESUMEN

Obtenemos un teorema de punto fijo para una clase de operadores. Este resultado es una extensión de un teorema similar devido a Constantin (1994).

Key words and phrases: *Fixed point theorem.*

Math. Subj. Class.: *47H10.*

Most fixed point theorems are proved either by examining the successive iterates of the operator, or by constructing an iteration scheme, such as that of Mann or Ishikawa. In this paper we consider the situation in which the operator is used on the successive iterates of a sequence.

In a recent paper, Constantin [3] obtained a fixed point theorem for a class of operators which are selfmaps of a Banach space X, and which satisfy the condition

$$\|Tx - Ty\| \le g(\|x-y\|, \|x-Tx\|, \|y-Ty\|) \tag{1}$$

for all $x, y \in X$, where $g : \mathbb{R}_+^3 \to \mathbb{R}_+, g$ is continuous, nondecreasing in each variable, and is such that, if $h(r) := g(r,r,r)$, then $r - h(r)$ is nonnegative and strictly increasing on $\mathbb{R}_+$.

A natural extension of (1) would be: Let Λ denote the set of all continuous functions $g : \mathbb{R}_+^5 \to \mathbb{R}_+$, nondecreasing in each variable and such that, if $h(r) := g(r,r,r,r,r)$, then $h(r) < r$ for each $r > 0$. Define T from X to X satisfying

$$\begin{aligned}\|Tx - Ty\| \le g(&\|x-y\|, \|x-Tx\|, \|y-Ty\|,\\ &\|x-Ty\|, \|y-Tx\|)\end{aligned} \tag{2}$$

for all $x, y \in X$, some $g \in \Lambda$.

However, a slightly more general extension of (1) is the following. Let X be a Banach space, $g : \mathbb{R}_+^3 \to \mathbb{R}_+, g$ continuous, nondecreasng, and satisfying $g(t) < t$ for each $t > 0$. Let T be a selfmap of X satisfying

$$\|Tx - Ty\| \le g(M(x,y)), \quad \text{for all} \quad x, y \in X, \tag{3}$$

where

$$\begin{aligned}M(x,y) := \max\{&\|x-y\|, \|x-Tx\|, \|y-Ty\|,\\ &\|x-Ty\|, \|y-Tx\|).\end{aligned}$$

Theorem 1. *Let A satisfy* (3) *and $\{x_n\} \subset X$. Then the following are equivalent:*

(i) $Tx_n - x_n \to 0$ *as* $n \to \infty$,

(ii) $\{Tx_n - x_n\}$ *is bounded, and* $\{x_n\}$ *converges to a point* p *which is the unique fixed point of* T.

Proof. (i) $\Rightarrow$ (ii). Define $y_n = Tx_n - x_n, \alpha_n = \sup_n\{\|x_m - x_n\| : m \ge n\}$, and $\beta_n = \sup_n\{\|y_m\| : m \ge n\}$. Then $\{\alpha_n\}$ and $\{\beta_n\}$ are nonincreasing nonnegative sequences. Hence $\lim \alpha_n = \alpha \ge 0$ and, from the hypotheses, $\lim y_n = 0$ and $\{y_n\}$ is bounded.

Assume that $\alpha > 0$. From (3), with $m \ge n$,

$$\begin{aligned}\|x_m - x_n\| \le \|Tx_m - y_m - (Tx_n - y_n)\| &\le \|Tx_m - Tx_n\|\\ &\quad + \|y_m - y_n\|\\ &\le g(\max\{\|x_m - x_n\|, \|y_m\|, \|y_n\|, \|x_m - Tx_n\|, \|x_n - Tx_m\|\})\\ &\quad + 2\beta_n\\ &\le g(\max\{\alpha_n, \beta_n, \beta_n, \alpha_n + \beta_n, \alpha_n + \beta_n\}) + 2\beta_n.\end{aligned}$$

Thus, $\alpha_n \leq g(\alpha_n+\beta_n)+2\beta_n$. Taking the limit as $n \to \infty$ yields $\alpha \leq g(\alpha) < \alpha$, a contradiction. Therefore $\alpha = 0$ and $\{x_n\}$ is Cauchy, hence convergent to some point p in X.

Since $Tx_n - x_n \to 0$ and $x_n \to p$, it follows that $Tx_n \to p$. Again using (3),

$$\|Tp - Tx_n\| \leq g(\max\{\|p - x_n\|, \|p - Tp\|, \|y_n\|, \|p - Tx_n\|, \|x_n - Tp\|\}).$$

Taking the limit as $n \to \infty$ yields

$$\|Tp - p\| \leq g(\max\{0, \|p - Tp\|, 0, 0, \|p - Tp\|\}) = g(\|p - Tp\|),$$

which implies that $\|p - Tp\| = 0$, or $Tp = p$.

To prove uniqueness, suppose that q is also a fixed point of T. Then, from (3),

$$\begin{aligned}\|p - q\| &= \|Tp - Tq\| \\ &\leq g(\max\{\|p - q\|, 0, 0, \|p - q\|, \|q - p\|\}) \\ &= g(\|p - q\|),\end{aligned}$$

which implies that $p = q$.

(ii) $\Rightarrow$ (i) Using (3),

$$\begin{aligned}\|Tx_n - x_n\| = \|Tx_n - Tp + Tp - x_n\| &\leq \|Tx_n - Tp\| + \|p - x_n\| \\ &\leq g(\max\{\|x_n - p\|, \|x_n - Tx_n\|, \|p - Tp\|, \\ &\qquad \|x_n - Tp\|, \|p - Tx_n\|\}) + \|p - x_n\|.\end{aligned}$$

Taking the lim sup of both sides, since $\{y_n\}$ is bounded, one obtains, with the identification $\gamma = \limsup \|y_n\|$,

$$\gamma \leq g(\max\{0, \gamma, 0, 0, \limsup \|p - Tx_n\|\}) + 0.$$

But $\limsup \|p - Tx_n\| \leq \limsup(\|p - x_n\| + \|x_n - Tx_n\|) = \gamma$. Therefore we have $\gamma \leq g(\gamma)$, which implies that $\gamma = 0$. $\square$

The special case of (3) with $g(t) := kt$ for some $0 \leq k < 1$, and X a metric space is that of Ćirić [2], which was shown in [7] to be one of the most general contractive definitions for which a unique fixed point exists.

In order to prove that a map satisfying (3) has a fixed point, it would be necessary to show that the orbit of some $x \in X$ is bounded, which cannot be implied from (3). However, the following is true.

Theorem 2. *Let X be a complete metric space, T a selfmap of X satisfying*

$$d(Tx, Ty) \leq g(M(x, y)), \qquad \text{for each} \quad x, y \in X, \tag{4}$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

If there exists a point $x_0 \in X$ with bounded orbit, then T has a unique fixed point in X.

Proof. For any $n \in \mathbb{N}$, $O(x,n) := \{x, Tx, T^2x, \ldots, T^nx\}$, and $\delta(A)$ denotes the diameter of a set A. Let $m, n \in \mathbb{N}, n < m$. Then, from (4), with $x = x_0$,

$$\begin{aligned} d(T^nx, T^mx) &= d(T(T^{n-1}x), T(T^{m-1}x)) \\ &\leq g(\max\{d(T^{n-1}x, T^{m-1}x), d(T^{n-1}x, T^nx), d(T^{m-1}x, T^mx), \\ &\qquad d(T^{n-1}x, T^mx), d(T^{m-1}x, T^nx)\} \\ &\leq g(\delta[O(T^{n-1}x, n-m+1)]) \\ &\leq g(g(\delta[O(T^{n-2}x, n-m+2)]) \\ &\quad \ldots \\ &\leq g^n(\delta[O(x,m)]). \end{aligned} \tag{5}$$

It is well known that the hypotheses on g imply that $\lim g^n(t) = 0$ for each $t \geq 0$. Since the orbit of $x = x_0$ is bounded, (5) implies that $\{T^nx\}$ is Cauchy, hence convergent to a point $p \in X$.

Suppose that $p \neq Tp$. Then, from (4),

$$\begin{aligned} d(p,Tp) &\leq d(p, T^{n+1}x) + d(T^{n+1}x, Tp) \\ &\leq d(p, T^{n+1}x) + g(\max\{d(T^nx, p), d(T^nx, T^{n+1}x), d(p,Tp), \\ &\qquad d(T^nx, p), d(p, T^{n+1}x)\}. \end{aligned}$$

Taking the limit of both sides of the above inequality as $n \to \infty$ yields

$$d(p,Tp) \leq g(d(p,Tp)) < d(p,Tp),$$

a contradiction, and $p = Tp$.

Suppose that p and q are fixed points of T, with $p \neq q$. Then, using (4),

$$\begin{aligned} d(p,q) &= d(Tp, Tq) \\ &\leq g(\max(\{d(p,q), 0, 0, d(p,q), d(q,p)\} \\ &= g(d(p,q)) < d(p,q), \end{aligned}$$

a contadiction. Therefore $p = q$. □

If T is continuous, then, even with X unbounded, Theorem 2 is a special case of Theorem 3.3 of [4]

If one replaces $M(x,y)$ with

$$m(x,y) := \max\{d(x,y), d(x,Tx), d(y,Ty), [d(x,Ty)+d(y,Tx)]/2\},$$

in Theorem 2, then Theorem 2 is true without the boundedness assumption. See, e.g., Theorem 2.2 of [1].

Most of the recent papers on fixed ont theory, which do not involve fixed point iterations, deal with four maps. For a survey of these results the reader may wish to consult [6] and the references therein.

Let $F(T)$ denote the fixed point set of a mapping T. In [5] it was conjectured that $F(T^n) = F(T)$ for every map T which satisfies a contractive condition that does not include nonexpansive maps. That conjecture was verified in [5] for many such maps. We shall now show that the same is true for maps satisfying (4).

Theorem 3. *Let X be a metric space, T a selfmap of X satisfying* (4) *with $F(T) \neq \emptyset$. Then $F(T^n) = F(T)$ for every integer $n \geq 1$.*

Proof. Since $F(T) \neq \emptyset$. $F(T^n) \neq \emptyset$. Clearly $F(T) \subseteq F(T^n)$. Suppose that $p \in F(T^n)$, for some positive integer n. We shall assume that $n > 1$, since the case for $n = 1$ is trivial. Let i, j be integers, $0 \leq i < j \leq n$. Then, using (4),

$$d(T^i p, T^j p) \leq g(M(T^{i-1}p, T^{j-1}p)) \leq g(\delta[(O(p,n)]).$$

Suppose that $\delta[(O(p,n)] > 0$. Then the above inequality implies that

$$\delta[(O(p,n)] \leq g(\delta[(O(p,n)]) < \delta[(O(p,n)],$$

a contradiction. Therefore $\delta[(O(p,n)] = 0$, and $p \in F(T)$. □

`Received: February 2008. Revised: March 2008.`

References

[1] R.P. Agarwal, Donal O'Regan, and M. Sambandham, *Random and deterministic fixed point theory for generalized contractive maps*, Applicable Analysis **83** (2004), 711–725.

[2] Lj. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 27–273.

[3] A. Constantin, *On the approximation of fixed points of operators*, Bull. Calcutta Math. Soc. **86** (1994), 323–326.

[4] J. Jachymski, *On common fixed point theorems for some families of maps*, Int. J. Pure & Appl. Math. **25** (1994), 925–937.

[5] G.S. Jeong and B.E. Rhoades, *Maps for which* $F(T) = F(T^n)$, Fixed Point Theory and Appl. **6** (2003), 87–131.

[6] Donal O'Regan, Naseer Shahzad, and Ravi P. Agarwal, *Common fixed point theory for compatible maps*, Nonlinear Analysis Forum **8** (2003), 179–22.

[7] B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Trans. Amer. Math. Soc. **226** (1977), 257–290.

CUBO A Mathematical Journal
Vol.10, $N^{\underline{o}}$ 04, (73–83). December 2008

An Intersection Theorem and its Applications

Mircea Balaj
Department of Mathematics,
University of Oradea, Romania
email: `mbalaj@uoradea.ro`
and
Donal O'Regan
Department of Mathematics,
National University of Ireland, Galway, Ireland
email: `donal.oregan@nuigalway.ie`

ABSTRACT

In this paper we obtain a very general intersection theorem for the values of a map. From this we derive existence theorems for two types of vectorial equilibrium problems, an analytic alternative and a minimax inequality involving three real functions.

RESUMEN

En este artículo obtenemos un teorema general de intersección para los valores de una aplicación. A través de este resultado deducimos teoremas de existencia para dos tipos de problemas de equilibrio vectoriales, una alternativa analítica y una desigualdad minimax envolviendo tres funciones reales.

Key words and phrases: *The better admissible class, fixed point, quasiconvex map, equilibrium problem.*

Math. Subj. Class.: *54C60, 49J35, 91B50.*

1. Introduction and preliminaries

A *multimap* (or simply a *map*) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of Y, that is a function with the *values* $T(x) \subseteq Y$ for $x \in X$. To a map $T : X \multimap Y$ we associate two other maps $T^c : X \multimap Y$ and $T^- : Y \multimap X$ defined by $T^c(x) = Y \setminus T(x)$, and respectively $T^-(y) = \{x \in X : y \in T(x)\}$ The values of T^- are called the fibers of T.

Let $T : X \multimap Y$ be a map. As usual the set $\{(x, y) \in X \times Y : y \in T(x)\}$ is called the *graph* of T. For $A \subseteq X$, and $B \subseteq Y$ let $T(A) = \bigcup_{x\in A} T(x)$ and $T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$.

For topological spaces X and Y a map $T : X \multimap Y$ is said to be: *upper semicontinuous* (u.s.c.) if for any closed set $F \subseteq Y$ the set $T^-(F)$ is closed in X; *lower semicontinuous* (l.s.c.) if for any open set $U \subseteq Y$ the set $T^-(U)$ is open in X; *compact* if $T(X)$ is contained in a compact subset of Y; *closed* if its graph is closed in $X \times Y$.

The following lemma collects known facts about u.s.c. or l.s.c. maps (see for example [7] for assertion (i), [16] for assertion (ii) and [9] for assertion (iii)).

Lemma 1 Let X and Y be topological spaces and $T : X \multimap Y$ be a map.

(i) If T has compact values, then it is u.s.c. if and only if for each $x \in X$, any net $\{x_t\}$ converging to x and any net $\{y_t\}$ with $y_t \in T(x_t)$ for all index t, there exists a subnet $\{y_{t'}\}$ of $\{y_t\}$ and $y \in T(x)$ such that $\{y_{t'}\}$ converges to y.

(ii) T is l.s.c. in $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_t\}$ converging to x, there exists a net $\{y_t\}$ converging to y, with $y_t \in T(x_t)$ for each t.

(iii) If Y is compact and T is closed, then T is u.s.c..

If X is a subset of a topological vector space we denote by coX and $\overline{X}$ the convex hull and the closure of X respectively.

Let Y be a convex set in a topological vector space and X be a topological space. The *better admissible class* $\mathcal{B}$ of mappings from Y into X (see [15]) is defined as follows:

$T \in \mathcal{B}(Y, X) \Leftrightarrow T : Y \multimap X$ is a mapping such that for any nonempty finite subset A of Y and any continuous mapping $p : T(co\ A) \to co\ A$ the composition $p \circ T_{|co\ A} : co\ A \multimap co\ A$ has a fixed point.

The class $\mathcal{B}(Y, X)$ includes many important classes of mappings, such as $U^k_c(Y, X)$ in [14], $KKM(Y, X)$ in [3] and $A(Y, X)$ in [2], as proper subclasses.

Definition 1. Let X be a convex set in a vector space and Y a vector space. A mapping $T : X \multimap Y$ is called:

(i) *quasiconvex*, if for every convex subset C of Y, $T^-(C)$ is a convex set;

(ii) *convex*, if for each $x_1, x_2 \in X$ and $\lambda \in (0, 1)$, $\lambda T(x_1) + (1 - \lambda)T(x_2) \subseteq T(\lambda x_1 + (1 - \lambda)x_2)$;

(iii) *concave*, if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$, $T(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda T(x_1) + (1-\lambda)T(x_2)$.

Lemma 2 If a map $T : X \multimap Y$ is convex then it is quasiconvex.

Proof. Let C be a convex subset of Y, $x_1, x_2 \in T^-(C)$ and $\lambda \in (0,1)$. If $y_1 \in T(x_1) \cap C$, $y_2 \in T(x_2) \cap C$, then

$$\lambda y_1 + (1-\lambda)y_2 \in (\lambda T(x_1) + (1-\lambda)T(x_2)) \cap C \subseteq T(\lambda x_1 + (1-\lambda)x_2) \cap C,$$

hence $\lambda x_1 + (1-\lambda)x_2 \in T^-(C)$. □

Let us describe in short the contents on the next sections. We obtain first a very general intersection theorem involving three maps, one of them from the class $\mathcal{B}$. Two types of applications of this result will be given in the last two sections.

The first one, offers existence theorems for the following types of vectorial equilibrium problems:

Let X be a topological space, Y be a convex set in a topological vector space, Z be a topological vector space and V be nonempty set. Let $F : Y \times Z \multimap V$, $C : Z \multimap V$ and $P : X \multimap Z$.

(I) Find $x_0 \in X$ such that $F(y,z) \subseteq C(z)$ for each $y \in Y$ and $z \in P(x_0)$;

and respectively,

(II) Find $x_0 \in X$ such that $F(y,z) \cap C(z) \neq \emptyset$ for each $y \in Y$ and $z \in P(x_0)$.

Finally, we obtain an analytic alternative and a minimax inequality involving three real functions.

From now all (topological) vector spaces will be assumed real and all topological (vector) spaces will be assumed Hausdorff.

2. An intersection theorem

Theorem 1. *Let X be a topological space, Y be a convex set in a topological vector space and Z be a nonempty set. Let $P : X \multimap Z$, $Q : Y \multimap Z$ two maps satisfying the following conditions:*

(i) for each $y \in Y$, $\{x \in X : P(x) \subseteq Q(y)\}$ is closed;

(ii) P has convex values and Q^c is quasiconvex;

(iii) there exists a compact mapping $T \in \mathcal{B}(Y,X)$ such that for each $y \in Y$, $P(T(y)) \subseteq Q(y)$.

Then there exists $x_0 \in X$ such that $P(x_0) \subseteq \bigcap_{y \in Y} Q(y)$.

Proof. Let $S : Y \multimap X$ be the map defined by

$$S(y) = \{x \in X : P(x) \nsubseteq Q(y)\}.$$

Suppose that the conclusion of theorem is false. Then $X = \bigcup_{y\in Y} S(y)$. Let $X_0 = \overline{T(Y)}$. Since X_0 is compact there exists a finite set $A = \{y_1, y_2, \dots, y_n\} \subseteq Y$ such that $X_0 = \bigcup_{i=1}^{n}(S(y_i) \cap X_0)$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a partition of unity on X_0 subordinated to the cover $\{S(y_i) \cap X_0 : 0 \leq i \leq n\}$. Recall that this means that

$$\begin{cases} \alpha_i : X_0 \to [0,1] \text{ is continuous, for each } i \in \{1, 2, \dots, n\}; \\ \alpha_i(x) > 0 \Rightarrow x \in S(y_i); \\ \sum_{i=1}^{n} \alpha_i(x) = 1 \text{ for each } x \in X_0. \end{cases}$$

Define $f : T(co\ A) \to co\ A$ by

$$f(x) = \sum_{i=1}^{n} \alpha_i(x) y_i \ \text{ for all } x \in T(co\ A).$$

Since f is continuous and $T \in \mathcal{B}(Y, X)$, $f \circ T_{|A} : coA \multimap coA$ has a fixed point. Hence there exists $\widetilde{y} \in coA$ such that $\widetilde{y} \in f(T(\widetilde{y}))$. Then, for some $\widetilde{x} \in T(\widetilde{y})$ we have $\widetilde{y} = f(\widetilde{x})$. Let $I = \{i \in \{1, \dots, n\} : \alpha_i(\widetilde{x}) > 0\}$. Then $\widetilde{y} = f(\widetilde{x}) \in co\{y_i : i \in I\}$. For each $i \in I$, $\widetilde{x} \in S(y_i)$, hence $P(\widetilde{x}) \cap Q^c(y_i) \neq \emptyset$. By (ii) it follows that $P(\widetilde{x}) \cap Q^c(\widetilde{y}) \neq \emptyset$, or equivalently, $P(\widetilde{x}) \nsubseteq Q(\overline{y})$. Since $\widetilde{x} \in T(\widetilde{y})$, we get $P(T(\overline{y})) \nsubseteq Q(\overline{y})$, which contradicts (iii). □

Proposition 2. *If Z is topological space, then condition (i) in Theorem 1 is fulfilled in any of the following cases:*

(i_1) P has open fibers;

(i_2) P is l.s.c. and Q has closed values;

Proof. If P has open values then for each $y \in Y$ the set $\{x \in X : P(x) \nsubseteq Q(y)\} = \bigcup_{z\in Q^c(y)} P^-(z)$ is open, hence $\{x \in X : P(x) \subseteq Q(y)\} = X \setminus \{x \in X : P(x) \nsubseteq Q(y)\}$ is closed.

By the definition of lower semicontinuity it follows that if (i_2) holds then each set $\{x \in X : P(x) \subseteq Q(y)\}$ is closed.

□

3. Equilibrium Theorems

In [5], [6], [10-13], for a suitable choice of the sets Y, Z and V and of the maps $F : Y \times Z \multimap V$ and $C : Z \multimap V$ the authors study, all or part of the following problems:

(I) Find $z_0 \in Z$ such that $F(y, z_0) \subseteq C(z_0)$ for all $y \in Y$;

(II) Find $z_0 \in Z$ such that $F(y, z_0) \cap C(z_0) \neq \emptyset$ for all $y \in Y$;

(III) Find $z_0 \in Z$ such that $F(y, z_0) \nsubseteq C(z_0)$ for all $y \in Y$;

(IV) Find $z_0 \in Z$ such that $F(y, z_0) \cap C(z_0) = \emptyset$ for all $y \in Y$.

Each existence result concerning problem (I) (respectively, (II)), yields an existence theorem for problem (IV) (respectively, (III)), if we take into account the following equivalences: $F(y, z) \subseteq C(z) \Leftrightarrow F(y, z) \cap C^c(z) = \emptyset$ and $F(y, z) \cap C(z) \neq \emptyset \Leftrightarrow F(y, z) \nsubseteq C^c(z)$. For this reason we can fix our attention on problems (I) and (II), only.

In this section we study equilibrium problems more general than (I) and (II):

Let X be a topological space, Y be a convex set in a topological vector space, Z be a topological vector space and V be a nonempty set. Let $F : Y \times Z \multimap V$, $C : Z \multimap V$ and $P : X \multimap Z$.

(V) Find $x_0 \in X$ such that $F(y, z) \subseteq C(z)$ for each $y \in Y$ and $z \in P(x_0)$;

and respectively,

(VI) Find $x_0 \in X$ such that $F(y, z) \cap C(z) \neq \emptyset$ for each $y \in Y$ and $z \in P(x_0)$.

Of course, when $X = Z$ and $P(z) = \{z\}$ for all $z \in Z$, problem (V) (respectively (VI)), reduces to problem (I) (respectively (II)).

Theorem 3. *Suppose that the maps F, C and P satisfy the following conditions:*

(i) one of the following two requirements is fulfilled:

(i_1) P has open fibers;

(i_2) P is l.s.c., C is closed map and for each $y \in Y$, $F(y, \cdot)$ is l.s.c.

(ii) F and C^c are convex maps, P has convex values;

(iii) there exists a compact mapping $T \in \mathcal{B}(Y, X)$ such that $F(y, z) \subseteq C(z)$, for each $y \in Y$ and $z \in P(T(y))$.

Then there exists $x_0 \in X$ such that $F(y, z) \subseteq C(z)$ for each $y \in Y$ and $z \in P(x_0)$.

Proof. Let $Q : Y \multimap Z$ be the map defined by

$$Q(y) = \{z \in Z : F(x, z) \subseteq C(z)\}.$$

We prove that if (i_2) holds, then Q has closed values. Let $y \in Y$ and $\{z_t\}_{t \in \Delta}$ be a net in $Q(y)$ converging to $z \in Z$. If $v \in F(y, z)$, since $F(y, \cdot)$ is l.s.c., there exists a net $\{v_t\}_{t \in \Delta}$ converging to v such that $v_t \in F(y, z_t)$, for all $t \in \Delta$. Since $z_t \in Q(y)$, $v_t \in F(y, z_t) \subseteq C(z_t)$. The map C is closed, hence $v \in C(z)$. Thus, $F(y, z) \subseteq C(z)$, hence $z \in Q(y)$. By Proposition 2, in both cases (i_1) and (i_2), condition (i) in Theorem 1 is satisfied.

We show next that the map Q^c is convex. Let $y_1, y_2 \in Y$, $\lambda \in (0.1)$ and $z \in \lambda Q^c(y_1) + (1-\lambda)Q^c(y_2)$. There exist $z_1, z_2 \in Z$ such that $z = \lambda z_1 + (1-\lambda)z_2$ and $v_1, v_2 \in V$ such that $v_i \in F(y_i, z_i) \cap C^c(z_i)$, for $i = 1, 2$. Since the maps $F$and C^c are convex,

$$\lambda v_1 + (1-\lambda)v_2 \in \lambda F(y_1, z_1) + (1-\lambda)F(y_2, z_2) \subseteq F(\lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2),$$

and similarly, $\lambda v_1 + (1-\lambda)v_2 \in C^c(\lambda z_1 + (1-\lambda)z_2)$. Thus, $\lambda v_1 + (1-\lambda)v_2 \in F(\lambda y_1 + (1-\lambda)y_2, z) \cap C^c(z)$, hence $z \in Q(\lambda y_1 + (1-\lambda)y_2)$.

Hence Q^c is convex and by Lemma 2, it is quasiconvex. It is clear that condition (iii) is equivalent to the requirement similarly denoted in Theorem 1, hence all requirements of this theorem are fulfilled. Consequently, there exists $x_0 \in X$ such that $P(x_0) \subseteq \bigcap_{y\in Y} Q(y)$, that is, $F(y,z) \subseteq C(z)$, for each $y \in Y$ and $z \in P(x_0)$. ☐

Theorem 4. *Suppose that the maps F, C and P satisfy the following conditions:*

- *(i) one of the following two requirements is fulfilled:*
- *(i_1) P has open fibers;*
- *(i_2) P is l.s.c., C is u.s.c. with compact values and for each $y \in Y$, $F(y,\cdot)$ is closed.*
- *(ii) F is concave map, C^c is convex map and P has convex values;*
- *(iii) there exists a compact mapping $T \in \mathcal{B}(Y,X)$ such that $F(y,z) \cap C(z) \neq \emptyset$, for each $y \in Y$ and $z \in P(T(y))$.*

Then there exists $x_0 \in X$ such that $F(y,z) \cap C(z) \neq \emptyset$ for each $y \in Y$ and $z \in P(x_0)$.

Proof. The proof is similar to that of Theorem 3. Let $Q : Y \multimap Z$ be the map defined by

$$Q(y) = \{z \in Z : F(x,z) \cap C(z) \neq \emptyset\}.$$

We show first that if (i_2) holds, then Q has closed values. Let $y \in Y$ and $\{z_t\}_{t\in\Delta}$ be a net in $Q(y)$ converging to $z \in Z$. Then, for each $t \in \Delta$, there exists $v_t \in F(y_t, z_t) \cap C(z_t)$. Since C is u.s.c. with compact values, by Lemma 1 (i), there exist a subnet $\{v_{t'}\}$ of $\{v_t\}$ and $v \in C(z)$ such that $v_{t'} \to v$. Since $F(y,\cdot)$ is closed, $v \in F(y,z)$. Therefore $F(y,z) \cap C(z) \neq \emptyset$, hence $z \in Q(y)$.

Let $y_1, y_2 \in Y$, $\lambda \in (0.1)$ and $z \in \lambda Q^c(y_1) + (1-\lambda)Q^c(y_2)$. There exist $z_1, z_2 \in Z$ such that $z = \lambda z_1 + (1-\lambda)z_2$ and $F(y_1, z_1) \subseteq C^c(z_1)$, $F(y_2, z_2) \subseteq C^c(z_2)$. By (ii) we infer that

$F(\lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2) \subseteq \lambda F(y_1, z_1) + (1-\lambda)F(y_2, z_2) \subseteq \lambda C^c(z_1) + (1-\lambda)C^(z_2) \subseteq C^c(\lambda z_1 + (1-\lambda)z_2)$.

It follows that $z \in Q^c(\lambda y_1 + (1-\lambda)y_2)$, hence the map Q^c is convex. The maps P and Q satisfy all the requirements of Theorem 1 and the desired conclusion follows from this theorem. ☐

4. Analytic alternative, minimax inequality

Definition 2. (see [1]). Let X and Y be convex sets in two vector spaces. We say that a function $q : Y \times Z \to \overline{\mathbb{R}}$ is (y, z)-*quasiconvex* if for any finite subset $\{(y_1, z_1), \ldots, (y_n, z_n)\}$ of $Y \times Z$, and each $y \in \text{co}\,\{y_1, \ldots, y_n\}$ there exists $z \in \text{co}\,\{z_1, \ldots z_n\}$ such that $q(y, z) \leq \max_{1 \leq i \leq n} q(y_i, z_i)$.

It is clear that any function $q : Y \times Z \to \overline{\mathbb{R}}$ quasiconvex on $Y \times Z$ is (y, z)-quasiconvex but Example 2 in [1] shows that the converse is not true.

Definition 3. Let X and Z be topological spaces. A function $p : X \times Z \to \mathbb{R}$ is said to be *marginally upper semicontinuous* in x (see [8]) if for every open subset U of Z the function $x \to inf_{z \in U} p(x, z)$ is upper semicontinuous on X.

Any function upper semicontinuous in x is marginally upper semicontinuous in x but the example given in [8], p.249 shows that the converse is not true.

Theorem 5. *Let X be topological space, Y and Z be convex sets in topological vector spaces, $p : X \times Z \to \mathbb{R}$, $q : Y \times Z \to \mathbb{R}$, $t : X \times Y \to \mathbb{R}$ be functions and α, β, λ be real numbers. Suppose that the following conditions are satisfied:*

(i) one of the following requirements is fulfilled:

(i_1) for each $z \in Z$ the set $\{x \in X : p(x, z) < \alpha\}$ is open;

(i_2) p is marginally upper semicontinuous in x and for each $y \in Y$ the set $\{z \in Z : q(y, z) \geq \beta\}$ is closed;

(ii) for each $x \in X$ the set $\{z \in Z : p(x, z) < \alpha\}$ is convex;

(iii) q is (y, z)-quasiconvex;

(iv) for $x \in X$, $y \in Y$ and $z \in Z$ the following implication holds: $p(x, z) < \alpha$ and $q(y, z) < \beta \Rightarrow t(x, y) < \lambda$;

(v) the map $T : Y \multimap X$ defined by $T(y) = \{x \in X : t(x, y) \geq \lambda\}$ is compact and belongs to the class $\mathcal{B}(Y, X)$.

Then at least one of the following assertions holds:

(a) There exists $x_0 \in X$ such that $p(x_0, z) \geq \alpha$, for all $z \in Z$.

(b) There exists $z_0 \in Z$ such that $q(y, z_0) \geq \beta$, for all $y \in Y$.

Proof. Define the maps $P : X \multimap Z$, $Q : Y \multimap Z$, $T : X \multimap Y$ by

$$P(x) = \{z \in Z : p(x, z) < \alpha\},\ Q(y) = \{z \in Z : q(y, z) \geq \beta\}, and$$

$$T(y) = \{x \in X : t(x, y) \geq \lambda\}.$$

If (i_1) holds, then P has open fibers, If (i_2) holds, then Q has closed values and we claim that P is l.s.c. Indeed, since p is marginally upper semicontinuous in x, for each open $U \subseteq Z$ the set

$$\{x \in X : P(x) \cap U \neq \emptyset\} = \{x \in X : inf_{z \in U} p(x, z) < \alpha\}$$

is open. Hence, according to Proposition 2, condition (i) in Theorem 1 holds.

Let C be a convex subset of Z, $y_1, y_2 \in Q^c(C)$ and $y \in co\{y_1, y_2\}$. Then there exist $z_1, z_2 \in C$ such that $q(y_1, z_1) < \beta$, $q(y_2, z_2) < \beta$. Since q is (y, z)-quasiconvex, there exists $z \in co\{z_1, z_2\} \subseteq C$ such that

$$q(y, z) \leq max\{q(y_1, z_1), q(y_2, z_2)] \ < \beta.$$

Thus $y \in Q^c(C)$, hence Q is quasiconvex. We prove that for each $y \in Y$, $P(T(y)) \subseteq Q(y)$. Suppose that for some $y \in Y$ there exists $x \in T(y)$ and $z \in P(x) \setminus Q(y)$. By $x \in T(y)$, we get $t(x, y) \geq \lambda$. On the other hand, since $z \in P(x) \setminus Q(y)$, we have $p(x, z) < \alpha$, $q(y, z) < \beta$ and, by (iv), we get $t(x, y) < \lambda$; a contradiction. Therefore the maps P, Q, T satisfy all the requirement of Theorem 1. According to this theorem there exists $x_0 \in X$ such that $P(x_0) \subseteq \bigcap_{y \in Y} Q(y)$. Suppose that both assertions in the conclusion of theorem are false. This means that:

(a') $P(x) \neq \emptyset$, for all $x \in X$;

(b') for each $z \in Z$ there exists $y \in Y$ such that $z \notin Q(y)$.

The following contradiction completes the proof:

$\emptyset \neq P(x_0) \subseteq \bigcap_{y \in Y} Q(y) = \emptyset$. □

Theorem 6. *Let X be a topological compact space, Y and Z be two convex sets in topological vector spaces and $p : X \times Z \to \mathbb{R}$, $q : Y \times Z \to \mathbb{R}$, $t : X \times Y \to \mathbb{R}$ functions. Suppose that the following conditions are fulfilled:*

(i) one of the following requirements is fulfilled:

(i_1) p is u.s.c. in x;

(i_2) p is marginally upper semicontinuous in x and q is u.s.c. in z;

(ii) p is quasiconvex in z;

(iii) q is (y, z)-quasiconvex;

(iv) for $x \in X$, $y \in Y$ and $z \in Z$ the following implication holds: $t(x, y) \leq p(x, z) + q(y, z)$;

(v) for each $\lambda < \inf_{y \in Y} \sup_{x \in X} \ t(x, y)$ the map $T : Y \multimap X$, defined by $T(y) = \{x \in X : t(x, y) \geq \lambda\}$ belongs to the class $\mathcal{B}(Y, X)$.

Then,

$\inf_{y\in Y}\sup_{x\in X}\ t(x,y)\le sup_{x\in X}inf_{z\in Z}\ p(x,z)+sup_{z\in Z}inf_{y\in Y}\ q(y,z),$

with the convention $\infty+(-\infty)=\infty$.

Proof. We may suppose that

$$\inf_{y\in Y}\sup_{x\in X}\ t(x,y)>-\infty,\ sup_{x\in X}inf_{z\in Z}\ p(x,z)<\infty,$$

$$sup_{z\in Z}inf_{y\in Y}\ q(y,z)<\infty.$$

By way of contradiction suppose that

$$\inf_{y\in Y}\sup_{x\in X}\ t(x,y)>sup_{x\in X}inf_{z\in Z}\ p(x,z)+sup_{z\in Z}inf_{y\in Y}\ q(y,z)$$

and choose $\alpha,\beta,\lambda\in\mathbb{R}$ such that $sup_{x\in X}inf_{z\in Z}\ p(x,z)<\alpha$, $sup_{z\in Z}inf_{y\in Y}\ q(y,z)<\beta$, $\lambda<\inf_{y\in Y}\sup_{x\in X}\ t(x,y)$, and $\alpha+\beta<\lambda$.

We prove that condition (iv) in Theorem 5 is fulfilled. Let $x\in X$, $y\in Y$ and $z\in Z$ such that $p(x,z)<\alpha$ and $q(y,z)<\beta$. Since $\alpha+\beta<\lambda$, by condition (iv) in the theorem that must be proved, we get $t(x,y)\le p(x,z)+q(y,z)<\alpha+\beta<\lambda$.

It is easy to see that all the requirements of Theorem 5 are fulfilled. We prove that none of assertions (a), (b) of the conclusion of Theorem 5 can take place.

If (a) happens, then

$\alpha\le inf_{z\in Z}\ p(x_0,z)\le sup_{x\in X}inf_{z\in Z}\ p(x,z)$; a contradiction.

If (b) happens, then

$\beta\le inf_{y\in Y}\ q(y,z_0)\le sup_{z\in Z}\ inf_{y\in Y}\ q(y,z)$; a contradiction. □

Corollary 7. *Let* X,Y *and* Z *be convex subsets of three topological vector spaces,* X *being compact and* $p:X\times Z\to\mathbb{R}$, $q:Y\times Z\to\mathbb{R}$, $t:X\times Y\to\mathbb{R}$ *three functions satisfying conditions (i), (ii), (iii), (iv) of Theorem 6 and*

(v') t is upper semicontinuous on $X\times Y$ and for each $y\in Y$, $t(.,y)$ is quasiconcave on X.

Then, $\inf_{y\in Y}\sup_{x\in X}\ t(x,y)\le sup_{x\in X}inf_{z\in Z}\ p(x,z)+sup_{z\in Z}inf_{y\in Y}\ q(y,z)$,

with the convention $\infty+(-\infty)=\infty$.

Proof. It suffices to prove that condition (v) in Theorem 6 is fulfilled. Obviously for each $\lambda<\inf_{y\in Y}\sup_{x\in X}\ t(x,y)$ the map T defined in condition (v) of Theorem 6 has nonempty values. Moreover, by (v') the values of T are convex. Since t is upper semicontinuous on $X\times Y$ the map T is closed. Since X is compact, by Lemma 1, T is upper semicontinuous with compact values. Consequently T is a Kakutani map. Since, $\mathbb{K}(Y,X)\subset\mathcal{B}(Y,X)$, it follows that condition (v) from Theorem 6 is satisfied. □

The results obtained in this section generalize Theorems 19, 20 and Corollary 21 in [1], where the corresponding map T, in each result, had the KKM property. Obviously, the condition $T \in \mathcal{B}(Y, X)$ is a weaker one.

`Received: February 2008. Revised: March 2008.`

References

[1] M. Balaj, *Coincidence and maximal element theorems and their applications to generalized equilibrium problems and minimax inequalities*, Nonlinear Anal., **68** (2008), 3962–3971.

[2] H. Ben-El-Mechaiekh, S Chebbi, M. Florenzano and J.-V. Llinares, *Abstract convexity and fixed points* J. Math. Anal. Appl. 222 (1998), 138–150.

[3] T.-H Chang and C.-L. Yen, *KKM property and fixed point theorems*, J. Math. Anal. Appl., **203** (1996), 224–235.

[4] X.P. Ding, *New H-KKM theorems and equilibria of generalized games*, Indian J. Pure Appl. Math., **27** (1996), 1057–1071.

[5] X.P. Ding and Y.J. Park, *Fixed points and generalized vector equilibrium problems in generalized convex spaces* Indian J. Pure Appl. Math., **34** (2003), 973–990.

[6] X.P. Ding and J.Y. Park, *Generalized vector equilibrium problems in generalized convex spaces*, J. Optim. Theory Appl., **120** (2004), 327–353.

[7] J.Y. Fu and A.H. Wan, *Generalized vector equilibrium problems with set-valued mappings*, Math. Methods Oper. Res., **56** (2002), 259–268.

[8] G.H. Greco and M.P. Moschen, *A minimax inequality for marginally semicontinuous functions* in Minimax Theory and Applications (B. Ricceri, S. Simons eds), Kluwer Acad. Publ., Dordrecht, 1998, pp. 41–50.

[9] M. Lassonde, *Fixed points for Kakutani factorizable multifunctions*, J. Math. Anal. Appl., **152** (1990), 46–60.

[10] L.J. Lin, Q.H. Qnsari and J.Y. Wu, *Geometric properties and coincidence theorems with applications to generalized vector equilibrium problems*, J. Optim. Theory Appl., **117** (2003), 121–137.

[11] L.J. Lin and H.L. Chen, *The study of KKM theorems with applications to vector equilibrium problems and implicit vector variational inequalities problems*, J. Global Optim., **32** (2005), 135–157.

[12] L.J. Lin, Z.T. Yu and G. Kassay, *Existence of equilibria for multivalued mappings and its application to vectorial equilibria*, J. Optim. Theory Appl., **114** (2002), 189–208.

[13] L.J. Lin and W.P. Wan, *KKM type theorems and coincidence theorems with applications to the existence of equilibrium*, J. Optim. Theory Appl., **123** (2004), 105–122.

[14] S. Park, *Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps*, J. Korean Math. Soc., **31** (1994), 493–519.

[15] S. Park, *Fixed points of the better admissible multimaps*, Math. Sci. Res. Hot-Line, **1(9)** (1997), 1–6.

[16] N.X. Tan and P.N. Tinh, *On the existence of equilibrium points of vector functions*, Numerical Functional Analysis Optimiz., **19** (1998), 141–156.

CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$04, (85–100). December 2008

A Disc-Cutting Theorem and Two-Dimensional Bifurcation of a Reaction-Diffusion System with Inclusions

Martin Väth[*]
University of Würzburg, Math. Institut
Am Hubland, D-97074 Würzburg, Germany
email: `vaeth@mathematik.uni-wuerzburg.de`

ABSTRACT

We provide a topological disc-cutting theorem which allows to prove that unilateral inclusions in a reaction-diffusion system of prey-predator type with a two-dimensional bifurcation parameter necessarily have a certain global branch of (global) bifurcation points.

RESUMEN

Presentamos un teorema “Disc-Cutting” topológico el cual permite probar que inclusiones unilaterales en un sistema de reación-difusión de tipo predador-presa con parametro de bifurcación 2-dimencional, necessariamente tiene una cierta rama global de puntos de bifucarción (global).

---

[*] This paper was written in the framework of a Heisenberg fellowship (Az. VA 206/1-2). Financial support by the DFG is gratefully acknowledged.
The author wants to thank E. Vogt for valuable comments and suggestions.

Key words and phrases: *Global bifurcation, two-dimensional bifurcation, elliptic equation, inclusion, Laplace operator.*

Math. Subj. Class.: *35B32, 35J60, 35K47, 47H04, 47H11.*

1 Introduction

Although we provide in Section 2 a general topological theorem about the existence of a global branch which is applicable to a large class of bifurcation problems with a parameter from a space of dimension at least 2, our main motivation for the result comes from the following particular problem.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary, and let measurable (possibly empty) subsets $\Omega_0 \subseteq \Omega$ and $\Gamma_0, \Gamma \subseteq \partial\Omega$ be fixed with $\operatorname{mes}(\Gamma_0 \cap \Gamma) = 0$. We consider the reaction-diffusion system

$$\begin{aligned} &u_t = d_1\Delta u + b_{11}u + b_{12}v + f_1(d,x,u,v,\nabla u,\nabla v) = 0 \qquad \text{on } \Omega,\\ &v_t \in d_2\Delta v + b_{21}u + b_{22}v + f_2(d,x,u,v,\nabla u,\nabla v) + \begin{cases}\{0\} & \text{on } \Omega\setminus\Omega_0,\\ m_0(d,x,u,v,\nabla u,\nabla v) & \text{on } \Omega_0,\end{cases}\end{aligned} \tag{1.1}$$

with the boundary conditions

$$\begin{cases} u = v = 0 & \text{on } \Gamma_0,\\ \dfrac{\partial u}{\partial n} = g_1(d,x,u,v) & \text{on } \partial\Omega\setminus\Gamma_0,\\ \dfrac{\partial v}{\partial n} = g_2(d,x,u,v) & \text{on } \partial\Omega\setminus(\Gamma_0\cup\Gamma),\\ \dfrac{\partial v}{\partial n} \in g_2(d,x,u,v) + m_1(d,x,u,v) & \text{on } \Gamma.\end{cases} \tag{1.2}$$

Here, $d = (d_1, d_2) \in \mathbb{R}^2_+$ is a bifurcation parameter, the nonlinearities f_i and g_i are small at $(u,v) = 0$, and m_i are nonnegative interval functions specified later. The scalar parameters b_{ij} are assumed to satisfy

$$\begin{aligned} &b_{11} > 0,\ b_{12} < 0,\ b_{21} > 0,\ b_{22} < 0,\\ &b_{11} + b_{22} < 0,\ b_{11}b_{22} - b_{12}b_{21} > 0,\end{aligned} \tag{1.3}$$

which means that system (1.1) is a special system of activator-inhibitor or prey-predator type such that in case $d_1 = d_2 = 0$ (i.e. without diffusion) the solution $(0,0)$ is stable. However, it is known (see e.g. [11] or [2, Appendix] or [1]) that the stability of (1.1)/(1.2) with classical data $m_0 = m_1 = 0$ depends on $d = (d_1, d_2)$. In fact, the domain D_S of those $d \in \mathbb{R}^2_+$ where this system is exponentially stable is the right-hand side of the "envelope" of the sequence of hyperbolas

$$C_n := \left\{(d_1,d_2) \in \mathbb{R}^2_+ : d_2 = \frac{b_{12}b_{21}/\kappa_n^2}{d_1 - b_{11}/\kappa_n} + \frac{b_{22}}{\kappa_n}\right\}. \tag{1.4}$$

where $\kappa_1 \leq \kappa_2 \leq \cdots \to \infty$ denotes the sequence of eigenvalues of $-\Delta$, i.e. for which a (weak) nontrivial solution of the problem

$$\begin{cases} -\Delta u = \kappa_n u & \text{on } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma_0 \end{cases} \tag{1.5}$$

exists.

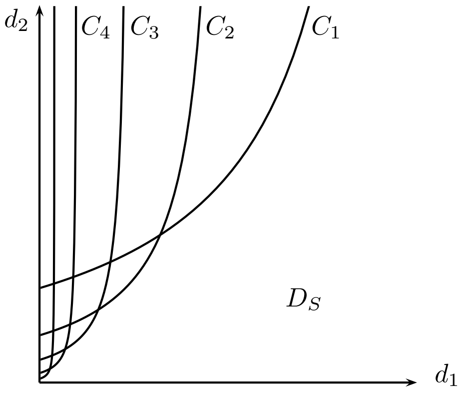

Figure 1: *Hyperbolas* (1.4) *determining* D_S

Using degree theory for multivalued maps, it was shown in [6] (for similar results and related systems, see also [2]–[5], [7]–[10]) that in the case of natural unilateral (possibly multivalued) functions m_0 and/or m_1 there is a destabilizing effect in the sense that even the stationary points admit a global bifurcation along certain paths in D_S. In this paper, we will show by a purely topological argument that results of such a type actually imply the existence of certain *global branches of bifurcation points* (i.e. not only the bifurcation branch is global but actually even the set of bifurcation points itself). Such phenomena can naturally arise only because our bifurcation parameter d is not only from a one-dimensional space.

We point out that although we concentrate only on the system (1.1)/(1.2), the same results (and proofs) hold for all systems for which corresponding results along paths are available. In particular, this is the case when we consider instead of (1.2) the Signorini type boundary conditions (see e.g. [9], [10])

$$\begin{cases} u = v = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \setminus \Gamma_0, \\ \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\Omega \setminus (\Gamma_0 \cup \Gamma), \\ \dfrac{\partial v}{\partial n} \geq 0,\ v \geq 0,\ \dfrac{\partial v}{\partial n} \cdot v = 0 & \text{on } \Gamma. \end{cases}$$

2 The Disc-Cutting Theorem

Our main topological tool is based on a generalization of the Whyburn lemma yielding global branches [12] and on the following result.

Theorem 2.1 (Boundaries connect squaresides)**.** *Let X be a topological space into which a square with (compact) "square-sides" A_1, A_2, A_3, A_4 and "square-interior" Q is homeomorphically embedded. Let $V \subseteq X$ be open such that $A_1 \subseteq V$ and $\overline{V} \cap A_3 = \varnothing$. Then there is a connected subset $C \subseteq Q \cap \partial V$ such that $\overline{C} \cap A_i \neq \varnothing$ for $i = 2, 4$.*

The afore mentioned generalization of the Whyburn lemma is the following (this version can be proved using only the [countable] axiom of dependent choices):

Theorem 2.2. *Let X be a regular space, $A \subseteq X$ compact, and $S \subseteq X$. Then for each open set $U \supseteq A$ for which $\overline{U} \cap S$ is compact and metrizable the following statements are equivalent:*

1. *For each open set $\Omega \supseteq A$ with $\overline{\Omega} \subseteq U$ there is some $x \in \partial\Omega$ in S.*
2. *There is a connected set $C \subseteq (S \cap U) \setminus A$ such that $\overline{C} \cap S$ intersects A and ∂U.*

For the proof of Theorem 2.2 we refer to [12]. Let us now use this result and some winding number theory to prove Theorem 2.1.

Proof of Theorem 2.1. We show first that it suffices to show the claim for the case $X = \mathbb{R}^2$ and $Q = (0,1) \times (0,1)$ with $A_1 := [0,1] \times \{0\}$, $A_2 := \{1\} \times [0,1]$, $A_3 := [0,1] \times \{1\}$, and $A_4 := \{0\} \times [0,1]$.

To see that then the general case holds also, assume that we have a homeomorphism f of $\overline{Q}$ onto a subset X_0 of a general space X. Note that X_0 is the union of the sets $\widetilde{Q} := f(Q)$ and $\widetilde{A}_i := f(A_i)$. Note that we do not assume that X is Hausdorff, so X_0 might not be closed, although it is compact. However, if $V \subseteq X$ is open with $\widetilde{A}_1 \subseteq V$ and $\overline{V} \cap \widetilde{A}_3 = \varnothing$, then $V_0 := f^{-1}(V \cap X_0)$ is open in $\overline{Q}$, and so there is some open $V_1 \subseteq \mathbb{R}^2$ with $V_1 \cap \overline{Q} = V_0$. Since $\overline{V}_0 \subseteq f^{-1}(\overline{V} \cap X_0)$ is disjoint from the compact set $f^{-1}(\widetilde{A}_3) = A_3$, and since $\mathbb{R}^2$ is regular, we thus find a closed neighborhood of A_3 which is disjoint from V_0. Eliminating this neighborhood from V_1 if necessary, we may thus assume without loss of generality that $\overline{V}_1 \cap A_3 = \varnothing$. By the special case of the Theorem, we thus find a connected set $C \subseteq Q \cap \partial V_1$ with $\overline{C} \cap A_i \neq \varnothing$ for $i = 2, 4$. Then $\widetilde{C} := f(C)$ is connected, and its closure contains $f(\overline{C})$ and thus intersects $\widetilde{A}_i = f(A_i)$ for $i = 2, 4$. Moreover, $\widetilde{C}$ is contained in $f(Q \cap \partial V_1)$. Note that, since Q is open, $\overline{V}_1 \cap Q = (\overline{V_1 \cap Q}) \cap Q = \overline{V}_0 \cap Q$, and so $\widetilde{C} \subseteq f(\overline{V}_0 \cap Q) \subseteq \widetilde{Q} \cap \overline{V}$. Moreover, $\widetilde{C}$ is disjoint from V, since $\widetilde{C} = f(C)$ is contained in X_0 and disjoint from $V \cap X_0 = f(V_0)$ in view of $C \cap V_0 = \varnothing$. Hence, we have indeed found a connected set $\widetilde{C} \subseteq Q \cap \overline{V} \setminus V = Q \cap \partial V$ whose closure intersects $\widetilde{A}_i$ for $i = 2, 4$. This proves the general case of the claim.

We prove now the claim in the special case $X := \mathbb{R}^2$, $Q := (0,1) \times (0,1)$ and A_i as above. Thus, let $V \subseteq \mathbb{R}^2$ be open with $A_1 \subseteq V$ and $\overline{V} \cap A_3 = \varnothing$. Without loss of generality, we may assume in addition that $\overline{V}$ is contained in $Q_0 := (-1,2)\times(-1,1)$. Indeed, otherwise, we could replace V by the intersection

$$V_0 := V \cap \{(x_1,x_2)\in(-1/2,3/2)\times(-1/2,1) : x_2 < 1-\operatorname{dist}(x_1,[0,1])\}$$

and note that $A_1 \subseteq V_0$, $\overline{V}_0 \cap A_3 = \varnothing$, and $Q\cap\partial V_0 = Q\cap\partial V$.

Put $S := \overline{Q}\cap\partial V$ and assume by contradiction that there is no connected subset $C\subseteq S\cap Q$ with $\overline{C}\cap A_i \neq \varnothing$ $(i=2,4)$. Applying Theorem 2.2 in the space $\overline{Q}$ with $A := A_2$ and $U := \overline{Q}\setminus A_4$, we find some open (in $\overline{Q}$) set $\Omega\subseteq\overline{Q}$ with $A_2\subseteq\Omega$ and $\overline{\Omega}\cap A_4=\varnothing$ such that the boundary B_0 of Ω with respect to $\overline{Q}$ contains no point from ∂V. By the compactness of these sets, we thus find some closed neighborhood $M\subseteq\mathbb{R}^2$ of ∂V which is disjoint from B_0. Since ∂V is a compact subset of Q_0 and $\mathbb{R}^2$ is regular, we may assume in addition that $M\subseteq Q_0$.

Covering the compact set $\overline{V}$ with sufficiently small open balls, we find an open set $G\subseteq V\cup M$ containing $\overline{V}$ such that the boundary of G consists of finitely many piecewise smooth closed curves. Fix some $a\in A_1$. Since $a\in V\subseteq G$, the argument principle of complex analysis (or, in other words, the well-known connection between the degree of the identity function with the winding number of the boundary) implies that at least one of these curves must have nonzero winding number with respect to a. We think of such a closed curve as a continuous map $\gamma\colon S^1\to Q_0$ (where S^1 denotes the unit circle). Note that this curve lies completely in $\partial G\subseteq M$. In particular, $\gamma(S^1)$ is disjoint from B_0 and thus contained in the union of the three sets

$$\Omega_2 := \Omega\cap Q,\quad \Omega_4 := (\overline{Q}\setminus\overline{\Omega})\cap Q = Q\setminus\overline{\Omega},\quad R := Q_0\setminus(Q\cup\{a\}).$$

Since Ω is open in $\overline{Q}$, it follows that Ω_2 is open in Q and thus open in $\mathbb{R}^2$. Analogously, also Ω_4 is open in $\mathbb{R}^2$. With the notation $\gamma=(\gamma_1,\gamma_2)$, we define now a homotopy $h\colon[0,1]\times S^1\to\mathbb{R}^2\setminus\{a\}$ by

$$h(t,s) := \begin{cases} ((1-t)\gamma_1(s),\gamma_2(s)) & \text{if } s\in\gamma^{-1}(\Omega_4),\\ ((1-t)\gamma_1(s)+t,\gamma_2(s)) & \text{if } s\in\gamma^{-1}(\Omega_2),\\ (\gamma_1(s),\gamma_2(s)) & \text{otherwise.}\end{cases}$$

This map is indeed continuous by the glueing lemma, because γ can cross the boundary of Ω_i only at A_i $(i=2,4)$. We thus have shown that γ is homotopic (in $\mathbb{R}^2\setminus\{a\}$) to a curve which assumes only values in R. Since R is obviously simply connected, γ is actually homotopic to a constant (in $\mathbb{R}^2\setminus\{a\}$). Hence, the homotopy invariance of the winding number shows that γ actually has winding number 0 around a. This is the required contradiction. □

Using Theorems 2.1 and 2.2, we can now prove the following disc-cutting theorem:

Theorem 2.3 (Disc-Cutting)**.** *Let X be a topological space into which a (compact) disc with "disc-interior" Q is homeomorphically embedded. Let the "disc-boundary" be the union of four nonempty disjoint connected sets A_1,A_2,A_3,A_4, enumerated in order along the boundary. Assume also that A_2 and A_4 both contain at least two points.*

Let $S \subseteq Q$ be closed in Q such that each compact smooth (via the embedding) injective path P in $Q \cup A_2 \cup A_4$ with $P \cap A_i \neq \varnothing$ $(i = 2, 4)$ contains some point from S. Then there is a connected subset $C \subseteq S$ such that $\overline{C} \cap \overline{A}_i \neq \varnothing$ for $i = 1, 3$.

Remark 2.1. One could also replace "smooth path" by "polygonal path" in the statement of Theorem 2.3 with the obvious modification in the following proof.

Proof. We show first that it suffices to show the claim for the case $X = \mathbb{R}^2$ and the unit disc Q. Indeed, if $f\colon \overline{Q} \to X$ is a homeomorphism onto a subset of a general space X, let $\widetilde{Q} := f(Q)$ and $\widetilde{A}_i := f(A_i)$. Let $S \subseteq \widetilde{Q}$ be as in the claim; in particular, S is closed in $\widetilde{Q}$. Then $S_0 := f^{-1}(S)$ is closed in Q. The hypothesis on S means that each smooth path connecting A_2 with A_4 in Q meets S_0. The special case of the result thus implies that there is a connected subset $C_0 \subseteq S_0$ such that $\overline{C}_0 \cap \overline{A}_i \neq \varnothing$ for $i = 1, 3$. Then $C := f(C_0) \subseteq S$ is connected, and $\overline{C} \cap \overline{\widetilde{A}_i} \supseteq f(\overline{C}_0) \cap f(\overline{A}_i) \neq \varnothing$. Hence, the statement holds also in the general case.

Thus, to prove the theorem, we may assume without loss of generality that $X = \mathbb{R}^2$ and that Q is the unit disc. Assume by contradiction that a set C as in the claim does not exist. We apply Theorem 2.2 in the space $\overline{Q}$ with $A := \overline{A}_1$, $U := \overline{Q} \setminus \overline{A}_3$, and $\overline{S}$ instead of S. Observing that $A \subseteq U$, because A_2 and A_4 are nondegenerate, we find some open in $\overline{Q}$ set $\Omega \supseteq \overline{A}_1$ with $\overline{\Omega} \cap \overline{A}_3 = \varnothing$ such that the boundary B_0 of Ω with respect to $\overline{Q}$ contains no element of $\overline{S}$. Note that B_0 is a closed subset of $\overline{Q}$ and thus compact. Note also that B_0 is disjoint from $\overline{S}$ and from $\overline{A}_i$ $(i = 1, 3)$. We thus find an open neighborhood $M \subseteq \mathbb{R}^2$ of B_0 which is disjoint from $\overline{S} \cup \overline{A}_1 \cup \overline{A}_3$. Moreover, if we let $\hat{A}_i$ $(i = 1, 3)$ be compact "intervals" of the circle boundary which contain the corresponding "intervals" $\overline{A}_i$ $(i = 1, 3)$ in their interior (with respect to the circle boundary) but still satisfy $\hat{A}_1 \subseteq \Omega$ and $\hat{A}_3 \subseteq \overline{Q} \setminus \overline{\Omega}$, and if we let $\hat{A}_i$ $(i = 2, 4)$ denote closure of the corresponding remaining intervals (contained in A_i), we can apply Theorem 2.1 with $V := \Omega$ and the four "square-sides" $\hat{A}_i$. We thus find a connected set $C \subseteq B_0$ such that there are points $a_i \in \overline{C} \cap \hat{A}_i$ for $i = 2, 4$, and so $a_i \in \overline{C} \cap A_i$ $(i = 2, 4)$. Since $\overline{C} \subseteq B_0$ is connected, if follows that a_2 and a_4 belong to the same connected component of B_0. Since $M \subseteq \mathbb{R}^2$ is an open neighborhood of B_0, we may thus connect a_2 and a_4 by a smooth injective path in M. Since M is disjoint from $\overline{A}_i$ $(i = 1, 3)$, we thus find a compact smooth injective path P in $M \cap (Q \cup A_2 \cup A_4)$ with $P \cap A_i \neq \varnothing$ $(i = 2, 4)$. Since $M \cap \overline{S} = \varnothing$, this path cannot contain a point from S, contradicting the hypothesis. $\square$

3 The Reaction-Diffusion System with Inclusions

3.1 Detailed Hypotheses

We will consider the weak formulation of the stationary problem corresponding to (1.1)/(1.2), i.e. we will consider the weak formulation of

$$\left.\begin{aligned} &d_1\Delta u + b_{11}u + b_{12}v + f_1(d, u, v, \nabla u, \nabla v) = 0 \\ &d_2\Delta v + b_{21}u + b_{22}v + f_2(d, u, v, \nabla u, \nabla v) \in -m_0(d, u, v, \nabla u, \nabla v) \end{aligned}\right\} \text{ in } \Omega \tag{3.1}$$

with boundary conditions

$$\begin{cases} u = v = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial n} = g_1(d,u,v) & \text{on } \partial\Omega \setminus \Gamma_0, \\ \dfrac{\partial v}{\partial n} \in g_2(d,u,v) + m_1(d,x,u,v) & \text{on } \partial\Omega \setminus \Gamma_0, \end{cases} \tag{3.2}$$

where we will assume that the (possibly multivalued) functions m_i have the form

$$m_0(d,x,u,v,w,z) := [\underline{c}_0(d)\underline{m}_0(x,u,v,w,z), \overline{c}_0(d)\overline{m}_0(x,u,v,w,z)]$$

and

$$m_1(d,x,u,v) := [\underline{c}_1(d)\underline{m}_1(x,u,v), \overline{c}_1(d)\overline{m}_1(x,u,v)],$$

and where we assumed for the simplicity of notation that $\underline{m}_0$, $\overline{m}_0$, $\underline{m}_1$ and $\overline{m}_1$ vanish for $x \notin \Omega_0$ or $x \notin \Gamma$, respectively, where $\Omega_0 \subseteq \Omega$ and $\Gamma \subseteq \partial\Omega \setminus \Gamma_0$ are measurable. In order to require nontrivial situations, we will assume that

$$\text{mes}\Omega_0 > 0 \text{ or } \text{mes}\Gamma > 0 \text{ (or both).} \tag{3.3}$$

For our considerations it will be crucial that

$$\text{mes}\Gamma_0 > 0 \tag{3.4}$$

so that we can equip the space $\mathbb{H}$ of all functions from $W^{1,2}(\Omega, \mathbb{R}^2)$ vanishing on Γ_0 with the scalar product

$$\langle U, V \rangle := \int_\Omega \langle \nabla U(x), \nabla V(x) \rangle \, dx,$$

which under hypothesis (3.4) generates the inherited topology, see e.g. [13, Theorem 4.8.1].

We assume (1.3), and by $D_S \subseteq \mathbb{R}^2_+$, we denote the (open) domain of stability mentioned in the introduction. Note that all points of $\mathbb{R}^2_+ \cap \partial D_S$ belong to some of the hyperbolas C_n defined by (1.4). We will assume that all of the above functions are at least defined for $d \in D_S \cup \{d^*\}$ where the point $d^* \in C_n \cap \partial D_S$ will be specified later on.

For $i = 0, 1$, we fix exponents p_i, q_i, and q_i^* according to the following restrictions.

$$\begin{cases} p_i \in [1,\infty),\ 1 \le q_i^* < q_i < \infty \text{ arbitrary} & \text{if } n \le 2, \\ p_0 := \frac{n}{n-2},\ p_1 := \frac{n-1}{n-2},\ \infty > q_0 > q_0^* := \frac{2n}{n+2},\ \infty > q_1 > q_1^* := \frac{2n-2}{n} & \text{if } n > 2. \end{cases}$$

Moreover, we assume the following hypothesis.

1. $\underline{c}_i, \overline{c}_i$ are continuous on $D_S \cup \{d^*\}$ and without zeros on D_S.
2. For each $d \in D_S \cup \{d^*\}$ the following holds: The functions $f_i(d,\cdot,u,v,w,z)$ and $g_i(d,\cdot,u,v)$ are measurable, and $f_i(d,x,\cdot,\cdot,\cdot,\cdot)$ and $g_i(d,x,\cdot,\cdot)$ are continuous for almost all x. Moreover, f_i and g_i satisfy the growth estimates

$$|f_i(d,x,u,v,w,z)| \le a_d(x) + b_d \cdot \left((|u|+|v|)^{p_0} + \|w\| + \|z\|\right)^{2/q_0},$$

and

$$|g_i(d,x,u,v,w,z)| \le \widetilde{a}_d(x) + \widetilde{b}_d \cdot \left((|u|+|v|)^{p_1}\right)^{2/q_1},$$

where the quantities $\|a_d\|_{L_{q_0}(\Omega)}$, $\|\widetilde{a}_d\|_{L_{q_1}(\partial\Omega\setminus\Gamma_0)}$, b_d, and $\widetilde{b}_d$ are locally bounded with respect to d.

3. For each $d_0 \in D_S \cup \{d^*\}$ there are estimates of the form

$$|f_i(d,x,u,v,w,z) - f_i(d_0,x,u,v,w,z)| \le$$
$$c_{d_0}(d)\left(a_{d_0,d}(x) + \left((|u|+|v|)^{p_0} + \|w\| + \|z\|\right)^{2/q_0^*}\right)$$

and

$$|g_i(d,x,u,v) - g_i(d_0,x,u,v)| \le$$
$$\widetilde{c}_{d_0}(d)\left(\widetilde{a}_{d_0,d}(x) + (|u|+|v|)^{2p_1/q_1^*}\right)$$

where $\|a_{d_0,d}\|_{L_{q_0^*}(\Omega)}, \|\widetilde{a}_{d_0,d}\|_{L_{q_1^*}(\partial\Omega\setminus\Gamma_0)} \le 1$ and $c_{d_0}(d), \widetilde{c}_{d_0}(d) \to 0$ as $d \to d_0$.

4. f_i and g_i become uniformly small at $(u,v)=0$ in the sense that for each sufficiently small ball B in D_S (and thus for each nonempty compact subset $B \subseteq D_S$) the following holds:

$$\sup_{w,z\in\mathbb{R}^n} \sup_{d\in B} |f_i(d,x,u,v,w,z)| \le c_B \max\left\{(|u|+|v|)^{2p_0/q_0}, |u|+|v|\right\}$$
$$\lim_{(u,v,w,z)\to 0} \sup_{d\in B} \frac{f_i(d,x,u,v,w,z)}{|u|+|v|+\|w\|+\|z\|} = 0$$
$$\sup_{w,z\in\mathbb{R}^n} \sup_{d\in B} |g_i(d,x,u,v)| \le c_B \max\left\{(|u|+|v|)^{2p_1/q_1}, |u|+|v|\right\}$$
$$\lim_{(u,v,w,z)\to 0} \sup_{d\in B} \frac{g_i(d,x,u,v)}{|u|+|v|} = 0$$

5. The functions $\underline{m}_0(\,\cdot\,,u,v,w,z)$ and $\overline{m}_0(\,\cdot\,,u,v,w,z)$ are measurable, $\underline{m}_0(x,\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is lower semicontinuous, $\overline{m}_0(x,\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is upper semicontinuous, and the corresponding superposition operators

$$\underline{M}_0(u,v,w,z)(x) := \underline{m}_0(x,u(x),v(x),w(x),z(x))$$

and

$$\overline{M}_0(u,v,w,z)(x) := \overline{m}_0(x,u(x),v(x),w(x),z(x))$$

send continuous (and thus measurable) functions into measurable functions. Moreover, we require for some $a_0 \in L_{q_0}(\Omega)$ and $b_0 < \infty$ the growth estimates

$$\max\left\{|\underline{m}_0(x,u,v,w,z)|, |\overline{m}_0(x,u,v,w,z)|\right\} \le$$
$$a_0(x) + b_0 \cdot \left((|u|+|v|)^{p_0} + \|w\| + \|z\|\right)^{2/q_0}.$$

6. The functions $\underline{m}_1(\,\cdot\,,u,v)$ and $\overline{m}_1(\,\cdot\,,u,v)$ are measurable, $\underline{m}_1(x,\,\cdot\,,\,\cdot\,)$ is lower semicontinuous, $\overline{m}_1(x,\,\cdot\,,\,\cdot\,)$ is upper semicontinuous, and the corresponding superposition operators

$$\underline{M}_1(u,v)(x) := \underline{m}_1(x,u(x),v(x))$$

and

$$\overline{M}_1(u,v)(x) := \overline{m}_1(x,u(x),v(x))$$

send continuous (and thus measurable) functions into measurable functions. Moreover, we require the following growth estimates for some $a_1 \in L_{q_1}(\Gamma)$ and $b_1 < \infty$:

$$\max\left\{\left|\underline{m}_1(x,u,v)\right|, \left|\overline{m}_1(d,x,u,v)\right|\right\} \le a_1(x) + b_1 \cdot (|u|+|v|)^{2p_1/q_1}.$$

7. The following unilateral conditions hold:

$$\begin{aligned}
0 = \underline{c}_0(d)\underline{m}_0(x,u,v,w,z) &= \overline{c}_0(d)\overline{m}_0(x,u,v,w,z) \quad \text{if } v>0,\\
0 = \underline{c}_0(d)\underline{m}_0(x,u,0,w,z) &\le \overline{c}_0(d)\overline{m}_0(x,u,0,w,z)\\
0 \le \underline{c}_0(d)\underline{m}_0(x,u,v,w,z) &\le \overline{c}_0(d)\overline{m}_0(x,u,v,w,z) \quad \text{if } v<0,\\
0 = \underline{c}_1(d)\underline{m}_1(x,u,v) &= \overline{c}_1(d)\overline{m}_1(x,u,v) \quad \text{if } v>0,\\
0 = \underline{c}_1(d)\underline{m}_1(x,u,0) &\le \overline{c}_1(d)\overline{m}_1(x,u,0)\\
0 \le \underline{c}_1(d)\underline{m}_1(x,u,v) &\le \overline{c}_1(d)\overline{m}_1(x,u,v) \quad \text{if } v<0.
\end{aligned}$$

$$\lim_{\substack{(u,v,w,z)\to 0\\ v<0}} \frac{\left|\underline{m}_0(x,u,v,w,z)\right|}{v} = -\infty \quad \text{for almost all } x \in \Omega_0,$$

$$\lim_{\substack{(u,v)\to 0\\ v<0}} \frac{\left|\underline{m}_1(x,u,v)\right|}{v} = -\infty \quad \text{for almost all } x \in \Gamma.$$

3.2 Definition of weak solutions

We consider the cone

$$K := \{U = (u_1,u_2) \in \mathbb{H} : u_2|_{\Omega_0} \ge 0 \text{ and } u_2|_{\Gamma} \ge 0\}$$

and define operators $A(d), G(d,\,\cdot\,), M(d,\,\cdot\,) \colon \mathbb{H} \to \mathbb{H}$ by

$$\begin{aligned}
\langle A(d)U,V\rangle &:= \int_\Omega \left\langle \begin{pmatrix} d_1^{-1}b_{11} & d_1^{-1}b_{12}\\ d_2^{-1}b_{21} & d_2^{-1}b_{22}\end{pmatrix} U(x), V(x)\right\rangle dx,\\
\langle G(d,U),V\rangle &:= \int_\Omega \left\langle \begin{pmatrix} d_1^{-1}f_1(d,U(x),\nabla U(x))\\ d_2^{-1}f_2(d,U(x),\nabla U(x))\end{pmatrix}, V(x)\right\rangle dx\\
&\quad + \int_{\partial\Omega\setminus\Gamma_0} \left\langle \begin{pmatrix} g_1(d,U(x))\\ g_2(d,U(x))\end{pmatrix}, V(x)\right\rangle dx,
\end{aligned}$$

and

$$
\begin{aligned}
M(d,U) := \bigcap_{V\in K} \Bigg\{ Z \in \mathbb{H} : \langle Z, V\rangle \in \int_{\Omega_0} \left\langle \begin{pmatrix} 0\cdot d_1^{-1} \\ d_2^{-1} m_0(d,x,U(x),\nabla U(x)) \end{pmatrix}, V(x) \right\rangle dx + \\
\int_\Gamma \left\langle \begin{pmatrix} 0 \\ m_1(d,x,U(x)) \end{pmatrix}, V(x) \right\rangle dx \Bigg\} := \\
\bigcap_{V=(\widetilde{v},v)\in K} \Bigg\{ Z = \begin{pmatrix} 0 \\ z \end{pmatrix} \in \mathbb{H} : \\
\int_{\Omega_0} d_2^{-1} \underline{c}_0(d) \underline{m}_0(x,U(x),\nabla U(x)) v(x)\, dx + \int_\Gamma \underline{c}_1(d) \underline{m}_1(x,U(x)) v(x)\, dx \le \\
\langle Z, V\rangle \le \int_{\Omega_0} d_2^{-1} \overline{c}_0(d) \overline{m}_0(x,U(x),\nabla U(x)) v(x)\, dx + \int_\Gamma \overline{c}_1(d) \overline{m}_1(x,U(x)) v(x)\, dx \Bigg\},
\end{aligned}
$$

respectively. We define weak solutions of problem (3.1)/(3.2) as solutions of the inclusion

$$U - A(d)U - G(d,U) \in M(d,U).$$

Our hypotheses imply in particular (see e.g. [6]):

Proposition 3.1. *$F(d,U) := A(d)U - G(d,U) - M(d,U)$ is an upper semicontinuous map with nonempty compact values. Moreover, F is compact in the sense that if $D_0 \subseteq D_S \cup \{d^*\}$ is compact and $B \subseteq \mathbb{H}$ is bounded then $F(D_0 \times B)$ is precompact.*

3.3 Local and Global Bifurcation Points

Note that $(d,0) \in D_S \times \mathbb{H}$ is always a solution of (3.1)/(3.2). We call a pair $(d,U) \in D_S \times \mathbb{H}$ a *nontrivial solution* if $U = (u,v) \neq 0$, and if (d,u,v) is a weak solution of (3.1)/(3.2). The *local bifurcation points* (in D_S) are the elements of the set

$$B_{\text{local}} := \{d \in D_S : \text{Each neighborhood of } (d,0) \in D_S \times \mathbb{H} \text{ contains a nontrivial solution}\}.$$

We call a point $d \in D_S$ a *global bifurcation point* (with respect to a point $d^* \in C_n \cap \partial D_S$) if there is a connected set $C \subseteq D_S \times (\mathbb{H} \setminus \{0\})$ consisting only of nontrivial solutions such that $(d,0) \in \overline{C}$ and such that C is a global branch in the sense that at least one of the following holds:

1. C is unbounded.
2. C reaches d^*, i.e. $\overline{C}$ contains some point (d^*,U) which is a weak solution of (3.1)/(3.2).

Note that in the second case, we do not exclude $U = 0$, i.e. C might return to the trivial branch at the hyperbola point $d^* \in C_n$. We denote the set of global bifurcation points (with respect to d^*) by $B_{\text{global}}(d^*)$.

Proposition 3.2. *Each global bifurcation point is a local bifurcation point. Moreover, B_{local} is closed in D_S. In particular, $\overline{B_{global}(d^*)} \cap D_S \subseteq B_{local}$.*

In our considerations an important role will be played by the vertical asymptote of the rightmost hyperbola

$$\{(d_1, d_2) \in D_S : d_1 = b_{11}/\kappa_1\} \tag{3.5}$$

and the corresponding part to the right of this asymptote, i.e.

$$H := \{(d_1, d_2) \in D_S : d_1 > b_{11}/\kappa_1\}. \tag{3.6}$$

The following has been shown in [6]:

Proposition 3.3. *$H \cap B_{local} = \varnothing$.*

We also need another terminology. We say that a point $d \in \partial D_S$ is *n-interior* if $d \in C_n$ and if there is some eigenfunction e of $-\Delta$ for the eigenvalue κ_n, i.e. $e = u$ is a weak solution of (1.5), such that, for some constant $\varepsilon > 0$,

$$\begin{gathered} e \geq \varepsilon > 0 \text{ almost everywhere on } \Omega_0 \text{ and} \\ e \geq \varepsilon > 0 \text{ almost everywhere on } \Gamma. \end{gathered} \tag{3.7}$$

Recall in this connection that we require (3.3)

For the case that Γ is a smooth manifold with boundary and $\Omega_0 = \varnothing$, we replace (3.7) by the milder requirement

$$e(x) > 0 \text{ for almost all } x \in \Gamma. \tag{3.8}$$

We say that $d \in \partial D_S$ is *(n,m)-interior* if $d \in C_n \cap C_m$ and if there is a function e which is a linear combination of eigenfunctions to the eigenvalues κ_n and κ_m such that (3.7) or (3.8) holds, respectively.

If $d \in C_n \cap C_m \cap \partial D_S$ and d is n-interior or m-interior then d is also (n,m)-interior. However, d might be (n,m)-interior without being n-interior or m-interior.

Using the main results from [6], we will prove now:

Lemma 3.1. *Let $d \in \partial D_S$ be n-interior or (n,m)-interior. Then there is an open neighborhood $U_0 \subseteq \mathbb{R}^2$ of d such that $U_0 \cap D_S \cap B_{local} = \varnothing$. Moreover, if the hypotheses are satisfied with $d^* = d$, then each continuous compact path γ in $D_S \cup \{d^*\}$ connecting $d^* = d$ with some point from (3.6) contains some point from $B_{global}(d^*) \subseteq D_S$.*

Lemma 3.1 would follow rather straightforwardly from the results of [6] if we would allow that the connected set C in the definition of global bifurcation points is contained in $(D_S \cup \{d^*\}) \times (\mathbb{H} \setminus \{0\})$. However, it might happen that $C \setminus (\{d^*\} \times \mathbb{H})$ fails to be connected. Therefore, we need some additional arguments. We use the following result which is actually a consequence of Theorem 2.2 (and can also be proved using only the [countable] axiom of dependent choices, see [12]):

Theorem 3.1. *Let X be a regular space, $A \subseteq X$ compact, and $S \subseteq X$ be closed. Suppose that S is locally compact, metrizable and σ-compact. Then for each open set $U \supseteq A$ the following statements are equivalent:*

1. *There is a connected set $C \subseteq S$ which intersects A and is either noncompact or intersects ∂U.*
2. *There is a connected set $C \subseteq (S \cap U) \setminus A$ such that $\overline{C} \cap S$ intersects A and is either noncompact or intersects ∂U.*

Proof of Lemma 3.1. Only the last claim is not immediately contained in some of the results from [6]. To see this last claim, we apply the main result from [6] first to show that there is a connected set $C_0 \subseteq (D_S \cup \{d^*\}) \times (\mathbb{H} \setminus \{0\})$ such that $\overline{C}_0$ intersects $(\gamma \cap D_S) \times \{0\}$ and such that either C_0 is unbounded or $\overline{C}_0$ intersects also $\{d^*\} \times \mathbb{H}$. Moreover, we will arrange it that, in the space $X := \mathbb{R}^2 \times \mathbb{H}$, C_0 has the additional property that closures of bounded subsets of $S := \overline{C}_0$ are compact and consist only of (weak) solutions and satisfies

$$\overline{C}_0 \cap (\mathbb{R}^2 \times \{0\}) = (\gamma \setminus (U_0 \cap D_S)) \times \{0\} . \tag{3.9}$$

Indeed, assume that $\gamma = \sigma([a,b])$ with some continuous $\sigma \colon [a,b] \to D_S \cup \{d^*\}$ satisfying $\sigma(a) = d^*$ and $\sigma(b) \in H$. We extend σ to a continuous $\sigma \colon [a,\infty) \to D_S \cup \{d^*\}$ with $\sigma(s) \in H$ for all $s \geq b$ such that both components of $\sigma(s)$ tend to ∞ as $s \to \infty$. For all sufficiently small $s_0 \in (a,b)$ we have $\sigma([a,s_0]) \subseteq U_0$, and by the main result from [6], we find some connected set $C_1 \subseteq [a,\infty) \times (\mathbb{H} \setminus \{0\})$ such that

$$C_0 := \{(\sigma(s), u) : (s,u) \in C_1\}$$

consists only of (nontrivial) weak solutions of (3.1)/(3.2) and such that $\overline{C}_1$ contains some point from $[s_0, b] \times \{0\}$ and such that either C_1 is unbounded or $\overline{C}_1$ contains some point from $\{a\} \times \mathbb{H}$ or from $([a,s_0) \cup (b,\infty)) \times \{0\}$

The set C_0 has all required properties. Indeed, since C_0 consists only of nontrivial solutions, the closure of $\sigma([a,\infty))$ is contained in $\gamma \cup (U_0 \cap D_S) \cup H$, and no point of $(U_0 \cap D_S) \cup H$ is a local bifurcation point, we obtain (3.9). The set C_0 is connected, because it is the image of the connected set C_1 under the continuous map $T(s,u) := (\sigma(s), u)$. The set $\overline{C}_0$ contains $T(\overline{C}_1)$ and thus intersects $T([s_0,b] \times \{0\}) \subseteq (\gamma \cap D_S) \times \{0\}$ and is either unbounded (by our choice of the extension of σ) or intersects $\{d^*\} \times \mathbb{H}$ or $(U_0 \cup H) \times \{0\}$. In the latter case, $\overline{C}_0$ actually intersects $\{d^*\} \times \{0\}$ by (3.9).

To see these remaining properties, recall that with F from Proposition 3.1 the weak solutions of (3.1)/(3.2) are the elements of

$$\{(d,u) \in (D_S \cup \{d^*\}) \times \mathbb{H} : u \in F(d,u)\} .$$

Since C_0 is contained in this set, F is upper semicontinuous, and the closure of $\sigma([a,\infty))$ is contained in $D_S \cup \{d^*\}$, also $S = \overline{C}_0$ is contained in this set. The compactness of F described in Proposition 3.1 and our choice of the extension of σ implies that closed bounded subsets of S are compact.

Hence, $S = \overline{C}_0$ has all required properties. In particular, S is locally compact and σ-compact. We apply Theorem 3.1 with $A := (\gamma \setminus U_0) \times \{0\}$ and $U := X \setminus (\{d^*\} \times \mathbb{H})$. The connected set $\overline{C}_0$ witnesses that the first statement of Theorem 3.1 is satisfied: Note that this set indeed intersects A in view of (3.9), because $\overline{C}_0$ intersects $(\gamma \cap D_S) \times \{0\}$.

Hence, also the second statement of Theorem 3.1 holds which means that there is a connected set $C \subseteq (S \cap U) \setminus A$ such that the set $\overline{C}$ contains some point $(d_0, 0)$ with $d_0 \in \gamma \setminus U_0 \subseteq \gamma \cap D_S$ and such that either $\overline{C}$ is noncompact (and thus unbounded) or intersects $\partial U = \{d^*\} \times \mathbb{H}$. Thus, $d_0 \in B_{\text{global}}(d^*)$. □

4 The main result

Theorem 4.1. *Let $D_0 := D_S \setminus H$ where H is from* (3.6).

Let $d^* \in \partial D_S$ be m-interior or (n,m)-interior $(n \leq m)$ and such that the hypotheses described at the beginning of Section 3 are satisfied with this d^*. Then there is a connected set $B \subseteq \overline{B_{global}(d^*)} \cap D_0 \subseteq B_{local}$ such that $\overline{B}$ intersects the d_1-axis or some hyperbola C_k "strictly under" d^*.

More precisely, we have $k \geq n$, and the case $C_k = C_m$ is only possible if d^* is an intersection point of two different hyperbolas. In all cases, the intersection $\overline{B} \cap C_k$ does not contain d^* (i.e. is strictly under *d^*).*

Moreover, this branch B satisfies in addition the following:

1. *If C_n is the right-most hyperbola (i.e. if $C_n = C_1$) then B is unbounded.*
2. *Otherwise (i.e. if $C_n \neq C_1$) the set B is unbounded, or $\overline{B}$ intersects some hyperbola C_k "strictly over" d^* (i.e. $k \leq n$, and the case $C_k = C_n$ is only possible if d^* is an intersection point of two different hyperbolas; $\overline{B} \cap C_k$ does not contain d^*).*

Moreover, for any k for which there is some k-interior point we have $\overline{B} \cap C_k = \varnothing$, and for any pair (k,ℓ) for which the intersection point is (k,ℓ)-interior point this intersection point is not contained in $\overline{B}$.

Figure 2 illustrates qualitatively the four possibilities of branches B of bifurcation points if there is some n-interior point with $C_n \neq C_1$; one of these possibilities must (qualitatively) occur according to Theorem 4.1. Similarly, Figure 3 illustrates the two possibilities of branches B if there is some 1-interior point.

In particular, if there are n-interior points for every n, then the last statement of Theorem 4.1 implies that only one possibility can occur: There must be a branch B which is unbounded and such that $\overline{B}$ intersects the d_1-axis (possibly at $(0,0)$).

We point out that (contrary to what the figures might suggest) the theorem does not state that the branch B is pathwise connected, i.e. it might look "weird" (but it is connected in the topological sense).

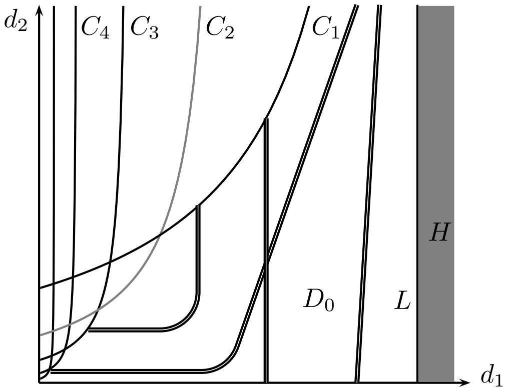

Figure 2: *The four qualitative different possible branches B of bifurcation points if there is some 2-interior point (one of these must occur)*

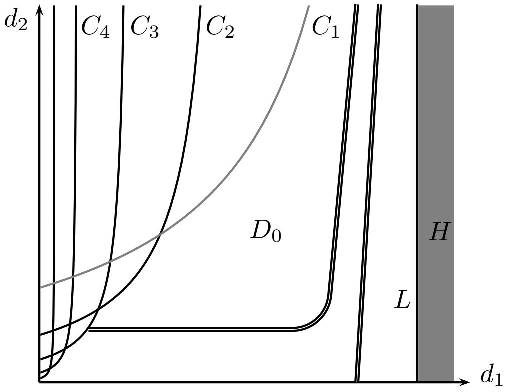

Figure 3: *The two qualitative different possible branches B of bifurcation points if there is some 1-interior point (one of these two must occur)*

Proof. The last statement of Theorem 4.1 is automatically satisfied by the first claim of Lemma 3.1, since $\overline{B_{\text{global}}(d^*)} \subseteq \overline{B_{\text{local}}}$ must be disjoint from any k-interior or (k,ℓ)-interior point.

Using this fact with d^*, we find some open neighborhood $U_0 \subseteq \mathbb{R}^2$ which is disjoint from $\overline{B_{\text{global}}(d^*)} \subseteq \overline{B_{\text{local}}}$.

Let $L_0 \subseteq H$ be some line which is parallel but strictly to the right of the line (3.5). Let $Q, H_0 \subseteq D_S$ be that parts to the left and right of this line L_0, respectively. Lemma 3.1 implies $\overline{B_{\text{local}}} \cap D_S \subseteq Q$.

Using the one-point compactification X of $\overline{Q}$, we consider Q as the disc-interior of some homeomorphically embedded disc, whose boundary corresponds to the union of the d_1-axis, the line L_0, the point ∞, and the "envelope" $E = \mathbb{R}^2_+ \cap \partial D_S$ of all of the hyperbolas C_n. Let A_2 be that part of the boundary which corresponds to L_0 (without the two "boundary points" at ∞ and at the d_1-axis), and let A_4 correspond to $U_0 \cap E$. Let A_1 and A_3 denote the ramining (compact) connected subsets of the boundary of the disc Q.

Now we can apply the disc-cutting theorem with $S = Q \cap \overline{B_{\text{global}}(d^*)}$. In fact, each continuous compact path in Q connecting A_2 with A_4 must intersect S by Lemma 3.1. Hence, the disc-cutting Theorem 2.3 implies the existence of a connected set $B \subseteq S$ with $\overline{B} \cap A_i \neq \varnothing$ for $i = 1, 3$. Since $\overline{B}$ cannot intersect L_0, the property $\overline{B} \cap A_3$ means that either B is unbounded or that $\overline{B}$ intersects some point of some C_k "strictly above" d^*. The property $\overline{B} \cap A_1$ means that $\overline{B}$ intersect some point of some C_k "strictly below" d^* or the d_1-axis. $\square$

`Received: March 2008. Revised: March 2008.`

References

[1] Drábek, P. and Kučera, M., *Reaction-diffusion systems: Destabilizing effect of unilateral conditions*, Nonlinear Anal., **12** (1988), no. 11, 1172–1192.

[2] Eisner, J., *Reaction-diffusion systems: Destabilizing effect of conditions given by inclusions*, Math. Bohem., **125** (2000), no. 4, 385–420.

[3] ______, *Critical and bifurcation points of reaction-diffusion systems with conditions given by inclusions*, Nonlinear Anal., **46** (2001), 69–90.

[4] ______, *Reaction-diffusion systems: Destabilizing effect of conditions given by inclusions II, examples*, Math. Bohem., **126** (2001), no. 1, 119–140.

[5] Eisner, J. and Kučera, M., *Spatial patterning in reaction-diffusion systems with nonstandard boundary conditions*, Fields Institute Communications, **25** (2000), 239–256.

[6] Eisner, J., Kučera, M. and Väth, M., *Global bifurcation of a reaction-diffusion system with inclusions*, (submitted).

[7] Kučera, M., *Bifurcation points of variational inequalities*, Czechoslovak Math. J., **32** (1982), 208–226.

[8] ———, *A new method for obtaining eigenvalues of variational inequalities. Multiple eigenvalues*, Czechoslovak Math. J., **32** (1982), 197–207.

[9] ———, *Influence of Signorini boundary conditions on bifurcation in reaction-diffusion systems*, Proceedings of the 5th International ISAAC Congress, Catania 2005, 2007, (to appear).

[10] Kučera, M. and Bosák, M., *Bifurcation for quasi-variational inequalities of reaction-diffusion type*, SAACM, **3** (1993), no. 2, 111–127.

[11] Turing, A.M., *The chemical basis of morphogenesis*, Phil. Trans. Roy. Soc. London, **B** (1952), 37–72.

[12] Väth, M., *Global solution branches and a topological implicit function theorem*, Ann. Mat. Pura Appl., **186** (2007), no. 2, 199–227.

[13] Ziemer, W.P., *Weakly differentiable functions*, Springer, New York, Berlin, Heidelberg, 1989.

CUBO A Mathematical Journal
Vol.10, $N^{\underline{o}}$04, (101–108). December 2008

Browder Convergence and Mosco Convergence for Families of Nonexpansive Mappings

TOMONARI SUZUKI[*]
Department of Mathematics, Kyushu Institute of Technology,
Tobata, Kitakyushu 804-8550, Japan
email: `suzuki-t@mns.kyutech.ac.jp`

ABSTRACT

We study the relationship between Browder's strong convergence and Mosco convergence of fixed-point set for families of nonexpansive mappings.

RESUMEN

Estudiamos la relación entre la convergencia fuerte de Browder y la convergencia de Mosco del conjunto de puntos fijos para familias de aplicaciones no espansivas.

Key words and phrases: *Nonexpansive mapping, nonexpansive semigroup, fixed point, Browder convergence, Mosco convergence.*

Math. Subj. Class.: *47H10, 47H09, 47H20.*

---

[*] The author is supported in part by Grants-in-Aid for Scientific Research from the Japanese Ministry of Education, Culture, Sports, Science and Technology.

1. Introduction

Let C be a subset of a Banach space E. A mapping T on C is called a *nonexpansive mapping* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. Using the results in Gossez and Lami Dozo [6] and Kirk [8], we can prove that $F(T)$ is nonempty in the case where C is weakly compact, convex and has the Opial property. See also [1, 5, 7] and others. In 1967, Browder [2] proved the following strong convergence theorem,

Theorem 1 (Browder [2]). *Let C be a bounded closed convex subset of a Hilbert space E and let T be a nonexpansive mapping on C. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ converging to 0. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by $x_n = (1-\alpha_n)Tx_n + \alpha_n u$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to Pu, where P is the metric projection from C onto $F(T)$.*

A family of mappings $\{T(t) : t \geq 0\}$ is called a *one-parameter strongly continuous semigroup of nonexpansive mappings* (*nonexpansive semigroup*, for short) on C if the following are satisfied:

(NS1) For each $t \geq 0$, $T(t)$ is a nonexpansive mapping on C.

(NS2) $T(s+t) = T(s) \circ T(t)$ for all $s, t \geq 0$.

(NS3) For each $x \in C$, the mapping $t \mapsto T(t)x$ from $[0, \infty)$ into C is strongly continuous.

Suzuki [15] proved that $\bigcap_t F(T(t))$ is nonempty provided C is bounded closed convex and every nonexpansive mapping on C has a fixed point. He also proved a semigroup version of Browder's convergence theorem in [11, 17].

Theorem 2 ([11, 17]). *Let E be a smooth Banach space with the Opial property such that the normalized duality mapping J of E is weakly sequentially continuous at zero. Let C be a weakly compact convex subset of E. Let $\{T(t) : t \geq 0\}$ be a nonexpansive semigroup on C. Let τ be a nonnegative real number. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ satisfying $0 < \alpha_n < 1$, $0 \leq \tau + t_n$ and $t_n \neq 0$ for $n \in \mathbb{N}$, and $\lim_n t_n = \lim_n \alpha_n / t_n = 0$. Fix $u \in C$ and define a sequence $\{x_n\}$ in C by $x_n = (1-\alpha_n)T(\tau + t_n)x_n + \alpha_n u$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to Pu, where P is the unique sunny nonexpansive retraction from C onto $\bigcap_t F(T(t))$.*

Motivated by Theorem 2, Suzuki [19] considered the Mosco convergence of $\{F(T(\tau + t_n))\}$. The following theorem is a corollary of the main result in [19].

Theorem 3 ([19]). *Let E, C and $\{T(t) : t \geq 0\}$ be as in Theorem 2. Let τ be a nonnegative real number and let $\{t_n\}$ be a sequence in $\mathbb{R}$ satisfying $0 \leq \tau + t_n$ and $t_n \neq 0$ for $n \in \mathbb{N}$, and $\lim_n t_n = 0$. Then $\{F(T(\tau + t_n))\}$ converges to $\bigcap_t F(T(t))$ in the sense of Mosco.*

Therefore we can guess that Browder convergence is strongly connected with Mosco convergence. In this paper, we study the relationship between Browder convergence and Mosco convergence for families of nonexpansive mappings.

2. Preliminaries

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

Let E be a Banach space and let $\{A_n\}$ be a sequence of subsets of E. Define two sets

$$\operatorname*{s\text{-}liminf}_{n\to\infty} A_n \quad \text{and} \quad \operatorname*{w\text{-}limsup}_{n\to\infty} A_n$$

as follows: $x \in \text{s-liminf}_n A_n$ if and only if there exist a sequence $\{x_n\}$ in E and $n_0 \in \mathbb{N}$ such that $\{x_n\}$ converges strongly to x and $x_n \in A_n$ for $n \in \mathbb{N}$ with $n \geq n_0$. $x \in \text{w-limsup}_n A_n$ if and only if there exists a sequence $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to x and $\{n \in \mathbb{N} : x_n \in A_n\}$ is an infinite subset of $\mathbb{N}$. It is obvious that $\text{s-liminf}_n A_n \subset \text{w-limsup}_n A_n$ holds. We say $\{A_n\}$ converges to a subset A of E in the *sense of Mosco* [9] if $A = \text{s-liminf}_n A_n = \text{w-limsup}_n A_n$. And we write

$$A = \operatorname*{M\text{-}lim}_{n\to\infty} A_n.$$

Let E be a Banach space. The *normalized duality mapping* J of E is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}.$$

E is said to be *smooth* if and only if $J(x)$ consists of one element for every $x \in E$. If E is smooth, then we can consider that J is a mapping from E into E^*. J is said to be *weakly sequentially continuous at zero* if for every sequence $\{x_n\}$ in E which converges weakly to $0 \in E$, $\{J(x_n)\}$ converges weakly* to $0 \in E^*$.

A nonempty subset C of a Banach space E is said to have the *Opial property* [10] if for each weakly convergent sequence $\{x_n\}$ in C with weak limit $z_0 \in C$,

$$\liminf_{n\to\infty} \|x_n - z_0\| < \liminf_{n\to\infty} \|x_n - z\|$$

holds for $z \in C$ with $z \neq z_0$. All nonempty compact subsets have the Opial property. Also, all Hilbert spaces, $\ell^p (1 \leq p < \infty)$ and finite dimensional Banach spaces have the Opial property. A Banach space with a duality mapping which is weakly sequentially continuous also has the Opial property [6]. We know that every separable Banach space can be equivalently renormed so that it has the Opial property [4].

Let C and K be subsets of a Banach space E. A mapping P from C into K is called sunny [3] if

$$P\big(Px + t\,(x - Px)\big) = Px$$

for $x \in C$ and $t \geq 0$ with $Px + t\,(x - Px) \in C$.

Let $\{S_n\}$ be a sequence of nonexpansive mappings on a closed convex subset C of a Banach space E and let $\{\alpha_n\}$ be a sequence in $(0, 1]$ with $\lim_n \alpha_n = 0$. $(E, C, \{S_n\}, \{\alpha_n\})$ is said to have *Browder's property* [16] if for each $u \in C$, a sequence $\{x_n\}$ defined by

$$x_n = (1 - \alpha_n)\, S_n x_n + \alpha_n\, u \tag{1}$$

for $n \in \mathbb{N}$ converges strongly. We note that $\{x_n\}$ is well defined because $x \mapsto (1-\alpha_n)\, S_n x + \alpha_n\, u$ is contractive. We know the following.

Lemma 1 ([16]). *Let $(E, C, \{S_n\}, \{\alpha_n\})$ have Browder's property. For each $u \in C$, put*

$$Pu = \lim_{n\to\infty} x_n, \tag{2}$$

where $\{x_n\}$ is a sequence in C defined by (1). Then P is a nonexpansive mapping on C.

Using P, we can rewrite Theorem 2 as follows.

Theorem 4. *Let E, C, $\{T(t) : t \geq 0\}$, τ, $\{\alpha_n\}$ and $\{t_n\}$ be as in Theorem 2. Then $\big(E, C, \{T(\tau + t_n)\}, \{\alpha_n\}\big)$ has Browder's property. Moreover a mapping P defined by (2) is the unique sunny nonexpansive retraction from C onto $\bigcap_t F(T(t))$.*

3. Main results

In this section, we prove our main results.

Theorem 5. *Let $(E, C, \{S_n\}, \{\alpha_n\})$ satisfy Browder's property. Assume that C has the Opial property. Define a mapping P on C by (2). Then* w-limsup$_n$ *$F(S_n) \subset F(P)$ holds.*

Proof. Fix $x \in$ w-limsup$_n\, F(S_n)$. Then there exist a subsequence $\{n_k\}$ of $\{n\}$ and a sequence $\{u_k\}$ in C such that $u_k \in F(S_{n_k})$ and $\{u_k\}$ converges weakly to x. We note that $\{u_k\}$ is bounded. Define a sequence $\{v_n\}$ in C by

$$v_n = (1-\alpha_n)\, S_n v_n + \alpha_n\, x.$$

Then from the assumption, $\{v_n\}$ converges strongly to Px. We have

$$\begin{aligned} \|u_k - v_{n_k}\| &\leq (1-\alpha_{n_k})\, \|u_k - S_{n_k} v_{n_k}\| + \alpha_{n_k}\, \|u_k - x\| \\ &\leq (1-\alpha_{n_k})\, \|u_k - v_{n_k}\| + \alpha_{n_k}\, \|u_k - x\| \end{aligned}$$

and hence $\|u_k - v_{n_k}\| \leq \|u_k - x\|$. So

$$\liminf_{k\to\infty} \|u_k - Px\| \leq \liminf_{k\to\infty} \big(\|u_k - v_{n_k}\| + \|v_{n_k} - Px\|\big) \leq \liminf_{k\to\infty} \|u_k - x\|.$$

From the Opial property, we obtain $Px = x$. □

As a direct consequence of Theorem 5, we obtain the following.

Theorem 6. *Let $(E, C, \{S_n\}, \{\alpha_n\})$ satisfy Browder's property. Define a mapping P on C by (2). Assume that C has the Opial property and $F(P) \subset F(S_n)$ for $n \in \mathbb{N}$. Then* M-lim$_n$ *$F(S_n) = F(P)$ holds.*

Proof. From the assumption, $F(P) \subset \text{s-liminf}_n F(S_n)$. So by Theorem 5, we obtain the desired result. $\square$

Remark. Using Theorems 2 and 6, we can prove Theorem 3.

We next apply Theorem 6 to infinite families of nonexpansive mappings. The following convergence theorem was proved in [12, 14].

Theorem 7 ([12, 14])**.** *Let E and C be as in Theorem 2. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $(0, 1/2)$ satisfying* $\lim_n t_n = \lim_n \alpha_n / {t_n}^{\ell} = 0$ *for $\ell \in \mathbb{N}$. Let $\{I_n\}$ be a sequence of nonempty subsets of $\mathbb{N}$ such that $I_n \subset I_{n+1}$ for $n \in \mathbb{N}$, and $\bigcup_n I_n = \mathbb{N}$. Define a sequence $\{S_n\}$ of nonexpansive mappings on C by*

$$S_n x = \left(\left(1 - \sum_{k \in I_n} {t_n}^k \right) T_1 x + \sum_{k \in I_n} {t_n}^k T_{k+1} x \right).$$

Then $\big(E, C, \{S_n\}, \{\alpha_n\}\big)$ has Browder's property. Moreover a mapping P defined by (2) *is the unique sunny nonexpansive retraction from C onto $\bigcap_n F(T_n)$.*

By Theorem 6, we obtain the following.

Theorem 8. *Let E and C be as in Theorem 2. Let $\{T_n : n \in \mathbb{N}\}$ be an infinite family of commuting nonexpansive mappings on C. Let $\{t_n\}$ be a sequence in $(0, 1/2)$ converging to 0. Let $\{I_n\}$ and $\{S_n\}$ be as in Theorem 7. Then $\{F(S_n)\}$ converges to $\bigcap_n F(T_n)$ in the sense of Mosco.*

Proof. Put $\alpha_n = {t_n}^n$. Then it is obvious that $\lim_n \alpha_n / {t_n}^{\ell} = 0$ holds for $\ell \in \mathbb{N}$. By Theorem 7, $\big(E, C, \{S_n\}, \{\alpha_n\}\big)$ has Browder's property and a mapping P defined by (2) is the unique sunny nonexpansive retraction from C onto $\bigcap_n F(T_n)$. Since $F(P) = \bigcap_n F(T_n)$, we have $F(P) \subset F(T_n)$. So by Theorem 6, we obtain the desired result. $\square$

We recall that a family of mappings $\{T(p) : p \in [0, \infty)^{\ell}\}$ is said to be an *ℓ-parameter nonexpansive semigroup* on a subset C of a Banach space E if the following are satisfied:

(ℓNS1) For each $p \in [0, \infty)^{\ell}$, $T(p)$ is a nonexpansive mapping on C.

(ℓNS2) $T(p + q) = T(p) \circ T(q)$ for all $p, q \in [0, \infty)^{\ell}$.

(ℓNS3) For each $x \in C$, the mapping $p \mapsto T(p)x$ from $[0, \infty)^{\ell}$ into C is continuous.

We denote by $\mathbb{Q}$ the set of all rational numbers. Using the result in [13], we obtain the following.

Theorem 9. *Let E and C be as in Theorem 2. Let $\{T(p) : p \in [0, \infty)^{\ell}\}$ be an ℓ-parameter nonexpansive semigroup on C. Let $p_1, p_2, \cdots, p_{\ell} \in [0, \infty)^{\ell}$ such that $\{p_1, p_2, \cdots, p_{\ell}\}$ is linearly independent in the usual sense. Let $\beta_1, \beta_2, \cdots, \beta_{\ell} \in \mathbb{R}$ such that $\{1, \beta_1, \beta_2, \cdots, \beta_{\ell}\}$ is linearly* *independent over* $\mathbb{Q}$*. Suppose* $p_0 := \beta_1 p_1 + \beta_2 p_2 + \cdots + \beta_\ell p_\ell \in [0,\infty)^\ell$*. Let* $\{t_n\}$ *be a sequence in* $(0, 1/2)$ *converging to* 0*. Define a sequence* $\{S_n\}$ *of nonexpansive mappings on* C *by*

$$S_n x = \left(1 - \sum_{k=1}^{\ell} {t_n}^k\right) T(p_0)x + \sum_{k=1}^{\ell} {t_n}^k \, T(p_k)x.$$

Then $\{F(S_n)\}$ *converges to* $\bigcap_p F(T(p))$ *in the sense of Mosco.*

4. Additional results

In this section, we observe Browder's property.

Proposition 1. *Let* $(E, C, \{T\}, \{\alpha_n\})$ *satisfy Browder's property. Define a mapping* P *on* C *by* (2)*. Then* P *is a nonexpansive retraction from* C *onto* $F(T)$*.*

Proof. We first fix $x \in C$ and define a sequence $\{u_n\}$ in C by $u_n = (1 - \alpha_n)\, T u_n + \alpha_n\, x$. Then since $\{u_n\}$ converges strongly to Px, we obtain $Px = TPx$, which implies $Px \in F(T)$. We next fix $y \in F(T)$ and define a sequence $\{v_n\}$ in C by $v_n = (1 - \alpha_n)\, T v_n + \alpha_n\, y$. Then since $y = (1 - \alpha_n)\, Ty + \alpha_n\, y$, we have $v_n = y$ and hence $Py = y$. This completes the proof. $\square$

Remark. Though it is not interesting, we have confirmed that M-$\lim_n F(T) = F(P)$ holds.

There is an example such that P is not a retraction. See also [18].

Example 1. Let E be the two dimensional real Hilbert space and put $C = E$. For $t \geq 0$, define a 2×2 matrices $T(t)$ by

$$T(t) = \begin{bmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{bmatrix}.$$

We can consider that $\{T(t) : t \geq 0\}$ is a linear nonexpansive semigroup on C. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences in $\mathbb{R}$ satisfying $0 < \alpha_n < 1$ and $0 < t_n$ for $n \in \mathbb{N}$, $\lim_n \alpha_n = \lim_n t_n = 0$ and $\eta := \lim_n t_n/\alpha_n \in (0, \infty)$. Then $(E, C, \{T(t_n)\}, \{\alpha_n\})$ satisfies Browder's property. However, a mapping P defined by (2) is not a retraction.

Proof. For $\alpha \in (0, 1)$ and $t \in (0, \infty)$, we put a 2×2 matrix $P(\alpha, t)$ by

$$P(\alpha, t) = \frac{\alpha}{4\,(1-\alpha)\, \sin^2(t/2) + \alpha^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix},$$

where $a = \alpha + 2\,(1-\alpha)\, \sin^2(t/2)$ and $b = (1-\alpha)\, \sin(t)$. It is easy to verify that for $u \in C$, $P(\alpha, t)u$ is the unique point satisfying $x = (1-\alpha)\, T(t)x + \alpha\, u$. We have

$$P := \lim_{n\to\infty} P(\alpha_n, t_n) = \frac{1}{\eta^2 + 1} \begin{bmatrix} 1 & -\eta \\ \eta & 1 \end{bmatrix} = \frac{1}{\sqrt{\eta^2 + 1}}\, T(\theta),$$

where $\theta := \arctan(\eta) \in (0, \pi/2)$. Hence $(E, C, \{T(t_n)\}, \{\alpha_n\})$ satisfies Browder's property. However, P does not satisfy $P^2 = P$. $\square$

We finally give an example such that $\text{M-}\lim_n F(S_n) \subsetneqq F(P)$.

Example 2. Let T be a nonexpansive mapping on a bounded closed convex subset C of a Banach space E. Assume that T is not the identity mapping on C. Define a sequence $\{S_n\}$ of nonexpansive mappings on C by

$$S_n x = (1 - t_n)\, x + t_n\, Tx,$$

where $\{t_n\}$ is a sequence in $(0, 1)$ converging to 0. Let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_n \alpha_n = 0$ and $\lim_n \alpha_n / t_n = \infty$. Then $(E, C, \{S_n\}, \{\alpha_n\})$ satisfies Browder's property, a mapping P defined by (2) is the identity mapping on C and $\text{M-}\lim_n F(S_n) \subsetneqq F(P)$ holds.

Proof. Fix $x \in C$ and define a sequence $\{u_n\}$ in C by $u_n = (1 - \alpha_n)\, S_n u_n + \alpha_n\, x$. We have

$$\begin{aligned} \|u_n - x\| &= (1 - \alpha_n)\, \|S_n u_n - x\| \\ &\leq (1 - \alpha_n)\,(1 - t_n)\, \|u_n - x\| + (1 - \alpha_n)\, t_n\, \|T u_n - x\| \end{aligned}$$

and hence

$$\lim_{n \to \infty} \|u_n - x\| \leq \lim_{n \to \infty} \frac{(1 - \alpha_n)\, t_n}{\alpha_n + t_n - \alpha_n\, t_n} \|T u_n - x\| = 0.$$

Thus, $\{u_n\}$ converges strongly to x. Therefore $(E, C, \{S_n\}, \{\alpha_n\})$ satisfies Browder's property and $Px = x$ holds. From the assumption, $F(T) \subsetneqq C = F(P)$. Since $F(S_n) = F(T)$, we have $\text{M-}\lim_n F(S_n) = F(T)$ and hence $\text{M-}\lim_n F(S_n) \subsetneqq F(P)$. $\square$

`Received: April 2008. Revised: April 2008.`

References

[1] F.E. Browder, *Fixed-point theorems for noncompact mappings in Hilbert space*, Proc. Nat. Acad. Sci. USA, **53** (1965), 1272–1276.

[2] ———, *Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces*, Arch. Ration. Mech. Anal., **24** (1967), 82–90.

[3] R.E. Bruck, *Nonexpansive retracts of Banach spaces*, Bull. Amer. Math. Soc., **76** (1970), 384–386.

[4] D. van Dulst, *Equivalent norms and the fixed point property for nonexpansive mappings*, J. London Math. Soc., **25** (1982), 139–144.

[5] J. García Falset, E. Llorens Fuster and E.M. Mazcuñán Navarro, *Uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings*, J. Funct. Anal., **233** (2006), 494–514.

[6] J.-P. Gossez and E. Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math., **40** (1972), 565–573.

[7] M. Kato and T. Tamura, *Weak nearly uniform smoothness and worth property of ψ-direct sums of Banach spaces*, Comment. Math. Prace Mat., **46** (2006), 113–129.

[8] W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72** (1965), 1004–1006.

[9] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Adv. Math., **3** (1969), 510–585.

[10] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73** (1967), 591–597.

[11] T. Suzuki, *On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces*, Proc. Amer. Math. Soc., **131** (2003), 2133–2136.

[12] ———, *Strong convergence theorems of Browder's type sequences for infinite families of nonexpansive mappings in Hilbert spaces*, Bull. Kyushu Inst. Technol., **52** (2005), 21–28.

[13] ———, *The set of common fixed points of an n-parameter continuous semigroup of mappings*, Nonlinear Anal., **63** (2005), 1180–1190.

[14] ———, *Browder's type strong convergence theorems for infinite families of nonexpansive mappings in Banach spaces*, Fixed Point Theory Appl., **2006** (2006), Article ID 59692, 1–16.

[15] ———, *Common fixed points of one-parameter nonexpansive semigroups*, Bull. London Math. Soc., **38** (2006), 1009–1018.

[16] ———, *Moudafi's viscosity approximations with Meir-Keeler contractions*, J. Math. Anal. Appl., **325** (2007), 342–352.

[17] ———, *Browder's type convergence theorems for one-parameter semigroups of nonexpansive mappings in Banach spaces*, Israel J. Math., **157** (2007), 239–257.

[18] ———, *Some comments about recent results on one-parameter nonexpansive semigroups*, Bull. Kyushu Inst. Technol., **54** (2007), 13–26.

[19] ———, *Mosco convergence of the sets of fixed points for one-parameter nonexpansive semigroups*, Nonlinear Anal. (2007), doi:10.1016/j.na.2007.04.026.

CUBO A Mathematical Journal
Vol.10, N$^{\underline{o}}$04, (109–117). December 2008

Positive Solutions for Elliptic Boundary Value Problems with a Harnack-Like Property

Toufik Moussaoui
Department of Mathematics, E.N.S.,
P.O. Box 92, 16050 Kouba, Algiers, Algeria
email: `moussaoui@ens-kouba.dz`
and
Radu Precup
Department of Applied Mathematics,
Babeş–Bolyai University, 400084 Cluj, Romania
email: `r.precup@math.ubbcluj.ro`

ABSTRACT

The aim of this paper is to present some existence results of positive solutions for elliptic equations and systems on bounded domains of $\mathbb{R}^N$ $(N \geq 1)$. The main tool is Krasnosel'skii's compression-expansion fixed point theorem.

RESUMEN

El objetivo de este artículo es presentar algunos resultados de existencia de soluciones positivas para ecuaciones elipticas y sistemas sobre dominios acotados de $\mathbb{R}^N$ $(N \geq 1)$. La principal herramienta es el teorema de punto fijo compresión-expansión de Krasnosel'skii.

Key words and phrases: *Positive solution, elliptic boundary value problem, elliptic systems, Harnack-like inequality, Krasnosel'skii's compression-expansion fixed point theorem.*

Math. Subj. Class.: *47H10, 35J65.*

1 Introduction

In this paper, we are concerned with the existence of positive solutions for the elliptic boundary value problem

$$\begin{cases} -\Delta u = \lambda\, f\,(x,u)\,, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and for the elliptic system

$$\begin{cases} -\Delta u = \alpha\, g\,(x,u,v)\,, & \text{in } \Omega, \\ -\Delta v = \beta\, h\,(x,u,v)\,, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Here Ω is a bounded regular domain of $\mathbb{R}^N$ $(N \geq 1)$, $f : \overline{\Omega} \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $g,\ h : \overline{\Omega} \times \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ are continuous functions, and λ, α and β are real parameters. By a *positive* solution of problem (1.1) we mean a function $u \in C^1\left(\overline{\Omega}, \mathbb{R}\right)$ which satisfies (1.1) (with Δu in the sense of distributions), and with $u\,(x) > 0$ for all $x \in \Omega$. A *positive* solution to problem (1.2) is a vector-valued function $(u,v) \in C^1\left(\overline{\Omega}, \mathbb{R}^2\right)$ satisfying (1.2), with $u, v \geq 0$ and $u + v > 0$ in Ω.

The main assumption will be a global weak Harnack inequality for nonnegative superharmonic functions. By a *superharmonic* function in a domain $\Omega \subset \mathbb{R}^N$ we mean a function $u \in C^1(\Omega, \mathbb{R})$ with $\Delta u \leq 0$ in the sense of distributions, i.e.,

$$\int_\Omega \nabla u \cdot \nabla v \geq 0 \quad \text{for every } v \in C_0^\infty(\Omega, \mathbb{R}) \quad \text{satisfying } v(x) \geq 0 \quad \text{on } \Omega.$$

We shall assume that the following *global weak Harnack inequality* holds:

$$\begin{cases} \text{There exists a compact set } K \subset \Omega \text{ and a number } \eta > 0 \\ \text{such that } u(x) \geq \eta \|u\|_0 \text{ for all } x \in K \\ \text{and every nonnegative superharmonic function} \\ u \in C^1(\overline{\Omega}, \mathbb{R}) \text{ with } u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.3}$$

Here by $\|u\|_0$ we denote the sup norm in $C\left(\overline{\Omega}, \mathbb{R}\right)$, i.e., $\|u\|_0 = \sup_{x \in \overline{\Omega}} |u(x)|$.

The connection between such type of inequalities and Krasnosel'skii's compression-expansion theorem when applied to boundary value problems was first explained in [4]. Also in [4] (see also [1]), several comments on weak Harnack type inequalities can be found.

By a cone in a Banach space E we mean a closed convex subset $\mathcal{C}$ of E such that $\mathcal{C} \neq \{0\}$, $\lambda\mathcal{C} \subset \mathcal{C}$ for all $\lambda \in \mathbb{R}_+$, and $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$.

Our main tool in proving the existence of positive solutions to problems (1.1) and (1.2) is Krasnosel'skii's compression-expansion theorem [3], [2]:

Theorem 1. *Let E be a Banach space, $\mathcal{C} \subset E$ a cone in E, and assume that $T : \mathcal{C} \longrightarrow \mathcal{C}$ is a completely continuous map such that for some numbers r and R with $0 < r < R$, one of the following conditions is satisfied:*

(i) $\|Tu\| \le \|u\|$ *for* $\|u\| = r$ *and* $\|Tu\| \ge \|u\|$ *for* $\|u\| = R$,

(ii) $\|Tu\| \ge \|u\|$ *for* $\|u\| = r$ *and* $\|Tu\| \le \|u\|$ *for* $\|u\| = R$.

Then T has a fixed point with $r \le \|u\| \le R$.

2 Existence results for Problem 1.1

In this section, E is the Banach space

$$C_0(\overline{\Omega}, \mathbb{R}) = \{u \in C(\overline{\Omega}, \mathbb{R}) :\ u = 0 \quad \text{on } \partial\Omega\}$$

endowed with norm $\|.\|_0$, and $\mathcal{C}$ is the cone

$$\mathcal{C} = \{u \in C_0(\overline{\Omega}, \mathbb{R}_+) :\ u(x) \ge \eta\|u\|_0 \quad \text{for all } x \in K\}. \tag{2.1}$$

In order to state our results we introduce the notation

$$\begin{aligned} f_0 &= \limsup_{y\to 0^+} \max_{x\in\overline{\Omega}} \frac{f(x,y)}{y} \quad \text{and} \quad \underline{f}_\infty = \liminf_{y\to\infty} \min_{x\in K} \frac{f(x,y)}{y} \\ \underline{f}_0 &= \liminf_{y\to 0^+} \min_{x\in K} \frac{f(x,y)}{y} \quad \text{and} \quad f_\infty = \limsup_{y\to\infty} \max_{x\in\overline{\Omega}} \frac{f(x,y)}{y}. \end{aligned}$$

Also, for a function $h : \overline{\Omega} \to \mathbb{R}$, by $h|_K$ we mean the function $h|_K\ (x) = h(x)$ if $x \in K$ and $h|_K$ $(x) = 0$ if $x \in \overline{\Omega} \setminus K$. For example, if 1 is the constant function 1 on $\overline{\Omega}$, then $1|_K(x) = 1$ if $x \in K$ and $1|_K(x) = 0$ for $x \in \overline{\Omega} \setminus K$.

Theorem 2. *Suppose* (1.3) *holds. Then for each λ satisfying*

$$\frac{1}{\underline{f}_\infty \eta \left\|(-\Delta)^{-1}\, 1|_K\right\|_0} < \lambda < \frac{1}{f_0\|(-\Delta)^{-1}1\|_0} \tag{2.2}$$

there exists at least one positive solution of problem (1.1).

Proof. Let λ be as in (2.2) and let $\epsilon > 0$ be such that

$$\frac{1}{(\underline{f}_\infty - \epsilon)\eta \left\|(-\Delta)^{-1}\, 1|_K\right\|_0} \le \lambda \le \frac{1}{(f_0 + \epsilon)\|(-\Delta)^{-1}1\|_0}. \tag{2.3}$$

We know that u is a solution of problem (1.1) if and only if

$$u=\lambda\,(-\Delta)^{-1}Fu$$

where $F:C(\overline{\Omega},\mathbb{R})\longrightarrow C(\overline{\Omega},\mathbb{R})$, $Fu(x)=f\,(x,u(x))$. Hence, a solution to problem (1.1) is a fixed point of the operator $T:\mathcal{C}\longrightarrow C_0(\overline{\Omega},\mathbb{R})$ given by

$$Tu=\lambda\,(-\Delta)^{-1}Fu.$$

We shall prove that the hypotheses of Theorem 1 are satisfied.

We have that the operator T satisfies

$$\begin{cases} -\Delta(Tu)=\lambda\,f\,(x,u)\,, & \text{in } \Omega,\\ Tu=0, & \text{on } \partial\Omega. \end{cases}$$

Then by the global weak Harnack inequality (1.3), one has $T(\mathcal{C})\subset\mathcal{C}$. Moreover, T is completely continuous by the Arzela-Ascoli Theorem.

Furthermore, by the definition of f_0, there exists an $r>0$ such that

$$f\,(x,u)\le(f_0+\epsilon)u\quad\text{for}\quad 0<u\le r \text{ and } x\in\overline{\Omega}. \tag{2.4}$$

Let $u\in\mathcal{C}$ with $\|u\|_0=r$. Then using (2.4), the monotonicity of operator $(-\Delta)^{-1}$ and of norm $\|.\|_0$, and (2.3), we obtain

$$\begin{aligned}\|Tu\|_0 &= \lambda\left\|(-\Delta)^{-1}Fu\right\|_0\\ &\le \lambda(f_0+\epsilon)\|u\|_0\left\|(-\Delta)^{-1}1\right\|_0\\ &\le \|u\|_0.\end{aligned}$$

Hence

$$\|Tu\|_0\le\|u\|_0\quad\text{for } \|u\|_0=r. \tag{2.5}$$

By the definition of $\underline{f}_\infty$, there is $R>r$ such that

$$f\,(x,u)\ge(\underline{f}_\infty-\epsilon)u\quad\text{for}\quad u\ge\eta R \text{ and } x\in K.$$

Then, if $u\in\mathcal{C}$ with $\|u\|_0=R$, we have

$$\begin{aligned}\|Tu\|_0 &= \lambda\left\|\,(-\Delta)^{-1}Fu\right\|_0\\ &\ge \lambda\left\|\,(-\Delta)^{-1}\,(Fu)|_K\right\|_0\\ &\ge \lambda(\underline{f}_\infty-\epsilon)\eta\|u\|_0\left\|(-\Delta)^{-1}\,1|_K\right\|_0\\ &\ge \|u\|_0.\end{aligned}$$

Hence

$$\|Tu\|_0\ge\|u\|_0\ \text{ for }\ \|u\|_0=R. \tag{2.6}$$

Inequalities (2.5) and (2.6) show that the expansion condition (i) in Theorem 1 is satisfied. Now Theorem 1 guarantees the existence of a fixed point u of T with $r\le\|u\|_0\le R$. □

Similarly, we have the following result:

Theorem 3. *Suppose* (1.3) *holds. Then for each* λ *satisfying*

$$\frac{1}{\underline{f}_0\eta\,\|(-\Delta)^{-1}\,1|_K\|_0} < \lambda < \frac{1}{f_\infty\|(-\Delta)^{-1}1\|_0} \tag{2.7}$$

there exists at least one positive solution of problem (1.1).

Proof. Let λ be as in (2.7) and let $\epsilon > 0$ be such that

$$\frac{1}{(\underline{f}_0-\epsilon)\eta\,\|(-\Delta)^{-1}\,1|_K\|_0} \le \lambda \le \frac{1}{(f_\infty+\epsilon)\|(-\Delta)^{-1}1\|_0}. \tag{2.8}$$

By the definition of $\underline{f}_0$, there exists an $r > 0$ such that

$$f(x,u) \ge (\underline{f}_0-\epsilon)u \ \text{ for } 0 < u \le r \text{ and } x \in K.$$

If $u \in \mathcal{C}$ and $\|u\|_0 = r$, then

$$\begin{aligned}\|Tu\|_0 &= \lambda\left\|\,(-\Delta)^{-1}Fu\right\|_0\\ &\ge \lambda\left\|\,(-\Delta)^{-1}\,(Fu)|_K\right\|_0\\ &\ge \lambda(\underline{f}_0-\epsilon)\eta\|u\|_0\,\left\|(-\Delta)^{-1}\,1|_K\right\|_0\\ &\ge \|u\|_0.\end{aligned}$$

Hence

$$\|Tu\|_0 \ge \|u\|_0 \ \text{ for } \ \|u\|_0 = r. \tag{2.9}$$

By the definition of f_∞, there is $R_0 > 0$ such that

$$f(x,u) \le (f_\infty+\epsilon)u \text{ for } u \ge R_0 \text{ and } x \in \overline{\Omega}.$$

Let M be such that $f(x,u) \le M$ for all $u \in [0,R_0]$ and $x \in \overline{\Omega}$, and let R be such that

$$R > r \text{ and } M \le (f_\infty+\epsilon)\,R.$$

If $u \in \mathcal{C}$ with $\|u\|_0 = R$, then $0 \le u(x) \le (f_\infty+\epsilon)\,R$ for all $x \in \overline{\Omega}$. Consequently, also using (2.8), we obtain

$$\begin{aligned}\|Tu\|_0 &= \lambda\left\|(-\Delta)^{-1}Fu\right\|_0\\ &\le \lambda(f_\infty+\epsilon)R\left\|(-\Delta)^{-1}1\right\|_0\\ &\le R\\ &= \|u\|_0.\end{aligned}$$

Hence

$$\|Tu\|_0 \le \|u\|_0 \ \text{ for } \ \|u\|_0 = R. \tag{2.10}$$

Inequalities (2.9) and (2.10) show that the compression condition (ii) in Theorem 1 is satisfied. Now Theorem 1 guarantees the existence of a fixed point u of T with $r \le \|u\|_0 \le R$. ☐

3 Existence results for Problem 1.2

In this section, we are concerned with the existence of positive solutions to the Dirichlet problem (1.2) for elliptic systems.

Here E will be the Banach space $C_0(\overline{\Omega},\mathbb{R}^2) := C_0(\overline{\Omega},\mathbb{R}) \times C_0(\overline{\Omega},\mathbb{R})$ endowed with the norm $\|(.,.)\|_0$ given by

$$\|(u,v)\|_0 = \|u\|_0 + \|v\|_0$$

and the cone in E will be $\mathcal{C} \times \mathcal{C}$, where $\mathcal{C}$ is given by (2.1).

In order to state our results in this section we introduce the notation

$$\begin{aligned} g_0 &= \limsup_{y+z\to 0^+} \max_{x\in\overline{\Omega}} \frac{g(x,y,z)}{y+z} \quad \text{and} \quad \underline{g}_\infty = \liminf_{y+z\to\infty} \min_{x\in K} \frac{g(x,y,z)}{y+z} \\ \underline{g}_0 &= \liminf_{y+z\to 0^+} \min_{x\in K} \frac{g(x,y,z)}{y+z} \quad \text{and} \quad g_\infty = \limsup_{y+z\to\infty} \max_{x\in\overline{\Omega}} \frac{g(x,y,z)}{y+z}. \end{aligned}$$

The limits $h_0, \underline{h}_0, h_\infty$ and $\underline{h}_\infty$ are defined similarly.

Theorem 4. *Suppose* (1.3) *holds. In addition assume that there are numbers* $p, q > 0$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *such that*

$$\frac{1}{\underline{g}_\infty \eta \left\|(-\Delta)^{-1} 1|_K\right\|_0} < \alpha < \frac{1}{p\, g_0 \|(-\Delta)^{-1}1\|_0} \tag{3.1}$$

and

$$\frac{1}{\underline{h}_\infty \eta \left\|(-\Delta)^{-1} 1|_K\right\|_0} < \beta < \frac{1}{q\, h_0 \|(-\Delta)^{-1}1\|_0}. \tag{3.2}$$

Then there exists at least one positive solution (u,v) *of problem* (1.2).

Proof. Let α, β be as in (3.1), (3.2) and let $\epsilon > 0$ be such that

$$\frac{1}{(\underline{g}_\infty - \epsilon)\eta \left\|(-\Delta)^{-1} 1|_K\right\|_0} \le \alpha \le \frac{1}{p\,(g_0+\epsilon) \|(-\Delta)^{-1}1\|_0}$$

and

$$\frac{1}{(\underline{h}_\infty - \epsilon)\eta \left\|(-\Delta)^{-1} 1|_K\right\|_0} \le \beta \le \frac{1}{q\,(h_0+\epsilon) \|(-\Delta)^{-1}1\|_0}.$$

It is easily seen that a vector-valued function (u,v) is a solution of problem (1.2) if and only if

$$\begin{aligned} u &= \alpha\,(-\Delta)^{-1} G\,(u,v) \\ v &= \beta\,(-\Delta)^{-1} H\,(u,v) \end{aligned}$$

where $G, H : C(\overline{\Omega},\mathbb{R}^2) \longrightarrow C(\overline{\Omega},\mathbb{R})$,

$$G(u,v)(x) = g\,(x,u(x),v(x))\,,\quad H\,(u,v)\,(x) = h\,(x,u\,(x)\,,v\,(x))\,.$$

Hence, (u,v) is a positive solution of (1.2) if it is a fixed point of the operator

$$T:\mathcal{C}\times\mathcal{C}\longrightarrow C_0(\overline{\Omega},\mathbb{R}^2),\ \ T=(T_1,T_2)$$

where

$$T_1\,(u,v)=\alpha\,(-\Delta)^{-1}G\,(u,v)\,,\ \ T_2\,(u,v)=\beta\,(-\Delta)^{-1}H\,(u,v)\,.$$

We shall prove that the hypotheses of Theorem 1 are satisfied.

Clearly the operator $T=(T_1,T_2)$ satisfies

$$\begin{cases} -\Delta(T_1u)=\alpha\,g\,(x,u,v)\,, & \text{in } \Omega,\\ -\Delta(T_2v)=\beta\,h\,(x,u,v)\,, & \text{in } \Omega,\\ T_1u=T_2v=0, & \text{on } \partial\Omega. \end{cases}$$

Then by the global weak Harnack inequality (1.3), we have $T(\mathcal{C}\times\mathcal{C})\subset\mathcal{C}\times\mathcal{C}$. Moreover, T is completely continuous by the Arzela-Ascoli Theorem.

By the definitions of g_0 and h_0, there exists an $r>0$ with

$$g\,(x,u,v)\le(g_0+\epsilon)(u+v)\quad\text{for}\quad u,v\ge0,\ 0<u+v\le r \text{ and } x\in\overline{\Omega}$$

and

$$h\,(x,u,v)\le(h_0+\epsilon)(u+v)\quad\text{for}\quad u,v\ge0,\ 0<u+v\le r \text{ and } x\in\overline{\Omega}.$$

Let $(u,v)\in\mathcal{C}\times\mathcal{C}$ with $\|(u,v)\|_0=r$. We have

$$\begin{aligned} \|T_1\,(u,v)\|_0 &= \alpha\left\|(-\Delta)^{-1}G\,(u,v)\right\|_0\\ &\le \alpha(g_0+\epsilon)\|u+v\|_0\left\|(-\Delta)^{-1}1\right\|_0\\ &\le \frac{1}{p}\|u+v\|_0\\ &\le \frac{1}{p}(\|u\|_0+\|v\|_0)\\ &= \frac{1}{p}\|(u,v)\|_0. \end{aligned}$$

Then $\|T_1\,(u,v)\,\|_0\le\frac{1}{p}\|(u,v)\|_0$. Similarly, we have

$$\begin{aligned} \|T_2\,(u,v)\|_0 &= \beta\left\|(-\Delta)^{-1}H\,(u,v)\right\|_0\\ &\le \beta(h_0+\epsilon)\|u+v\|_0\left\|(-\Delta)^{-1}1\right\|_0\\ &\le \frac{1}{q}\|u+v\|_0\\ &\le \frac{1}{q}(\|u\|_0+\|v\|_0)\\ &= \frac{1}{q}\|(u,v)\|_0. \end{aligned}$$

Thus $\|T_2(u,v)\|_0 \le \frac{1}{q}\|(u,v)\|_0$. Combining the above two inequalities, we obtain

$$\|T(u,v)\|_0 = \|T_1(u,v)\|_0 + \|T_2(u,v)\|_0 \le (\frac{1}{p}+\frac{1}{q})\|(u,v)\|_0 = \|(u,v)\|_0.$$

Next by the definitions of $\underline{g}_\infty$ and $\underline{h}_\infty$, there is $R>0$ such that

$$g(x,u,v) \ge (\underline{g}_\infty - \epsilon)(u+v) \quad \text{for} \quad u,v\ge 0,\ u+v\ge \eta R \text{ and } x\in K$$

and

$$h(x,u,v) \ge (\underline{h}_\infty - \epsilon)(u+v) \quad \text{for} \quad u,v\ge 0,\ u+v\ge \eta R \text{ and } x\in K.$$

Let $(u,v)\in \mathcal{C}\times\mathcal{C}$ with $\|(u,v)\|_0 = R$. Then for each $x\in K$, $u(x)\ge \eta\|u\|_0$ and $v(x)\ge\eta\|v\|_0$. Hence $(u+v)(x)\ge \eta(\|u\|_0+\|v\|_0)$, that is $(u+v)(x)\ge\eta R$ for all $x\in K$. Consequently,

$$G(u,v)(x) \ge (\underline{g}_\infty-\epsilon)(u+v)(x) \quad \text{for all } x\in K.$$

Furthermore

$$\begin{aligned} \|T_1(u,v)\|_0 &= \alpha\left\|(-\Delta)^{-1}G(u,v)\right\|_0 \\ &\ge \alpha\left\|(-\Delta)^{-1}\,G(u,v)|_K\right\|_0 \\ &\ge \alpha(\underline{g}_\infty-\epsilon)\left\|(-\Delta)^{-1}\,(u+v)|_K\right\|_0 \\ &\ge \alpha(\underline{g}_\infty-\epsilon)\left\|(-\Delta)^{-1}\,u|_K\right\|_0 \\ &\ge \alpha(\underline{g}_\infty-\epsilon)\eta\|u\|_0\left\|(-\Delta)^{-1}\,1|_K\right\|_0 \\ &\ge \|u\|_0. \end{aligned}$$

Similarly, we have

$$\|T_2(u,v)\|_0 \ge \|v\|_0.$$

The above two inequalities give

$$\|T(u,v)\|_0 \ge \|(u,v)\|_0.$$

Thus condition (i) in Theorem 1 is satisfied. Now Theorem 1 guarantees the existence of a fixed point (u,v) of T with $r\le\|(u,v)\|_0\le R$. □

In a similar way, one can prove:

Theorem 5. *Suppose* (1.3) *holds. In addition assume that there are numbers* $p,q>0$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *such that*

$$\frac{1}{\underline{g}_0\eta\,\|(-\Delta)^{-1}\,1|_K\|_0} < \alpha < \frac{1}{p\,g_\infty\|(-\Delta)^{-1}1\|_0}$$

and

$$\frac{1}{\underline{h}_0\eta\,\|(-\Delta)^{-1}\,1|_K\|_0} < \beta < \frac{1}{q\,h_\infty\|(-\Delta)^{-1}1\|_0}.$$

Then there exists at least one positive solution (u,v) *of problem* (1.2).

Received: April 2008. Revised: April 2008.

References

[1] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 1983.

[2] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer, New York, 2003.

[3] M.A. Krasnoselskii, *Topological Methods in the Theory of Nonlinear Integral Equations*, Cambridge University Press, New York, 1964.

[4] R. Precup, *Positive solutions of semi-linear elliptic problems via Krasnoselskii type theorems in cones and Harnack's inequality*, Mathematical Analysis and Applications, AIP Conf. Proc., 835, Amer. Inst. Phys., Melville, NY, 2006, 125–132.

CUBO
A Mathematical Journal

Submitting: By Submitting a paper to this journal, authors certify that the manuscript has not been previously published nor is it under consideration for publication by another journal or similar publication. Work submitted to **CUBO** will be refereed by specialist appointed by the Editorial Board of the journal. The acceptance (or not) of the paper will be communicated to the corresponding author by the Editor in Chief.

Manuscript: Manuscript should be written in English or French and submitted in duplicate to any one of the editors or to the Editorial Office. Authors should send two copies in their final form, printed on one side of the paper with generous margins. The first page should contain a short descriptive title, the name(s) of the author(s), the institutional affiliation and complete address (including e-mail) of each author. Paper should be accompanied by a short abstract, key words and phrases and Mathematical Subject Classification. References are indicated in the text by consecutive arabic numeral enclosed in square bracket. The full list should be collected and typed at the end of the paper in numerical order.

Press requirement: The abstract, no longer than 100 words, must be send in two languages, being english one of these. The **CUBO** Journal strongly encourages the use of LaTeX(or one of the TeXvariants: AMSTeX or AMSLaTeX) for manuscript preparation. References should be headed numerically, alphabetically organized and complete in all its details. Authors' initial should precede their name; journal title abbreviations should follow the style of Mathematical Reviews Index.

Reprint: Electronic reprints(PDF) will be supplied free of charge to the corresponding author.

Exchanges: The **CUBO** Journal accepts exchanges with other mathematical publications. More information can be obtained from the Editor.

For any technical question about **CUBO**, please send a e-mail to: Oscar Neto(oscar@dmat.ufpe.br).