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SUMMARY

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Preface

Analysis and geometry in Clifford algebras, in associated structures like Cayley-Dickson type algebras or commutative quaternions are being studied in many places all over the world. Such hypercomplex structures are widely used to investigate phenomena in physics, natural sciences and even in engineering. Starting point for all these considerations are methods of complex analysis. The Swiss mathematician R. Fueter and his followers like H.G. Haefeli as well as the Romanian mathematicians G. Moisil and N. Theodorescu in the thirties of the last century considered properties the solutions of corresponding three and four dimensional generalizations of Cauchy-Riemann systems. Nowadays much more sophisticated and intricated problems are beeing fields of research. Discrete and continuous hypercomplex structures can be applied advandageousely to problems in gauge theories, mathematical physics, signal and image processing, robotics as well as natural scienes and engineering. New classes of function spaces, boundary and initial boundary value problems of partial differential equations and adequate numerical methods are in the focus of the research. Articles in this volume represent the diversity research activities.

In the contribution of the Clifford Research Group at the Ghent University (here represented by F. Brackx, H. De Schepper, F. Sommen, L. Van de Voorde) is worked out a Clifford algebra framework in which discrete Dirac operators and discrete function theories can be studied. Suitable definitions of topological concepts lead to general function theoretioc results and a geometric interpretation of curvature vectors. With the aim to solve boundary value problems of partial differential equations numerically K. Gürlebeck developed in 1988 a discrete version of a function theory of quaternions in his professorial dissertation titled " Grundlagen einer diskreten räumlich verallgemeinerten Funktionen-Theorie und ihrer Anwendungen" (TU Karl-Marx-Stadt, 1988).

A. Wiman and G. Valiron studied the asymptotic growth behaviour of holomorphic and meromorphic functions in the complex analysis. The authors D. Constales (Ghent) R. De Almeida (Vila Real) and R.S. Kraußhar (Leuven) obtained in their paper a generalization of the Wiman-Valiron theory to the Clifford analysis setting. In order to be able to formulate multidimensional growth behaviour results a proper notion of growth and growth type of entire functions as well as entire and monogenic functions was necessary to determine. First results were obtained by M.A. Abul-Ez and D. Constales in 1990.

K. Gürlebeck and J. Morais (both Weimar) study geometric mapping properties of monogenic functions. The main result is a generalization of the famous classical Bohr's theorem from 1914. Among other results is proved: Let be given a Clifford algebra valued monogenic function such that $f(x) - f(0)$ is orthogonal to the so-called *hyperholomorphic constants* with respect to a corresponding inner product with $|f(x)| < 1$ in the unit ball. Then the series of the absolute values of the Fourier expansion is smaller than 1 in a ball of the radius $0 \leq r < 0.004..$ Furthermore, Hadamard's real part theorems from 1892 are generalized to quaternionic monogenic square integrable functions. So called M-conformal mappings are studied.

In cooperation with Le Thu Hoai (Hanoi) we introduce in our contribution the very general notion of L-holomorphy, which also includes functions defined on lattices. Taylor-Gontcharov formulae are deduced and used for the solution of boundary value problems of higher order system of elliptic partial differential equations.

S. V. Ludkovsky (Moscow) describes so-called wrap groups of fibre bundels over quaternions and octonians and proved their existence. Such groups have a structure of an infinite dimensional Lie group. Loop groups of spheres and fibres with parallel transport structures are generalized by wrap groups. Geometric loop groups have for instance important applications in quantum physics, in gauge theories and for the decription of branes in the superstring theory.

A. Perotti (Trento) investigated in his article the action of the conformal group of the one-point compactification of the real quaternions. He studies so called *hyperholomorphic functions*, which appear after identification of $\mathbb{H}$ by $\mathbb{C}^2$. By consideration of a general complex structure $J_p - p$ an imaginary unit in S^2 – a Cauchy–Riemann operator

$$\overline{\partial}_p = \frac{1}{2}(d + pJ_p^+ \circ d) \tag{1}$$

is defined. In order to deduce criterions for holomorphicity a energy-minimum property of such holomorphic maps is used. The results can be applied to describe rotations in three dimension by two three-dimensional biregular rotations.

Finally, J. Tolksdorf (Freiberg) deals with two different geometrical aspects of Maxwell equations. In the center of his attention are the so called " Dirac type Gauge theories". A universal Dirac action is fundamental for his theory. Well-known first order differential equations like Maxwell equations and the Einstein equation are special cases. In terms of a Hermitean Clifford modulus the Yang-Mills functional is deduced in a natural way. New general Lichnerowicz formulae belong to the highlights of his paper. The Dirac and the Majorana equation can now be studied by common geometric observations.

Freiberg, Germany Wolfgang Sprößig

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A Generalization of Wiman and Valiron's theory to the Clifford analysis setting

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ABSTRACT

The classical notions of growth orders, maximum term and the central index provide powerful tools to study the asymptotic growth behavior of complex-analytic functions.

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This leads to much insight into the structure of the solutions to many two dimensional partial differential equations that are related to boundary value problems from harmonic analysis in the plane. In this overview paper we show how the classical techniques and results from Wiman and Valiron can be extended to the Clifford analysis setting in order to treat successfully analogous higher dimensional problems.

RESUMEN

Las nociones clásicas de orden de crecimiento, término máximo y de índice central proporcionan herramientas poderosas para estudiar el comportamiento de crecimiento asintótico de funciones complejas analíticas. Esto nos revela la estructura de las soluciones de varias ecuaciones diferenciales parciales de dimensión dos que son relacionadas con problemas de valores en la frontera venidos de análisis armónico en el plano. Mostramos como las técnicas clásicas y resultados de Wiman y Valiron pueden ser extendidas al contexto de análisis de Clifford para tratar con éxito problemas análogos de dimensión grande.

Key words and phrases: *monogenic functions, growth orders, growth type, maximum term, central index, Valiron's inequalities, asymptotic growth, partial differential equations.*

Math. Subj. Class.: *30G35, 30D15.*

1 Introduction

The study of the asymptotic growth behavior of holomorphic and meromorphic functions in one and several complex variables is one of the central topics in complex analysis. This line of investigation started with early works of E. Lindelöf [22], A. Pringsheim [24], A. Wiman [26] and G. Valiron [25] and had its major breakthrough in the 1920s by works of R. Nevanlinna [23] and his school. Their results turned out to be very useful in the study of complex partial differential equations, see e.g. [20, 21] and elsewhere.

This provides a strong motivation to also develop analogous methods for other function classes and in higher dimensions. One natural higher dimensional generalization of complex analysis is Clifford analysis. In this context one considers Clifford algebra valued solutions of the generalized Cauchy-Riemann system

$$Df := \frac{\partial f}{\partial x_0} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i = 0. \tag{1}$$

Solutions to this system are often called monogenic or Clifford holomorphic. Many classical theorems from complex analysis, such as for instance the Cauchy integral formula, the residue theorem, Laurent expansion theorems, etc. carry over to the higher dimensional context using this operator, see for instance [8, 5, 7]. Nevertheless, as far as we know, questions concerning possible generalizations of Wiman-Valiron theory remained untouched for a long time.

In [1] M.A. Abul-Ez and the first author introduced the notion of the growth order and the type for a particular subclass of entire Clifford holomorphic functions. See also the follow-up papers [2, 3]. In our recent papers [8, 9, 11, 13] we developed the basics for a generalized Wiman-Valiron theory for general entire monogenic functions and for monogenic Taylor series of finite convergence radius. We also managed to extend these techniques to the context of more general systems of partial differential equations, such as higher dimensional iterated Cauchy-Riemann systems [6, 7] and to polynomial Cauchy-Riemann systems equations with complex coefficients [10]. In this paper we give a concise overview over our results concerning the entire monogenic case. We show how the notions of growth orders, growth type, maximum term and the central index can be reasonably generalized to the Clifford analysis context. We exhibit how these tools can be applied to get insight in the asymptotics of related function classes and in the structure of solutions to related higher dimensional partial differential equations. This line of investigation should be regarded as a starting point to develop analogous methods for larger classes of functions that are in kernels of elliptic differential operators. We hope to get more insight in the structure of the solutions to larger classes of higher dimensional partial differential equations.

2 Preliminaries

We begin by introducing the basic notions and concepts. For detailed information about Clifford algebras and their function theory we refer for example to [8, 1] and [7].

2.1 Clifford algebras

By $\{e_1, e_2, \ldots, e_n\}$ we denote the canonical basis of the Euclidean vector space $\mathbb{R}^n$. The attached real Clifford algebra Cl_{0n} is the free algebra generated by $\mathbb{R}^n$ modulo the relation

$$\mathbf{x}^2 = -\|\mathbf{x}\|^2 e_0,$$

where $\mathbf{x} \in \mathbb{R}^n$ and e_0 is the neutral element with respect to multiplication of the Clifford algebra Cl_{0n}. In the Clifford algebra Cl_{0n} the following multiplication rules hold

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0, \quad i, j = 1, \cdots, n,$$

where δ_{ij} is the Kronecker symbol. A basis for the Clifford algebra Cl_{0n} is given by the set $\{e_A : A \subseteq \{1, \cdots, n\}\}$ with $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \le l_1 < \cdots < l_r \le n$, $e_\emptyset = e_0 = 1$. Each $a \in Cl_{0n}$ can be written in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$. Two examples of real Clifford algebras are the complex number field $\mathbb{C}$ and the Hamiltonian skew field $\mathbb{H}$.

The conjugation anti-automorphism in the Clifford algebra Cl_{0n} is defined by $\overline{a} = \sum_A a_A \overline{e}_A$, where $\overline{e}_A = \overline{e}_{l_r}\overline{e}_{l_{r-1}} \cdots \overline{e}_{l_1}$ and $\overline{e}_j = -e_j$ for $j = 1, \cdots, n$, $\overline{e}_0 = e_0 = 1$. The linear subspace $span_{\mathbb{R}}\{1, e_1, \cdots, e_n\} = \mathbb{R} \oplus \mathbb{R}^n \subset Cl_{0n}$ is the so-called space of paravectors $z = x_0 + x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ which we simply identify with $\mathbb{R}^{n+1}$. The term $x_0 =: Sc(z)$ is called the scalar part of the paravector z and $\mathbf{x} := x_1 e_1 + \cdots + x_n e_n =: Vec(z)$ its vector part.

A scalar product between two Clifford numbers $a, b \in Cl_{0n}$ is defined by $\langle a, b \rangle := Sc(a\overline{b})$ and the Clifford norm of an arbitrary $a = \sum\limits_A a_A e_A$ is $\|a\| = (\sum\limits_A |a_A|^2)^{1/2}$.

Any paravector $z \in \mathbb{R}^{n+1} \backslash \{0\}$ has an inverse element in $\mathbb{R}^{n+1}$ given by $z^{-1} = \overline{z}/\|z\|^2$.

In order to present the calculations in a more compact form, the following notations will be used, where $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ is an n-dimensional multi-index:

$$\mathbf{x}^{\mathbf{m}} \ := \ x_1^{m_1} \cdots x_n^{m_n}, \ \ \mathbf{m}! \ := \ m_1! \cdots m_n!, \quad |\mathbf{m}| \ := \ m_1 + \cdots + m_n.$$

By $\tau(i)$ we denote the multi-index $(m_1, \ldots, m_n)$ with $m_j = \delta_{ij}$ for $1 \leq j \leq n$.

2.2 Clifford analysis

One way to generalize complex function theory to higher dimensional hypercomplex spaces is offered by the Riemann approach which considers Clifford algebra valued functions defined in $\mathbb{R}^{n+1}$ that are annihilated by the generalized Cauchy-Riemann operator

$$D := \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}. \tag{2}$$

If $U \subset \mathbb{R}^{n+1}$ is an open set, then a real differentiable function $f : U \rightarrow Cl_{0n}$ is called left (right) monogenic or Clifford holomorphic at a point $z \in U$ if $Df(z) = 0$ (or $fD(z) = 0$). Functions that are left monogenic in the whole space are also called left entire.

The notion of left (right) monogenicity in $\mathbb{R}^{n+1}$ provides indeed a powerful generalization of the concept of complex analyticity to Clifford analysis. Many classical theorems from complex analysis could be generalized to higher dimensions by this approach, we refer e.g. to [8]. One important tool is the generalized Cauchy integral formula.

Let us denote by A_{n+1} the n-dimensional surface "area" of the $(n+1)$-dimensional unit ball, and by $q_{\mathbf{0}}(z) = \frac{\overline{z}}{\|z\|^{n+1}}$ the Cauchy kernel function.

Then every function f that is left monogenic in a neighbourhood of the closure $\overline{\mathcal{D}}$ of a domain $\mathcal{D}$ satisfies

$$f(z) = \frac{1}{A_{n+1}} \int_{\partial \mathcal{D}} q_{\mathbf{0}}(z - w)\, d\sigma(w)\, f(w), \tag{3}$$

where $d\sigma(w)$ is the paravector-valued outer normal surface measure, i.e.,

$$d\sigma(w) = \sum_{j=0}^{n} (-1)^j e_j dw_0 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_n$$

with $\widehat{dw_j} = dw_0 \wedge \cdots \wedge dw_{i-1} \wedge dw_{i+1} \wedge \cdots \wedge dw_n$. It is important to mention that the set of left (right) monogenic functions forms only a Clifford right (left) module for $n > 1$.

In contrast to complex analysis, the ordinary powers of the hypercomplex variables are not null-solutions to the generalized Cauchy-Riemann system. In Clifford analysis these are substituted by the Fueter polynomials. These are defined by

$$\mathcal{P}_{\mathbf{m}}(z) = \frac{1}{|\mathbf{m}|!} \sum_{\pi \in perm(\mathbf{m})} z_{\pi(m_1)} \cdots z_{\pi(m_n)}$$

where $perm(\mathbf{m})$ is the set of all permutations of the sequence $(m_1, \ldots, m_n)$ and $z_i := x_i - x_0 e_i$ for $i = 1, \ldots, n$ and $\mathcal{P}_{\mathbf{0}}(z) := 1$.

In this paper we prefer to work with the slightly modified Fueter polynomials

$$V_{\mathbf{m}}(z) := \mathbf{m}! \mathcal{P}_{\mathbf{m}}(z) \tag{4}$$

which turns out to be more convenient in our calculations.

These polynomials play the analogous role of the complex power functions in the Taylor series representation of a monogenic function. More precisely, if f is a left monogenic function in a ball $\|z\| < R$, then for all $\|z\| \leq r$ with $0 < r < R$

$$f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} V_{\mathbf{m}}(z) a_{\mathbf{m}},$$

where the elements $a_{\mathbf{m}}$ are Clifford numbers which — as a consequence of Cauchy's integral formula (3) — are uniquely defined by

$$a_{\mathbf{m}} = \frac{1}{\mathbf{m}! A_{n+1}} \int_{\|z\|<r} q_{\mathbf{m}}(\zeta) d\sigma(\zeta) f(\zeta)$$

where

$$q_{\mathbf{m}}(z) = \frac{\partial^{m_0+m_1+\cdots+m_n}}{\partial x_0^{m_0} \partial x_1^{m_1} \cdots \partial x_n^{m_n}} q_{\mathbf{0}}(z) \tag{5}$$

are the generalized negative power functions. An optimal upper bound estimate for the general negative power functions (5) is given in (cf. [5]) by

$$\|q_{\mathbf{m}}(z)\|_{\|z\|=r} \leq \frac{n(n+1)\cdots(n+|\mathbf{m}|-1)}{r^{|\mathbf{m}|+n}}. \tag{6}$$

We thus have the following upper bound estimate on the Taylor coefficients

$$\|a_{\mathbf{m}}\| \leq M(r,f)\frac{c(n,\mathbf{m})}{r^{|\mathbf{m}|}}.$$

Here, and in all that follows,

$$M(r,f) := \max_{\|z\|r}\{\|f(z)\|\}$$

denotes the maximum modulus of the function f in the closed ball with radius r and

$$c(n,\mathbf{m}) := \frac{n(n+1)\cdots(n+|\mathbf{m}|-1)}{\mathbf{m}!}. \tag{7}$$

3 Order of growth of monogenic functions in $\mathbb{R}^{n+1}$

In this overview paper we restrict us to treat entire monogenic functions. These are represented by Taylor series with infinite convergence radius. Many of the results that we are going to present here can be adapted to the case of monogenic Taylor series of finite convergence radius. However, this requires in many circumstances a much more technical treatment. We refer the reader who is interested in this particular topic to our recent paper [11].

The starting point for the following investigation is that monogenic functions $f : \mathbb{R}^{n+1} \to Cl_{0n}$ satisfy a maximum principle (cf. e.g. [8]). As a consequence, the function

$$M(r,f) := \max_{\|z\|=r}\{\|f(z)\|\}, \qquad r \geq 0 \tag{8}$$

is a well-defined, continuous and strictly monotonic increasing function whenever f is non-constant. From Cauchy's inequality one can deduce a direct generalization of the classical Liouville theorem (cf. [14, 16]). This states that every left entire monogenic function that is bounded in $\mathbb{R}^{n+1}$ is a constant. Cauchy's inequality furthermore permits us to deduce the following more general version of Liouville's theorem (see [13])

Theorem 1. *Suppose that $f : \mathbb{R}^{n+1} \to Cl_{0n}$ is left entire. If there exists an index $\mathbf{s} \in \mathbb{N}_0^n$ with $|\mathbf{s}| > 0$ satisfying*

$$\liminf_{r\to\infty} \frac{M(r,f)}{r^{|\mathbf{s}|}} = L < \infty, \tag{9}$$

then

$$f(z) = \sum_{|\mathbf{m}|=0}^{|\mathbf{s}|} V_{\mathbf{m}}(z)a_{\mathbf{m}}.$$

In order to characterize larger classes of monogenic functions by their asymptotic growth behavior it turned out to be convenient to introduce growth orders for monogenic functions [1, 3, 13]. For convenience we recall its definition. First we need, cf. e.g. [20]:

Definition 1. *Let $\alpha \geq 0$. Then the plus logarithm is defined by*

$$\log^+(\alpha) := \max\{0, \log(\alpha)\}. \tag{10}$$

In the same way as in the planar case (see [20]) one introduces in the Clifford analysis setting (see also [2, 13]):

Definition 2. *(Order of growth)*
Let $f : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire function. Then

$$\rho(f) = \rho := \limsup_{r\to\infty} \frac{\log^+(\log^+ M(r,f))}{\log(r)}, \quad 0 \leq \rho \leq \infty \tag{11}$$

is called the order of growth of the function f. We further introduce

$$\lambda(f) = \lambda : \liminf_{r\to\infty} \frac{\log^+(\log^+ M(r,f))}{\log(r)}, \quad 0 \leq \lambda \leq \infty \tag{12}$$

as the inferior order of growth of f.
If $\rho = \lambda$, then we say that f is a function of regular growth. If $\rho > \lambda$ then f has irregular growth.

To get a finer classification of the growth behavior within the set of monogenic functions that have the same growth order, one further introduces the growth type of a monogenic function as follows, cf. [9].

Definition 3. *For an entire monogenic function $f : \mathbb{R}^{n+1} \to Cl_{0n}$ of order ρ $(0 < \rho < \infty)$ the (growth) type is defined by*

$$\tau(f) = \tau := \limsup_{r\to\infty} \frac{\log^+ M(r,f)}{r^\rho}.$$

Let us start with discussing some particular examples. In [13] we have proved that the following higher dimensional generalizations of the exponential function all have growth order equal to 1:

(i) The monogenic plane wave exponential function from [1] defined for $\mathbf{m} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ by

$$f_1(\mathbf{m}, z) := (|\mathbf{m}| + i\mathbf{m})e^{-|\mathbf{m}|x_0}e^{i<\mathbf{m},\mathbf{x}>},$$

(ii) The monogenic generalization exponential function from [8]

$$\begin{aligned} f_2(z) = \exp(x_0, x_1, \ldots, x_n) &= e^{x_1+\cdots+x_n}\cos(x_0\sqrt{n}) \\ &- e^{x_1+\cdots+x_n}\frac{1}{\sqrt{n}}(e_1 + \cdots + e_n)\sin(x_0\sqrt{n}) \end{aligned}$$

(iii) The quaternion-valued 3-fold periodic exponential function from [17] given by

$$f_3(z) := Exp_0(z) + e_1 Exp_1(z) + e_2 Exp_2(z) + e_3 Exp_3(z)$$

where

$$Exp_0(z) = e^{x_0}(\cos(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) - \sin(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}}))$$

$$Exp_1(z) = e^{x_0}\frac{\sqrt{3}}{3}(\sin(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) + \cos(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}}))$$

$$Exp_2(z) = e^{x_0}\frac{\sqrt{3}}{3}(\cos(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) + \sin(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}}))$$

$$Exp_3(z) = e^{x_0}\frac{\sqrt{3}}{3}(\sin(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) + \cos(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}})).$$

However, not all of these higher dimensional analogues of the exponential function turn out to be of the same *type*.

For the first and the second example we can determine the value of $M(r, f)$ exactly.

We obtain that $M(r, f_1) = \||\mathbf{m}| + i\mathbf{m}\|e^{|\mathbf{m}|r}$, thus $\tau(f_1) = |\mathbf{m}|$. For f_2 we obtain that $\|f_2(z)\| = e^{x_1+\cdots+x_n}$ which implies that $M(r, f_2)e^{nr}$ and therefore $\tau(f_2) = n$.

For the third example we are able to establish a useful lower and upper bound estimate for the maximum modulus. By a direct calculation we obtain that $\frac{\sqrt{3}}{3}e^r \leq M(r, f_3) \leq e^r$ so that $\tau(f_3) = 1$. When $|\mathbf{m}| = 1$, f_1 and f_3 thus share the same growth order and growth type.

After having discussed some concrete examples, let us now turn to the more general framework. As a consequence of Cauchy's integral formula we can establish, cf. [13]:

Theorem 2. *Let f be a left entire function in $\mathbb{R}^{n+1}$. By f_i we denote the function $f_i := \frac{\partial}{\partial x_i} f$ and $M_i(r) := \max_{\|z\|=r}\left\{\|f_i(z)\|\right\}$ where $r > 0$ and $i \in \{0, \ldots, n\}$. Then*

$$\rho(f) = \rho'(f) \text{ and } \lambda(f) = \lambda'(f),$$

where

$$\rho'(f) := \limsup_{r\to\infty} \frac{\log^+(\log^+(M'(r)))}{\log(r)}$$

$$\lambda'(f) := \liminf_{r\to\infty} \frac{\log^+(\log^+(M'(r)))}{\log(r)},$$

for $M'(r) := \max\limits_{i=0,1,\ldots,n}\{M_i(r)\}$.

Proof. We consider an arbitrary rectifiable curve from the origin to z. Then

$$f(z) = f(0) + \int_0^1 \sum_{i=0}^n x_i\ f_i(tz)dt.$$

For $z \in \mathbb{R}^{n+1}$ with $\|z\| = r$ we get

$$\begin{aligned} \|f(z)\| &\leq \|f(0)\| + r\sum_{i=0}^{n} M_i(r) \\ &\leq \|f(0)\| + r(n+1)M'(r). \end{aligned}$$

Therefore

$$M(r) \leq \|f(0)\| + r(n+1)M'(r).$$

Applying some properties of $\log^+$ we obtain that

$$\log^+(M(r,f)) \leq \log^+(\|f(0)\|) + \log^+(r(n+1)) + \log^+(M'(r)) + \log(2).$$

This in turn leads to

$$\rho(f) \leq \rho'(f) \text{ and } \lambda(f) \leq \lambda'(f).$$

To show the inequality in the other direction, we apply on f_i Cauchy's integral formula:

$$f_i(z) = \frac{1}{A_{n+1}} \int_{\|\zeta - z\| = R-r} q_{\tau(i)}(\zeta - z)d\sigma(\zeta)f(\zeta). \tag{13}$$

Applying the estimate (6) to (13) we hence obtain

$$\|f_i(z)\| \leq \frac{1}{A_{n+1}} \int_{\|\zeta - z\| = R-r} \frac{n}{(R-r)^{n+1}} M(R)dS$$

from which we then infer that

$$M_i(r) \leq \frac{n}{(R-r)} M(R,f).$$

In particular, for $M'(r) := \max_{i=0,1,\ldots,n}\{M_i(r)\}$ we have

$$M'(r) \leq \frac{n}{(R-r)} M(R,f). \tag{14}$$

Replacing $R = 2r$ into (14) yields:

$$M'(r) \leq \frac{n}{r} M(2r,f).$$

Thus,

$$\log^+ M'(r) \leq \log^+ M(2r,f) + \log^+\left(\frac{n}{r}\right).$$

For what follows we may assume without loss of generality that $r > n$. Hence,

$$\log^+ M'(r) \leq \log^+ M(2r,f).$$

Furthermore,

$$\begin{aligned} \frac{\log^+(\log^+ M'(r))}{\log(r)} &\leq \frac{\log^+(\log^+ M(2r,f))}{\log(r)} \\ &= \frac{\log^+(\log^+ M(2r,f))}{\log(2r)} \frac{\log^+(2r)}{\log(r)} \end{aligned}$$

Thus, we have

$$\frac{\log^+(\log^+ M'(r))}{\log(r)} \leq \frac{\log^+(\log^+ M(2r,f))}{\log(2r)}\left(\frac{\log 2}{\log(r)}+1\right)$$

from which we can infer directly that

$$\rho(f) \geq \rho'(f) \text{ and } \lambda(f) \geq \lambda'(f). \quad \Box$$

After having computed the growth order $\rho(f)$ resp. $\lambda(f)$ of a monogenic function f, we know the maximal value of the growth order of all partial derivatives.

Notice that Cauchy's integral formula was an important ingredient in the proof of this statement. To establish these types of results in a more general framework it is thus indeed important to work in classes of functions that are in the kernel of a differential operator that satisfy a Cauchy type integral formula. See also our paper [7] where we treated more general solutions to higher dimensional iterated Cauchy-Riemann and Dirac operators. The class of monogenic functions actually provides us with the canonical and easiest example of a function class which satisfies a Cauchy type integral formula.

4 Generalizations of some theorems from Valiron to Clifford analysis

In this section we present some generalizations of some classical theorems from G. Valiron to the Clifford analysis setting.

To this end we first define the *maximum term* and *central index* which are associated to the Taylor series of a monogenic function.
Let us consider a left entire function

$$f(z) = \sum_{|\mathbf{l}|=0}^{+\infty} V_{\mathbf{l}}(z)a_{\mathbf{l}}.$$

Let $r > 0$ be a fixed real. If f is transcendental, i.e. infinitely many $a_{\mathbf{l}} \neq 0$, then

$$\lim_{|\mathbf{l}|\to\infty} \|a_{\mathbf{l}}\| r^{|\mathbf{l}|} = 0.$$

The following expression thus is well-defined:

Definition 4. *(Maximum term)*
Let $f : \mathbb{R}^{n+1} \to Cl_{0n}$ be a left entire function with the Taylor series representation $f(z) \sum_{|\mathbf{l}|=0}^{+\infty} V_{\mathbf{l}}(z)a_{\mathbf{l}}$. Furthermore, let $r > 0$ be a fixed real. Then the associated maximum term is defined by

$$\mu(r) := \mu(r,f) := \max_{|\mathbf{l}|\geq 0}\{\|a_{\mathbf{l}}\| r^{|\mathbf{l}|}\}. \tag{15}$$

We further introduce

Definition 5. *(Central indices)*
Let $f(z) = \sum_{|\mathbf{l}|=p}^{+\infty} V_{\mathbf{l}}(z) a_{\mathbf{l}}$ *be a left entire function. For* $r > 0$ *the index (or the indices)* $\mathbf{m}$ *with maximal length* $|\mathbf{m}|$ *with* $\mu(r)\|a_{\mathbf{m}}\|r^{|\mathbf{m}|}$ *is (are) called central index (indices) which shall be denoted by* $\nu(r) = \nu(r, f) = \mathbf{m}$. *By* $\nu(0)$ *we denote the indices* $\mathbf{l}$ *which satisfy* $|\mathbf{l}| = p$.

The following theorem proved in [13], providing us with a direct generalization of Valiron theorem, states a relation between the *maximum term*, *central index* and the *maximum modulus.*

Theorem 3. *If* $f : \mathbb{R}^{n+1} \to Cl_{0n}$ *is a left entire function, then for all* $0 < r < R$

$$M(r) \leq \mu(r)\Big[|\nu(R)|(1 + |\nu(R)|)^{n-1} + \Big(\frac{R}{R - r}\Big)^n\Big]. \tag{16}$$

Proof. The function f is assumed to be left entire. Thus, it can be represented by

$$f(z) = \sum_{|\mathbf{l}|=0}^{+\infty} V_{\mathbf{l}}(z) a_{\mathbf{l}}$$

where infinitely many $a_{\mathbf{l}} \neq 0$, since f is transcendental. From the maximum modulus theorem for monogenic functions we infer that for $0 < r < R$:

$$\begin{aligned} M(r) &\leq \sum_{|\mathbf{l}|=0}^{+\infty} \|a_{\mathbf{l}}\| r^{|\mathbf{l}|} = \sum_{|\mathbf{l}|=0}^{|\nu(R)|-1} \|a_{\mathbf{l}}\| r^{|\mathbf{l}|} + \sum_{|\mathbf{l}|=|\nu(R)|}^{+\infty} \|a_{\mathbf{l}}\| r^{|\mathbf{l}|} \\ &\leq \sum_{|\mathbf{l}|=0}^{|\nu(R)|-1} \mu(r) + \sum_{|\mathbf{l}|=|\nu(R)|}^{+\infty} \|a_{\mathbf{l}}\| r^{|\mathbf{l}|}. \end{aligned} \tag{17}$$

In view of

$$\begin{aligned} \sum_{|\mathbf{l}|=0}^{|\nu(R)|-1} 1 &= \sum_{|\mathbf{l}|=0} 1 + \sum_{|\mathbf{l}|=1} 1 + \cdots + \sum_{|\mathbf{l}|=|\nu(R)|-1} 1 \\ &= 1 + \frac{((n-1)+1)!}{(n-1)!1!} + \cdots + \frac{[(n-1) + (|\nu(R)-1)]!}{(n-1)!(|\nu(R)|-1)!} \\ &\leq |\nu(R)| \left[\frac{[(n-1) + |\nu(R)| - 1]!}{(n-1)!(|\nu(R)| - 1)!}\right] \end{aligned}$$

where we use that for all $n \geq 1$ the inequality

$$\frac{(n-1+k)!}{(n-1)!k!} \leq \frac{(n-1+(k+1))!}{(n-1)!(k+1)!}$$

holds, which itself can be verified by a straightforward induction over k. Further,

$$\begin{aligned}
& |\nu(R)|\left[\frac{[(n-1)+|\nu(R)|-1]!}{(n-1)!(|\nu(R)|-1)!}\right] \\
= & |\nu(R)|\left[\frac{(|\nu(R)|+n-2)(|\nu(R)|+n-3)\cdots(|\nu(R)|+1)|\nu(R)|}{(n-1)!}\right] \\
= & |\nu(R)|\left[\frac{|\nu(R)|+n-2}{n-1}\cdot\frac{|\nu(R)|+n-3}{n-2}\cdot\cdots\cdot\frac{|\nu(R)|+1}{2}\cdot\frac{|\nu(R)|}{1}\right] \\
\leq & |\nu(R)|\left[\underbrace{(1+\frac{|\nu(R)|}{n-1})}_{\leq 1+|\nu(R)|}\underbrace{(1+\frac{|\nu(R)|}{n-2})}_{\leq 1+|\nu(R)|}\cdot\cdots\cdot\underbrace{(1+\frac{|\nu(R)|}{1})}_{=1+|\nu(R)|}\right] \\
\leq & |\nu(R)|\left[(1+|\nu(R)|)^{n-1}\right].
\end{aligned}$$

Inserting these results into (17) leads to

$$\begin{aligned}
M(r) \leq & \mu(r)|\nu(R)|\left[(1+|\nu(R)|)^{n-1}\right]+\sum_{|\mathbf{l}|=|\nu(R)|}^{+\infty}\|a_{\mathbf{l}}\|r^{|\mathbf{l}|}\frac{\|a_{\nu(r)}\|r^{|\nu(r)|}R^{|\mathbf{l}+\nu(R)|}}{\|a_{\nu(r)}\|r^{|\nu(R)|}R^{|\mathbf{l}+\nu(R)|}} \\
= & \mu(r)|\nu(R)|\left[(1+|\nu(R)|)^{n-1}\right]+\mu(r)\sum_{|\mathbf{l}|=|\nu(R)|}^{+\infty}\frac{\|a_{\mathbf{l}}\|R^{|\mathbf{l}|}R^{|\nu(R)|}r^{|\mathbf{l}|}}{\|a_{\nu(R)}\|R^{|\nu(R)|}R^{|\mathbf{l}|}r^{|\nu(R)|}} \\
\leq & \mu(r)|\nu(R)|\left[(1+|\nu(R)|)^{n-1}\right]+\mu(r)\sum_{|\mathbf{l}|=|\nu(R)|}^{+\infty}\left(\frac{r}{R}\right)^{|\mathbf{l}|-|\nu(R)|} \\
= & \mu(r)\Big[|\nu(R)|\left[(1+|\nu(R)|)^{n-1}\right]+\left(\frac{R}{R-r}\right)^{n}\Big]. \quad \square
\end{aligned}$$

G. Valiron has also proved that an entire complex-analytic function f of finite order shows the asymptotic behavior $\log(M(r,f)) \approx \log(M'(r))$ where M' is the maximum modulus of the derivative. The classical proof is based on the fact that one has the relation $\mu(r) \leq M(r,f)$ for a complex-analytic function in one complex variable. In the framework of working with Clifford algebra valued monogenic Taylor series which are built with the Fueter polynomials, we have a more complicated upper bound estimate of the form

$$\mu(r) \leq \frac{n(n+1)\cdots(n+|\nu(r)|-1)}{\nu(r)!}M(r,f)$$

for a central index $\nu(r)$. This is a consequence of the higher dimensional Cauchy's inequality. Notice that this is a sharp upper bound, cf. [5]. Adapting the classical methods based on Cauchy's inequality to the higher dimensional case provides us only with a weaker result in the Clifford analysis setting. In [13] we proved that

Proposition 1. *For a left entire function $f : \mathbb{R}^{n+1} \to Cl_{0n}$ of order ρ and inferior order λ set*

$$\rho_1 := \limsup_{r\to\infty} \frac{\log^+ \log^+ \mu(r)}{\log(r)}, \qquad \rho_2 := \limsup_{r\to\infty} \frac{\log^+ |\nu(r)|}{\log(r)}, \tag{18}$$

and

$$\lambda_1 := \liminf_{r\to\infty} \frac{\log^+ \log^+ \mu(r)}{\log(r)}, \qquad \lambda_2 : \liminf_{r\to\infty} \frac{\log^+ |\nu(r)|}{\log(r)}. \tag{19}$$

Then $\rho \leq \rho_1 = \rho_2$ and $\lambda \leq \lambda_1 = \lambda_2$.

Remark: In the two-dimensional complex case where we have $\mu(r) \leq M(r)$ these methods allow one to establish the stronger result $\rho = \rho_1 = \rho_2$ and $\lambda = \lambda_1 = \lambda_2$, as shown for instance in [20, Theorem 4.5].

With this proposition we may establish the following theorem. It provides us with a weaker analogy of Valiron's asymptotic result on the growth of the logarithm of the derivative of a given analytic function:

Theorem 4. *If $f : \mathbb{R}^{n+1} \to Cl_{0n}$ is left entire with $\rho_2(f) < \infty$, then*

$$\limsup_{r\to\infty} \frac{\log M_i(r)}{\log \mu(r)} \leq 1 \tag{20}$$

where $M_i(r) := \max_{\|z\|=r} \left\{ \left\| \frac{\partial}{\partial x_i} f(z) \right\| \right\}$ for $i = 1, \ldots, n$.

5 The growth behavior and the Taylor coefficients of a monogenic function

In general it is difficult to determine the precise value of the maximum modulus. In many cases it is even complicated to just get a useful estimate on $M(r, f)$ from below. In this section we present an explicit relation between the Taylor coefficients and the growth order and the *type* of an entire monogenic function. This allows us to compute the growth type directly on the knowledge of the Taylor coefficients without any knowledge on the maximum modulus of the function. Notice that Taylor series actually are a natural method to construct and to define entire monogenic functions. Recall, that the product of two monogenic functions is not monogenic anymore in general. Hence it is natural to construct entire monogenic functions in an additive way, for instance by its Taylor series.

The following two theorems provide us with higher dimensional generalizations in the Clifford analysis setting of two theorems proved by Lindelöf and Pringsheim for complex analytic functions. In [8] we established

Theorem 5. *For an entire monogenic function $f : \mathbb{R}^{n+1} \to Cl_{0n}$, with Taylor series representation $f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} V_{\mathbf{m}}(z) a_{\mathbf{m}}$ let*

$$\Pi = \limsup_{|\mathbf{m}| \to +\infty} \frac{|\mathbf{m}| \log |\mathbf{m}|}{-\log \|\frac{1}{c(n,\mathbf{m})} a_{\mathbf{m}}\|}. \tag{21}$$

Then we have $\rho(f) = \Pi$.

Remark: In the cases where $\|a_{\mathbf{m}}\| = 0$ one puts $\limsup_{|\mathbf{m}| \to +\infty} \frac{|\mathbf{m}| \log |\mathbf{m}|}{-\log \|\frac{1}{c(n,\mathbf{m})} a_{\mathbf{m}}\|} := 0$.

The following theorem, proved in [9], also relates the growth type with the Taylor coefficients of an entire monogenic function:

Theorem 6. *Let $f : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire monogenic function with Taylor series expansion $f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} V_{\mathbf{m}}(z) a_{\mathbf{m}}$ with order ρ $(0 < \rho < +\infty)$ and*

$$\Theta = \limsup_{|\mathbf{m}| \to +\infty} |\mathbf{m}| \Big(\|a_{\mathbf{m}}\| \Big)^{\frac{\rho}{|\mathbf{m}|}}. \tag{22}$$

Then $\Theta = \tau e \rho$, where τ is the type of f.

In turn, Theorem 6 allows us to construct immediately examples of entire monogenic Taylor series of non-zero finite growth order ρ of any arbitrary real growth type $0 \le \tau \le +\infty$. Recalling from [9], we start with

Proposition 2. *Suppose that $f : \mathbb{R}^{n+1} \to Cl_{0n}$ is an entire monogenic function. If $\rho(f) = 0$, then $\tau(f) = \infty$ or f is a constant.*

Proof. If $\rho(f) = 0$, then

$$\tau(f) = \limsup_{r \to +\infty} \log^+ M(r, f).$$

If $\tau(f) \neq \infty$, then

$$\limsup_{r \to +\infty} M(r, f) = e^{\tau},$$

which implies that

$$\|f(z)\| \le e^{\tau} \quad \text{for all } \ z \in \mathbb{R}^{n+1}.$$

As a consequence of Theorem 1, f must be a constant. □

Example: Consider $P(z)$ to be an arbitrary left monogenic polynomial, i.e. there exist Clifford numbers $a_{\mathbf{m}} \in Cl_{0n}$ and $N \in \mathbb{N}_0$ such that $P(z) \sum_{|\mathbf{m}|=0}^{N} V_{\mathbf{m}}(z) a_{\mathbf{m}}$. From [13, Theorem 3.1] we know that

$$\|P(z)\| \le \left(\frac{(n-1+N)!}{(n-1)!N!} + \varepsilon \right) \|a_{\mathbf{N}}\| r^N, \tag{23}$$

where $\mathbf{N}$ is the index of length N for which $\|a_{\mathbf{N}}\| \geq \|a_{\mathbf{m}}\|$ for all $|\mathbf{m}| = N$, with an arbitrarily small $\varepsilon > 0$ for r sufficiently large. Thus, it follows with $C(N) : \left(\frac{(n-1+N)!}{(n-1)!N!} + \varepsilon\right)\|a_{\mathbf{N}}\|$ that

$$\lim_{r\to\infty} \frac{\log^+(\log^+(M(r,P))}{\log(r)} \leq \lim_{r\to\infty} \frac{\log^+(\log^+(C(N)r^N)}{\log(r)} 0.$$

Thus, all monogenic polynomials satisfy $\rho(P) = \lambda(P) = 0$, like in the complex case. In view of Proposition 2 the growth type τ equals $+\infty$.

More generally, we could establish, cf. [9]:

Proposition 3. *Let $0 < \delta < +\infty$ and $0 < \lambda < +\infty$ be arbitrary real numbers. Then*

$$f(z) = \sum_{|\mathbf{m}|=1}^{+\infty} c(n,\mathbf{m})|\mathbf{m}|^{-\frac{|\mathbf{m}|}{\delta}} V_{\mathbf{m}}\Big(\big(\frac{\lambda e\delta}{n^\delta}\big)^{\frac{1}{\delta}} z\Big) \tag{24}$$

is an entire monogenic function of growth order $\rho = \delta$ and growth type $\tau = \lambda$.

Proof. By applying Hadamard's formula, one may directly conclude that the convergence radius of (24) is $+\infty$. Since the Fueter polynomials $V_{\mathbf{m}}$ are homogeneous polynomials of total degree $|\mathbf{m}|$, f can directly be rewritten in the form $f(z) = \sum_{|\mathbf{m}|=1}^{+\infty} V_{\mathbf{m}}(z)a_{\mathbf{m}}$ with $a_{\mathbf{m}} = c(n,\mathbf{m})|\mathbf{m}|^{-\frac{|\mathbf{m}|}{\delta}}\left(\frac{\lambda e\delta}{n^\delta}\right)^{\frac{|\mathbf{m}|}{\delta}}$.
According to Theorem 5, the growth order of f therefore equals

$$\begin{aligned}
\rho(f) &= \limsup_{|\mathbf{m}|\to+\infty} \frac{|\mathbf{m}|\log|\mathbf{m}|}{-\log\|\frac{1}{c(n,\mathbf{m})}a_{\mathbf{m}}\|} = \limsup_{|\mathbf{m}|\to+\infty} \frac{|\mathbf{m}|\log|\mathbf{m}|}{-\log\left||\mathbf{m}|^{-\frac{|\mathbf{m}|}{\delta}}\left(\frac{\lambda e\delta}{n^\delta}\right)^{\frac{|\mathbf{m}|}{\delta}}\right|}\\
&= \limsup_{|\mathbf{m}|\to+\infty} \delta\frac{\log|\mathbf{m}|}{\log|\mathbf{m}| - \log(\frac{\lambda e\delta}{n^\delta})} = \delta.
\end{aligned}$$

By Theorem 6 we indeed furthermore obtain that

$$\begin{aligned}
\tau(f) &= \frac{1}{e\delta}\limsup_{|\mathbf{m}|\to+\infty} c(n,\mathbf{m})^{\frac{\delta}{|\mathbf{m}|}}\frac{\lambda e\delta}{n^\delta}\\
&= \frac{\lambda}{n^\delta}\limsup_{M\to+\infty}\max_{|\mathbf{m}|=M} c(n,\mathbf{m})^{\frac{\delta}{M}}\\
&= \frac{\lambda}{n^\delta}\limsup_{M\to+\infty}\left[\frac{(n+M-1)!}{(n-1)!(\frac{M}{n})!^n}\right]^{\frac{\delta}{M}}\\
&= \frac{\lambda}{n^\delta}\limsup_{M\to+\infty}\Big[\frac{1}{\Big[(n-1)!\Big]^{\frac{\delta}{M}}}\frac{\left((n+M-1)^{n+M-1+\frac{1}{2}}e^{-(n+M-1)}\right)^{\frac{\delta}{M}}}{\left(\left(\frac{M}{n}\right)^{\frac{M}{n}+\frac{1}{2}}e^{-\frac{M}{n}}\right)^{\frac{n\delta}{M}}}\\
&= \frac{\lambda}{n^\delta}\limsup_{M\to+\infty}\left(\frac{n+M-1}{\frac{M}{n}}\right)^{\delta}\lambda.
\end{aligned}$$

By analogous calculations one can further show that

Proposition 4. *Let $0 < \rho < \infty$. The functions*

$$\begin{aligned} g(z) &= \sum_{|\mathbf{m}|=2}^{+\infty} c(n,\mathbf{m}) \Big[\frac{\log|\mathbf{m}|}{|\mathbf{m}|}\Big]^{\frac{|\mathbf{m}|}{\rho}} V_{\mathbf{m}}(z) \\ h(z) &= \sum_{|\mathbf{m}|=2}^{+\infty} c(n,\mathbf{m}) \Big[\frac{1}{|\mathbf{m}|\log|\mathbf{m}|}\Big]^{\frac{|\mathbf{m}|}{\rho}} V_{\mathbf{m}}(z) \end{aligned}$$

are entire monogenic functions of growth order ρ and $\tau(g) = +\infty$ and $\tau(h) = 0$.

6 Applications to partial differential equations

In this section we show how the notions of the maximum term and the central indices can be applied to obtain some information on the structure of the solutions of certain class of higher dimensional partial differential equations.

To proceed in this direction it turns out to be useful to first establish a relation between the asymptotic behavior of the maximum term of a monogenic function and that of their iterated radial derivatives. In [13] we established:

Theorem 7. *Let $f : \mathbb{R}^{n+1} \to Cl_{0n}$ be a left entire function. Then for all $k \in \mathbb{N}$ holds asymptotically*

$$\frac{1}{|\nu(r)|^k}\Big\|[E^k]f(z) - f(z)\Big\| \le C\mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon}, \quad r \notin F \tag{25}$$

where $E := \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$ is the Euler operator on $\mathbb{R}^{n+1}$, C is a real positive constant and F is a set of finite logarithmical measure.

As a direct consequence of Theorem 7 one obtains

Proposition 5. *Let $0 < \delta < \frac{1}{2}$. We assume that $\|z\| = r$ and that r be sufficiently large. Suppose further that the relation*

$$\|f(z)\| > \mu(r)|\nu(r)|^{-\frac{1}{2}+\delta} \tag{26}$$

is satisfied for all those z that belong to a neighborhood $\mathcal{V}_{z_0}$ of a point z_0 in which we have $\|z_0\| = r$ and $\|f(z_0)\| = \max_{\|z\|=r}\{\|f(z)\|\}$. Then for all $k \in \mathbb{N}$ holds asymptotically

$$\frac{1}{|\nu(r)|^k}[E^k]f(z) - f(z) = o(1)f(z), \quad r \notin F, \tag{27}$$

where F is again a set of finite logarithmical measure.

Remark: This statement provides us with an analogy in the context of Clifford analysis of the classical result [20, Theorem 21.3] which states that entire complex-analytic functions that satisfy

$$\|f(z)\| > M(r,f)[\nu(r)]^{-\frac{1}{4}+\delta}$$

have the asymptotic behavior

$$f^{(m)}(z) = \Big(\frac{\nu(r)}{z}\Big)^m (1+o(1))f(z).$$

In the Clifford analysis setting one thus obtains a similar asymptotic result when substituting the complex operator $z\frac{d}{dz}$ by the higher dimensional Euler operator E.

With these tools in hand we can study the structure of the solutions to some classes of partial differential equations. As a concrete example we present the following special case of an unpublished result from [12]:

Theorem 8. *Let f be an entire monogenic function of finite order $\rho_2 < \infty$. Let $\|z\| = r$ and assume that r is sufficiently large. Suppose further that the relation*

$$\|f(z)\| > \mu(r)|\nu(r)|^{-\frac{1}{2}+\delta}, \quad r \notin F$$

is satisfied for all those z that belong to a neighborhood $\mathcal{V}_{z_0}$ of a point z_0 in which we have $\|z_0\| = r$ and $\|f(z_0)\| = \max_{\|z\|=r}\{\|f(z)\|\}$. Let

$$M_j[f] = a_j \prod_{i=0}^{k} (E^i(f))^{n_i},$$

where a_j are polynomials of degree j, and $M_j[f]$ has degree $\gamma_{M_j} = \sum_{i=0}^{k} n_i$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k} in_i$. Let

$$Q[f] = \sum_{j=0}^{s} M_j[f]$$

be of degree γ_Q and weight Γ_Q. If $\gamma_Q \gamma_{M_0}$ then the differential equation $Q[f] = 0$ has no transcendental entire solutions.

Proof. If $Q[f] = 0$, then $M_0[f] = -\sum_{j=1}^{s} M_j[f]$. >From the definition of M_j it follows that

$$a_0 \left[\prod_{i=0}^{k} (E^i(f))^{n_i}\right]_{M_0} = -\sum_{j=1}^{s} \left[a_j \prod_{i=0}^{k} (E^i(f))^{n_i}\right]_{M_j}.$$

Applying Proposition 5, we obtain that

$$\|a_0\| |\nu(r)|^{\Gamma_{M_0}} \|f(z)\|^{\gamma_{M_0}} \le \sum_{j=1}^{s} \left(\|a_j\| |\nu(r)|^{\Gamma_{M_j}} \|f(z)\|^{\gamma_{M_j}} \right).$$

Since a_0 is a non zero constant and a_j are polynomials of degree j, taking the maximum over the norm, and applying (23) leads to

$$\begin{aligned} |\nu(r)|^{\Gamma_{M_0}} M(r,f)^{\gamma_{M_0}} &\leq |\nu(r)|^{\Gamma_Q} M(r,f)^{\gamma_Q - 1} \sum_{j=1}^{s} \max_{\|z\| r} \frac{\|a_j\|}{\|a_0\|} \\ &\leq |\nu(r)|^{\Gamma_Q} M(r,f)^{\gamma_Q - 1} r^{\alpha}. \end{aligned} \tag{28}$$

Therefore, in view of $\gamma_Q = \gamma_{M_0}$ one has

$$M(r,f) \leq |\nu(r)|^{\Gamma_Q - \Gamma_{M_0}} r^{\alpha}. \tag{29}$$

For $\Gamma_Q - \Gamma_{M_0} < 0$ it follows

$$\liminf_{r\to\infty} \frac{M(r,f)}{r^{\alpha}} \leq \liminf_{r\to\infty} |\nu(r)|^{\Gamma_Q - \Gamma_{M_0}} = 0$$

which implies that f is a polynomial, as a consequence of Theorem 1.

Let us now consider the case where $\Gamma_Q - \Gamma_{M_0} > 0$. Since $\rho_2 < \infty$, we have that $|\nu(r)| < r^{\rho_2+\epsilon}$ for $\epsilon > 0$. Therefore, there exists a $\beta > (\Gamma_Q - \Gamma_{M_0})(\rho_2 + \epsilon)$ such that

$$\begin{aligned} \liminf_{r\to\infty} \frac{M(r,f)}{r^{\beta+\alpha}} &\leq \liminf_{r\to\infty} \frac{|\nu(r)|^{\Gamma_Q - \Gamma_{M_0}}}{r^{\beta}} \\ &\leq \liminf_{r\to\infty} r^{(\Gamma_Q - \Gamma_{M_0})(\rho_2+\epsilon)-\beta} \quad = \quad 0 \end{aligned}$$

which implies that f is a polynomial, as a consequence of Theorem 1. □

Concluding remarks: One can apply these techniques to obtain analogous statements for much more general classes of partial differential equations. In our recent paper [10] we were able to prove analogous statements for far more general systems involving polynomial expressions of the Cauchy-Riemann operator with arbitrary complex coefficients and radial differential operators. This paper gives a first impression in how one can extend the classical techniques from Wiman and Valiron from complex analysis to study much larger classes of higher dimensional elliptic operators and summarizes the first basic results.

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Dirac Type Gauge Theories – Motivations and Perspectives

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ABSTRACT

We summarize the geometrical description of a specific class of gauge theories, called "of Dirac type", in terms of Dirac type first order differential operators on twisted Clifford bundles. We show how these differential operators may be geometrically considered as being the images of sections of a specific principal fibering naturally associated with twisted Clifford bundles. Based on the notion of real Hermitian vector bundles, we discuss the most general real Dirac type operator on "particle-anti-particle" modules over an arbitrary (orientable) semi-Riemannian manifold of even dimension. This setting may be appropriate for a common geometrical description of both the Dirac and the Majorana equation.

RESUMEN

Nosotros resumimos la descripción geométrica de una clase específica de teoría gauge, llamada "de tipo Dirac", en términos del tipo de Dirac de operadores diferenciales de primer orden sobre fibrados de Clifford twisted. Mostramos como esos operadores pueden ser geométricamente considerados como siendo imágenes de secciones de una fibra principal específica naturalmente asociada con el fibrado de Clifford twisted. Basado en la noción de fibrado vectorial Hermitiano real, discutimos el más general operador de tipo Dirac real sobre módulos "partícula-anti-partícula" sobre una variedad semi-Riemanniana (orientable) arbitraria de dimensión par. Este contexto puede ser apropiado para una descripción geométrica común para las ecuaciones de Dirac y de Majorana.

Key words and phrases: *Dirac Type Differential Operators, Real Clifford Modules, General Relativity, Gauge Theories, Majorana equation*

Math. Subj. Class.: *53C05, 53C07, 70S05, 70S15, 83C05.*

1 Synopsis

In a nutshell, Dirac type gauge theories are based on the following "universal (Dirac) action functional":

$$\mathcal{I}_{\mathrm{D}} : \int_M [\langle\psi, \not{D}\psi\rangle_{\mathcal{E}} + \mathrm{tr}_\gamma curv(\not{D})] \; dvol_{\mathrm{M}} \,. \tag{1}$$

Here, $\not{D} \in \mathcal{D}(\mathcal{E})$ denotes the most general Dirac type first order differential operator, acting on the $\mathcal{C}^\infty(M)-$module of smooth sections $\psi \in \mathfrak{S}ec(M, \mathcal{E})$ on a Hermitian Clifford module bundle $\pi_{\mathcal{E}} : \mathcal{E} \longrightarrow M$ over a smooth orientable semi-Riemannian manifold (M, g_{M}) of even dimension $n = 2k \geq 2$. The notation $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ denotes a chosen Hermitian form on $\mathcal{E}$ and

$$curv(\not{D}) \in \Omega^2(M, \mathrm{End}(\mathcal{E})) \tag{2}$$

is the *curvature* of $\not{D}$.

A detailed general discussion of the geometrical background of the functional (1) and how it is related to the well-known general Lichnerowicz decomposition (c.f. [5] and [3])

$$\not{D}^2 = -\triangle_{\mathrm{B}} + V_{\mathrm{D}} \tag{3}$$

can be found in [22] and [23]. Note that in contrast to what has been stated in the latter Reference, however, the "Dirac potential" V_{D} actually reads:

$$V_{\mathrm{D}} \;=\; \gamma(curv(\not{D})) - \mathrm{ev}_g(\omega_{\mathrm{D}}^2) + \mathrm{ev}_g(\partial_{\mathrm{D}}\omega_{\mathrm{D}}) \tag{4}$$

where the "Dirac form" $\omega_{\mathrm{D}} \in \Omega^1(M, \mathrm{End}(\mathcal{E}))$ is a certain one-form canonically associated with $\not{D}$ and "ev_g" means evaluation with respect to the metric g_{M} (c.f. below). In our discussion presented here we shall omit the quadratic term. The third term in (4) only contributes as a boundary term. As a consequence, the functional (1) differs from the "universal Dirac action" that is discussed in Reference [22] by the quadratic term (and the boundary term) in (4). To calculate the curvature of a general Dirac type operator is generally more involved than for a connection. In contrast, explicit formulae are available for the corresponding Dirac potential. Hence, (4) may be used to also obtain the curvature of a general $\not{D}$ via (neglecting the boundary term)

$$\gamma(curv(\not{D})) = V_{\mathrm{D}} + \mathrm{ev}_g(\omega_{\mathrm{D}}^2) \,. \tag{5}$$

We follow this line of reasoning to calculate the curvature of the most general Dirac type operator on "particle-anti-particle" modules in section four.

In the following we focus on a summary of some of the basic features of the universal Dirac action (1). A detailed discussion of its motivation is presented, whereby we put emphasize to its "universality" and its relation to various partial differential equations well-known from physics and geometry. In the particular case of twisted Clifford bundles we discuss how Dirac type first order differential operators can be geometrically considered as being images of sections of a principal fibering that is naturally associated with the geometry of Clifford modules. Finally, we discuss a specific class of real Hermitian Clifford modules. For these we present an explicit formula for the universal Dirac action.

Our work is organized as follows: The second section is addressed to present some detailed discussion of the motivation for the Dirac action and how it is related to various well-known "field equations", like Yang-Mills and Einstein's equation of gravity. In the third section we discuss how the Dirac action may be regarded as a functional of the metric and (endomorphism valued) super fields. The fourth section is addressed to the Dirac action on the geometrical background of (a specific class of) real Hermitian Clifford modules which may allow to incorporate the geometrical description of the Majorana equation in terms of the universal Dirac action. Finally, in the fives section we present some outlook.

2 Motivation: Four equations and one action

To get started, let us call in mind that the two most profound equations in classical physics are provided by the Maxwell equations of electrodynamics:

$$dF = 0\,, \tag{6}$$

$$d{*}F = j_{\rm elm} \tag{7}$$

and the Einstein equation of gravity:

$$Ric(g_{\rm M}) - \frac{1}{2}scal(g_{\rm M}) = \lambda_{\rm grav}\tau\,. \tag{8}$$

Here, the "electromagnetic Field strength" is geometrically represented by a (closed) two-form $F \in \Omega^2(M)$ on a given four-dimensional, orientable semi-Riemannian manifold $(M, g_{\rm M})$ with index of $g_{\rm M}$ equals ± 2. Accordingly, the two-form $*F$ denotes the Hodge-dual of F with respect to $g_{\rm M}$ and a chosen orientation of M. Moreover, the differential operator d is the usual exterior derivative. We stress, that in the case of the Maxwell equations (6–7) the metric structure $g_{\rm M}$ on the manifold M is supposed to be fixed.

In contrast, in the case of Einstein's theory of gravity the gravitational field is supposed to be fully described in terms of the metric structure $g_{\rm M}$ on M. However, only those metric structures are physically admissible which satisfy Einstein's field equation (8). The tensor $Ric \in \mathfrak{Sec}(M, {\rm End}TM)$ denotes the "Ricci-tensor" and $scal \in \mathcal{C}^\infty(M)$ its trace the so-called "Ricci-scalar". For $TM \twoheadrightarrow M$

being the tangent bundle of M, the bundle $\mathrm{End}TM \twoheadrightarrow M$ is the associated bundle of endomorphisms on TM (over the identity on M).

The right-hand side of the Maxwell equation (7) denotes the "electrically charged matter current". Similarly, the right-hand side of the Einstein equation (8) is the "energy-momentum current" associated with any form of energy and matter. The numerical constant λ_{grav} is called the "gravitation coupling constant". It carries a physical dimension in contrast to the electromagnetic coupling constant which is purely numerical (approximately 1/137).

In classical physics these source terms for non-trivial electromagnetic field strength and gravitational fields are supposed to be given objects, reflecting the physical situation at hand. Of course, as a special case one may consider the physical situation where (part of) space-time (M, g_{M}) is filled only with an electromagnetic field that is generated by electrically charged matter whose support is outside the considered region of space-time. Then, within this region j_{elm} vanishes identically and τ is a unique function of F such that the pair (g_{M}, F) is physically admissible provided it is a solution of the coupled Einstein-Maxwell equations.

In a so-called "semi-classical" description of matter (i.e. within a certain approximation of a full quantum description), the classical Maxwell and Einstein equations are supplemented by the (gauge covariant) Dirac equation

$$(i\partial\!\!\!/_{\mathrm{A}} - m)\psi = 0\,. \tag{9}$$

In particular, the electromagnetic current

$$\begin{aligned} *j_{\mathrm{elm}} &= \sum_{\mu,\nu=0}^{3} q_{\mathrm{elm}}\langle\psi, \gamma(e^\mu)\psi\rangle_{\mathcal{E}}\, g_{\mathrm{M}}(e_\mu, e_\nu)e^\nu \\ &\equiv q_{\mathrm{elm}}\langle\psi, \gamma(e^\mu)\psi\rangle_{\mathcal{E}}\, g_{\mathrm{M}}(e_\mu, e_\nu)e^\nu \end{aligned} \tag{10}$$

becomes a (quadratic) function of $\psi \in \mathfrak{Sec}(M, \mathcal{E})$ such that the triple $(g_{\mathrm{M}}, F, \psi)$ is physically admissible if and only if it fulfills the now coupled Einstein-Maxwell-Dirac equation (6–9). Here, when appropriate units are used the parameters $(m, q_{\mathrm{elm}}) \in \mathbb{R}_+ \times \mathbb{Z}$ are physically interpreted as "mass" and "electric charge" of the matter described in terms of the "matter field" ψ (e.g. an electron).

In (10), $e_0, \ldots, e_3 \in \mathfrak{Sec}(M, TM)$ is a local orthonormal frame with respect to g_{M} and $e^0, \ldots, e^3 \in \mathfrak{Sec}(M, T^*M)$ its (local) dual frame.

Note that henceforth we will make use of Einstein's summation convention whenever local expressions come up like in (10).

Geometrically, the matter field ψ is usually considered as a section of a twisted spinor bundle

$$\pi_{\mathcal{E}} : \mathcal{E} = S \otimes W \longrightarrow M \tag{11}$$

over a semi-Riemannian spin-manifold $(M, g_{\mathrm{M}}, \Lambda_{\mathrm{Spin}})$ with Λ_{Spin} being a chosen spin structure on M. The Hermitian vector bundle $W = P \times_\rho V \twoheadrightarrow M$ is an associated vector bundle of a given principal G-bundle $G \hookrightarrow P \twoheadrightarrow M$ that represents the so-called "internal gauge degrees of freedom" of matter. Here, $\rho : G \rightarrow GL(V)$ is a unitary representation of G on a Hermitian vector space V which servers a the typical fiber of the twisting bundle $W \twoheadrightarrow M$.

In the case of electromagnetism the (semi-simple real) Lie group G equals the unitary group $U(1)$ with Lie-algebra Lie$G = i\mathbb{R}$. Accordingly, the gauge covariant Dirac operator

$$i\partial\!\!\!/_{\rm A} \quad \equiv \quad i\gamma \circ (\partial_{\rm A}) i\gamma \circ (\partial^{\rm S} \otimes {\rm Id}_{\rm W} + {\rm Id}_{\rm S} \otimes \partial^{\rm W}) \tag{12}$$

is given by

- The covariant derivative with respect to the spin connection on the spinor bundle $S \twoheadrightarrow M$:

$$\partial^{\rm S}_\mu \overset{\rm loc.}{=} \partial_\mu + \frac{1}{4}[\gamma^a, \gamma^b]\, \omega^{\rm LC}_{\mu ab} \tag{13}$$

  with $\omega^{\rm LC} \in \Omega^1(M, so(p,q))$ being the Levi-Civita form that is determined by $g_{\rm M}$;

- The gauge covariant derivative on the Hermitian vector bundle $W \twoheadrightarrow M$:

$$\partial^{\rm W}_\mu \overset{\rm loc.}{=} \partial_\mu + \rho'(A_\mu) \tag{14}$$

  with $\rho'(A) \equiv \rho' \circ A \in \Omega^1(M, {\rm End}(W))$ being a (local) $U(1)-$gauge potential represented on W. The Lie-algebra representation ρ' : Lie$G \rightarrow {\rm End}(V)$ is the derived representation with respect to the underlying group representation ρ.

Hence, locally the gauge covariant Dirac operator reads:

$$i\partial\!\!\!/_{\rm A} \overset{\rm loc.}{=} i\gamma^\mu \left(\partial_\mu + \frac{1}{4}\omega^{\rm LC}_{\mu ab}\, [\gamma^a, \gamma^b] \otimes {\rm Id}_{\rm W} + {\rm Id}_{\rm S} \otimes \rho'(A_\mu) \right). \tag{15}$$

Here and in the expression (10) the symbol "γ" denotes a Clifford mapping, i.e.

$$\begin{array}{rcl} \gamma : T^*M & \longrightarrow & {\rm End}(\mathcal{E}) \\ \omega & \mapsto & \gamma(\omega) \end{array} \tag{16}$$

satisfying $\gamma(\omega)^2 g_{\rm M}(\omega,\omega)\, {\rm Id}_{\mathcal{E}}$. In the following we will suppress the identity mappings ${\rm Id}_{\mathcal{E}}, {\rm Id}_{\rm S}, {\rm Id}_{\rm W}$ on $\mathcal{E}, S, W$ whenever this will not cause any confusion. Also, we will not make a distinction in our notation with respect to the metric on the tangent and the co-tangent bundle $T^*M \twoheadrightarrow M$. Finally, $\gamma^a \equiv \gamma(e^a)$ are the usual "gamma matrices" and $[\cdot, \cdot]$ is the ordinary commutator.

Note that the Lie algebra $so(p,q)$ is isomorphic to the Lie algebra $spin(p,q)$ of the spin group and $\frac{1}{2}[\gamma^a, \gamma^b]$ $(0 \le a \ne b \le 3)$ are the corresponding generators of the "spinor representation" of $so(p,q)$. Indeed, the Clifford action γ on a twisted spinor bundle (11) is simply given by the regular left action of the Clifford bundle

$$Cl_{\rm M} \twoheadrightarrow M \tag{17}$$

on $S \subset Cl_{\rm M}$. Here, the Clifford bundle is the algebra bundle over $(M, g_{\rm M})$ whose typical fiber is given by the Clifford algebra $Cl_{p,q}$ that is generated by the Minkowski space $\mathbb{R}^{p,q} \equiv (\mathbb{R}^4, \eta)$, where

$$\eta(\mathfrak{e}_\mu, \mathfrak{e}_\nu) : \begin{cases} \pm 1 & \text{for} \quad \mu = 0\,, \\ \mp 1 & \text{for} \quad 1 \leq \mu = \nu \leq 3\,, \\ 0 & \text{for} \quad 0 \leq \mu \neq \nu \leq 3 \end{cases} \tag{18}$$

and $\mathfrak{e}_0, \ldots, \mathfrak{e}_3 \in \mathbb{R}^4$ the standard basis.

Therefore, on a twisted spinor bundle the Clifford action γ is uniquely determined by the metric $g_{\rm M}$. If γ denotes the Clifford action with respect to the metric $g_{\rm M}$, then $i\gamma$ is the Clifford action with respect to the metric $-g_{\rm M}$. Likewise, if the Clifford action is "even", i.e. $\gamma(\omega)^t = -\gamma(\omega)$ for all $\omega \in T^*M$, then the Clifford action given by $i\gamma$ is "odd": $i\gamma(\omega)^t = i\gamma(\omega)$ and vice verse.

Apparently, the geometrical background of the equations (6– 9) seems quite different like the equations themselves. To summarize: The geometrical background of the Maxwell equations is given by the Grassmann bundle $\Lambda_{\rm M} \twoheadrightarrow M$ over a given orientable semi-Riemannian manifold $(M, g_{\rm M})$. In contrast, the geometrical background of the Einstein equation is provided by so-called $SO(p, q)-$reductions of the frame bundle $F_{\rm M} \twoheadrightarrow M$. That is, the geometrical background is given by the fiber bundle

$$\mathcal{E}_{\rm EH} := F_{\rm M} \times_{\rm Gl(4)} GL(4)/SO(p, q) \longrightarrow M\,. \tag{19}$$

In fact, a section of this associated bundle with typical fiber $GL(4)/SO(p, q)$ is in one-to-one correspondence to a semi-Riemannian structure $g_{\rm M}$ of signature (p, q). We thus do not make a distinction between a section of the Einstein-Hilbert bundle (19) and the corresponding metric. We denote both by the same symbol. Finally, the geometrical background of the Dirac equation is provided by a Clifford module $(\mathcal{E}, \gamma)$ over a given orientable semi-Riemannian (spin-)manifold $(M, g_{\rm M})$.

Apparently, Maxwell's equations, Einstein's equation and Dirac's equation are rather different equations. Nonetheless, one may ask for a common geometrical root of these three equations which play such a fundamental role in physics and mathematics.

An appropriate hint is provided by the gauge covariant Dirac equation (9) and the geometrical interpretation of the Maxwell equations (6–7) in terms of gauge theory. For this one may regard the electromagnetic field strength F as a section of the complexified Grassmann bundle

$$\Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C} \twoheadrightarrow M \tag{20}$$

which corresponds to the curvature of a $U(1)-$connection on $U(1) \hookrightarrow P \twoheadrightarrow M$. We emphasize that with respect to this geometrical interpretation of the electromagnetic field strength the Maxwell equation (6) becomes an identity (the "Bianci-identity"). If A denotes a local gauge potential of the curvature, then (7) is read as the $U(1)-$Yang-Mills equation:

$$d_{\rm A} {*} F_{\rm A} = j\,. \tag{21}$$

Here, respectively, $F_{\rm A} = iF \in \Omega^2(M, i\mathbb{R})$ is, again, the curvature of a $U(1)-$connection and $\mathrm{d}_{\rm A}$ its gauge covariant exterior derivative, locally given by the first order differential operator $d + A$ and $A \in \Omega^1(M, i\mathbb{R})$. Clearly, the adjoint action is trivial, for $U(1)$ is abelian. Hence,

$$\begin{aligned} \mathrm{d}_{\rm A}{*}F_{\rm A} &\overset{\text{loc.}}{=} d{*}F_{\rm A} + [A, {*}F_{\rm A}] \\ &= d{*}F_{\rm A}\,. \end{aligned} \tag{22}$$

If $j \equiv ij_{\rm elm}$, then the Yang-Mills equation (21) is equivalent to (7).

Note that $F_{\rm A} = \mathrm{d}_{\rm A}^2 \overset{\text{loc.}}{=} dA$. That is, the square of the first order differential operator $\mathrm{d}_{\rm A}$ is a zero order differential operator taking values in $\Omega^2(M, i\mathbb{R})$.

Let $\delta_{\rm A}$ be the formal adjoint of $\mathrm{d}_{\rm A}$ with respect to the pairing $\int_M \alpha^c \wedge *\beta$ for all compactly supported $\alpha, \beta \in \Omega(M, \mathbb{C}) \equiv \mathfrak{S}ec(M, \Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C})$. By α^c we denote the complex conjugate of α with respect to the canonical real structure on $\Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C}$ that is given by $\alpha^c := e^\mu \otimes \overline{\lambda}_\mu$ for $\alpha \overset{\text{loc.}}{=} e^\mu \otimes \lambda_\mu \in \Omega^1(M, \mathbb{C})$. It follows that $\delta_{\rm A} = \pm * \mathrm{d}_{\rm A} *$, where the sign depends on the signature of $g_{\rm M}$ and the degree of the form the operator acts on. Then, the equation (21) may be rewritten as

$$\delta_{\rm A} F_{\rm A} = \pm j \tag{23}$$

and thus the original Maxwell equations become equivalent to

$$(\,\mathrm{d}_{\rm A} + \delta_{\rm A})F_{\rm A} = \pm j\,. \tag{24}$$

The point to be stressed here is, that the (complexified) Grassmann bundle serves as a canonical Clifford module with respect to the Clifford action

$$\begin{aligned} \gamma : T^*M &\longrightarrow \mathrm{End}(\Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C}) \\ \omega &\mapsto \left\{ \begin{array}{ccc} \Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C} & \longrightarrow & \Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C} \\ \alpha & \mapsto & -i(\mathrm{ext}_\omega(\alpha) - \mathrm{int}_\omega(\alpha))\,. \end{array} \right. \end{aligned} \tag{25}$$

Here, $\mathrm{ext}_\omega(\alpha) := \omega \wedge \alpha$ and $\mathrm{int}_\omega(\alpha) := \alpha(\omega^\sharp, \cdot)$ with $\omega^\sharp \in TM$ is the metric dual with respect to $g_{\rm M} : \beta(\omega^\sharp) := g_{\rm M}(\omega, \beta)$ for all $\beta \in T^*M$.

As consequence,

$$\mathrm{d}_{\rm A} + \delta_{\rm A} i\partial\!\!\!/_{\rm A}\,, \tag{26}$$

with

$$i\partial\!\!\!/_{\rm A} \overset{\text{loc.}}{=} i\gamma^\mu(\partial_\mu + \frac{1}{4}\omega^{\rm LC}_{\mu ab}\,[\gamma^a, \gamma^b] + A_\mu)\,. \tag{27}$$

The Maxwell equations for purely imaginary $F_{\rm A} \in \mathfrak{S}ec(M, \Lambda_{\rm M} \otimes_{\mathbb{R}} \mathbb{C})$ can thus be brought into a form analogous to the Dirac equation for $\psi \in \mathfrak{S}ec(M, \mathcal{E})$:

$$i\partial\!\!\!/_{\rm A} F_{\rm A} = \pm j\,. \tag{28}$$

The similarity between the Dirac equation (9) and (28) can be made even more close by noting that $\Lambda_{\rm M}\otimes_{\mathbb{R}}\mathbb{C}\simeq Cl_{\rm M}\otimes_{\mathbb{R}}\mathbb{C}\simeq {\rm End}(S_{\mathbb{C}})$, where $S_{\mathbb{C}}\equiv S\otimes_{\mathbb{R}}\mathbb{C}$. Hence, $\Lambda_{\rm M}\otimes\mathbb{C}\simeq S_{\mathbb{C}}\otimes S^*_{\mathbb{C}}$ and the (complexified) spinor bundle $S_{\mathbb{C}}\twoheadrightarrow M$ (with respect to a chosen spin structure) can be regarded as a sub-vector bundle of the Grassmann bundle:

$$S_{\mathbb{C}}\hookrightarrow\Lambda_{\rm M}\otimes\mathbb{C}\twoheadrightarrow M\,. \tag{29}$$

Geometrically, the complexified Grassmann bundle $\Lambda_{\rm M}\otimes_{\mathbb{R}}\mathbb{C}$ is but a special *twisted Grassmann bundle*

$$\Lambda_{\rm M}\otimes L\longrightarrow M \tag{30}$$

with $L:=M\times\mathbb{C}\to M$ being the trivial complex line bundle over M. The Hermitian Clifford module

$$\pi_{\Lambda,\,E}:\mathcal{E}_{\Lambda,\,E}\equiv\Lambda_{\rm M}\otimes E\longrightarrow M\,, \tag{31}$$

with $E:=L\oplus W\twoheadrightarrow M$ being the Whitney sum of the two Hermitian vector bundles $L\twoheadrightarrow M$ and $E\twoheadrightarrow M$, actually provides a common geometrical setting for the Dirac and Maxwell equations.

Obviously, all of this can be immediately generalized to arbitrary twisted Grassmann bundles parameterized by arbitrary Hermitian vector bundles $E\twoheadrightarrow M$ over $(M,g_{\rm M})$. In this case, one only has to replace the covariant derivative $\partial^{\rm S}$ that corresponds to a chosen spin structure on M by the covariant derivative ∂^{Λ} of the Levi-Civita connection on $\Lambda_{\rm M}\twoheadrightarrow M$ with respect to the induced metric $g_{\Lambda{\rm M}}$. Then, (12) is replaced by the *twisted Gauss-Bonnet* like operator

$$\begin{aligned} i\partial\!\!\!/_{\rm A} &= i\gamma\circ(\partial^{\Lambda}\otimes{\rm Id}_{\rm E}+{\rm Id}_{\Lambda}\otimes\partial^{\rm E})\\ &= {\rm d}_{\rm A}+\delta_{\rm A}\,. \end{aligned} \tag{32}$$

Note that locally there is no distinction between the first order operators (32) and (12). This is, because the bundle of homomorphisms ${\rm End}(\mathcal{E})\twoheadrightarrow M$ of any Clifford module $(\mathcal{E},\gamma)$ over an even dimensional semi-Riemannian manifold $(M,g_{\rm M})$ globally decomposes as

$${\rm End}(\mathcal{E})\simeq(Cl_{\rm M}\otimes_{\mathbb{R}}\mathbb{C})\otimes{\rm End}_{\rm Cl}(\mathcal{E})\,. \tag{33}$$

Here, ${\rm End}_{\rm Cl}(\mathcal{E})$ denote the sub-algebra of $\gamma-$invariant endomorphisms on $\mathcal{E}\twoheadrightarrow M$. The fundamental isomorphism (33) can be inferred from the two Wedderburn Theorems about "invariant linear mappings" (c.f. [9] and [3]). In fact, the use of this global decomposition forces the dimension of M to be even such that $Cl_{p,q}$ is simple.

Finally, nothing basically chances even in the case the Maxwell equations are replaced by general Yang-Mills equations, i.e. the abelian structure group $U(1)$ is replaced by an arbitrary (semi-simple, real and compact) Lie group G. In this case, one only has to replace the (trivial) line bundle $L\twoheadrightarrow M$ by the adjoint bundle $\mathfrak{a}d(P):=P\times_{\rm ad}{\rm Lie}G\twoheadrightarrow M$.

Like in the particular case of a spinor bundle, the Clifford action (25) is uniquely determined by the metric $g_{\rm M}$. Actually, both Clifford actions coincide on their common domain. Hence, with respect to twisted Grassmann bundles we may consider γ and $g_{\rm M}$ as being basically the same. Accordingly, the Einstein equation is seen to provide a physical constraint on the possible Clifford module structures to which (31) refers to. Note the change of the meaning of the metric when the Maxwell equations are written similar to the Dirac equation.

Once we have established a common geometrical setup for the Dirac and the Maxwell (resp. Yang-Mills) equations we proceed to show that this common setup also provides an appropriate geometrical background for the Einstein equation. For this we remark that on the one hand side the Maxwell and Dirac equations make use of a given metric $g_{\rm M}$ (i.e. a fixed Clifford module structure of the underlying twisted Grassmann bundle). On the other hand, the Einstein equation are considered as differential equations determining $g_{\rm M}$. In particular, the (Levi-Civita) connection which fixes the first order operator ∂^Λ is fully determined by $g_{\rm M}$. In contrast to the Maxwell equations (resp. Yang-Mills equations), the gravitational gauge potential has thus an underlying geometrical structure given by the metric $g_{\rm M}$ from which the connection is derived. For this matter the Einstein-Hilbert functional, from which the Einstein equation can be derived as Euler-Lagrange equation, is *linear* in the curvature. In contrast, the Yang-Mills functional, which yields the (homogeneous) Maxwell equation (7) in the case $G = U(1)$, is *quadratic* in the curvature:

$$\mathcal{I}_{\rm EH}(g_{\rm M}) \;:=\; \lambda^{-1}_{\rm grav}\int_M scal(g_{\rm M})\, dvol_{\rm M}\,, \tag{34}$$

$$\mathcal{I}_{\rm YM}(g_{\rm M};A) \;:=\; \lambda^{-1}_{\rm elm}\int_M g_{\Lambda{\rm M}}(F_{\rm A},F_{\rm A})\, dvol_{\rm M}\,. \tag{35}$$

Note that the variation of $\mathcal{I}_{\rm YM}(g_{\rm M};A)$ with respect to the metric $g_{\rm M}$ gives rise to the energy-momentum current $\tau \in \mathfrak{Sec}(M, {\rm End}(TM))$ as a function of $F_{\rm A}$ as mentioned before.

To get a relation between these seemingly different functionals (34) and (35) we notice that in contrast to the square ${\rm d}^2_{\rm A} = F_{\rm A}$ of the first order operator ${\rm d}_{\rm A}$, the square of the associated Dirac operator $i\partial\!\!\!/_{\rm A}$ has the well-known Lichnerowicz decomposition into a specific second order differential operator and a specific zero order operator:

$$\begin{aligned} i\partial\!\!\!/^2_{\rm A} &= ({\rm d}_{\rm A} + \delta_{\rm A})^2 \overset{G=U(1)}{=} d\circ\delta + \delta\circ d \\ &= -\triangle_{\rm B} + V_{\rm D} \end{aligned} \tag{36}$$

with $\triangle_{\rm B} \overset{\rm loc.}{=} -g^{\mu\nu}(\nabla_\mu\circ\nabla_\nu - \Gamma^\sigma_{\mu\nu}\nabla_\sigma)$ being the *Bochner-Laplacian* and $\nabla_\mu \equiv \partial_\mu + \frac{1}{4}\omega^{\rm LC}_{\mu ab}\,[\gamma^a,\gamma^b] + A_\mu$. The local functions $\Gamma^\sigma_{\mu\nu}$ are the usual Christoffel symbols with respect to $g_{\rm M}$ and a chosen coordinate frame.

The *"Dirac potential"* has the specific form:

$$V_{\rm D} = \frac{1}{4}scal(g_{\rm M}) + \gamma(F_{\rm A}) \in \mathfrak{Sec}(M, {\rm End}(\mathcal{E}_{\Lambda,\,E})) \tag{37}$$

where locally $\gamma(F_{\rm A})\frac{1}{2}\,\gamma^\mu\gamma^\nu\otimes F_{\mu\nu}$. The tensor product refers to the fundamental decomposition (33). As a consequence, the zero order operator $\gamma(F_{\rm A})$ is always a trace-less operator: ${\rm tr}_{\mathcal{E}}(\gamma(F_{\rm A})) \equiv 0$, where the trace is taken in ${\rm End}(\mathcal{E}_{\Lambda,\,E})$.

Therefore, the Einstein-Hilbert functional may be expressed in terms of $i\not{\partial}_{\rm A}$ as

$$\mathcal{I}_{\rm EH}(g_{\rm M}) = \lambda'^{-1}_{\rm grav} \int_M {\rm tr}_{\mathcal{E}} V_{\rm D}\, dvol_{\rm M}\,. \tag{38}$$

Note that the Dirac potential is uniquely determined by $i\not{\partial}_{\rm A}$.

We notice that the trace-less zero order operator $\gamma(F_{\rm A}) \in \mathfrak{S}ec(M, {\rm End}(\mathcal{E}_{\Lambda,\,E}))$ is indeed the "square root" of the Yang-Mills Lagrangian, for

$$\mathcal{I}_{\rm YM}(g_{\rm M}; A) = \lambda'^{-1}_{\rm elm} \int_M {\rm tr}_{\mathcal{E}}(\gamma(F_{\rm A})^2)\, dvol_{\rm M}\,. \tag{39}$$

However, also in this form the Yang-Mills action is still quadratic in the curvature in contrast to the Einstein-Hilbert action.

The question then is whether the Yang-Mills Lagrangian can be "linearized" such that it becomes most similar to the Einstein-Hilbert Lagrangian without violating the second order character of the Yang-Mills equations. Note that both the Einstein and the Yang-Mills equations are of second order. Hence, one cannot simply try to square the Einstein-Hilbert Lagrangian to bring it into a form similar to the Yang-Mills Lagrangian without obtaining higher order differential equations for $g_{\rm M}$.

In order to appropriately linearize the integrand of (39) one may take into account that $i\not{\partial}_{\rm A}$ also determines a specific curvature on the bundle (31), denoted by $curv(i\not{\partial}_{\rm A}) \in \Omega^2(M, {\rm End}(\mathcal{E}_{\Lambda,\,E}))$ (c.f. [22] and [23]). Explicitly it reads

$$\begin{aligned} curv(i\not{\partial}_{\rm A}) &= R_g \otimes {\rm Id}_{\rm E} + {\rm Id}_\Lambda \otimes F_{\rm A} \\ &\equiv R_g + F_{\rm A}\,. \end{aligned} \tag{40}$$

Again, this is due to the fundamental decomposition (33). Here, $R_g \in \Omega^2(M, {\rm End}(\mathcal{E}_{\Lambda,\,E}))$ is the Riemannian curvature with respect to the induced metric $g_{\Lambda{\rm M}}$ on the Grassmann bundle over M. Locally, it reads: $R_g \stackrel{\rm loc.}{=} \frac{1}{2} e^\mu \wedge e^\nu \otimes \frac{1}{4}[\gamma^a, \gamma^b] R_{ab\mu\nu}$, where the local functions $R_{ab\mu\nu} \equiv g_{\rm M}(e_a, e_c)\, e^c({\rm Riem}(e_\mu, e_\nu) e_b)$ and ${\rm Riem} \in \Omega^2(M, {\rm End}(TM))$ denotes the (semi-)Riemann curvature tensor with respect to $g_{\rm M}$. Note again that Einstein's summation convention is employed in local formulae.

Therefore, the Yang-Mills curvature (especially the electromagnetic field strength) may be expressed in terms of $i\not{\partial}_{\rm A}$. In fact, it is but the *"relative curvature"* of the curvature of $i\not{\partial}_{\rm A}$ (again, neglecting the identity mappings):

$$F_{\rm A} = curv(i\not{\partial}_{\rm A}) - R_g \in \Omega^2(M, {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))\,. \tag{41}$$

This geometrical interpretation of $F_{\rm A}$ in terms of $i\not{\partial}_{\rm A}$ yields a different interpretation of the Yang-Mills (resp. of the Maxwell) equations. The latter are considered to yield a constraints for $i\not{\partial}_{\rm A}$. Of course, this simply means constraints for $\partial^{\rm E}$ and thus does not yield anything new in comparison with the usual description of Yang-Mills type gauge theories in terms of G principal bundles. However, the strength of the presented geometrical viewpoint of the Yang-Mills equations in terms of $i\not\partial_{\rm A}$ has a powerful potential for a straightforward generalization. This is, because the geometrical point of view can be immediately generalized to arbitrary Dirac type first order differential operators. In other word, there are much more general Dirac type operators on (31) than those given by $i\not\partial_{\rm A}$. In fact, the latter are only very specific Dirac type operators. They are fully characterized by the decomposition (37) and the fact that $F_{\rm A} \in \Omega^2(M, {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))$ is $\gamma-$invariant. This may provide a sufficient motivation to consider the form (38) of the Einstein-Hilbert function as more profound than the form (34). In fact, the former should be considered as a functional of a specific class of Dirac type operators on (31) and hence as a specific restriction of a much more general functional (c.f. our discussion in the next section).

As discussed in ([23]), the trace of the Dirac potential (37) can be recast into the geometrical form (neglecting boundary terms):

$$\mathrm{tr}_{\mathcal{E}} V_{\rm D} = \mathrm{tr}_\gamma curv(i\not\partial_{\rm A})\,. \tag{42}$$

Therefore,

$$\begin{aligned}\mathcal{I}_{\rm EH}(g_{\rm M}) \equiv \mathcal{I}_{\rm EH}(i\not\partial_{\rm A}) &= \lambda'^{-1}_{\rm grav}\int_M \mathrm{tr}_\gamma(curv(i\not\partial_{\rm A}))\, dvol_{\rm M}\\ &\equiv \lambda'^{-1}_{\rm grav}\int_M \mathrm{tr}_\gamma(curv(\,\mathrm{d}_{\rm A} + \delta_{\rm A}))\, dvol_{\rm M}\end{aligned} \tag{43}$$

where $\mathrm{tr}_\gamma(curv(i\not\partial_{\rm A})) \equiv \mathrm{tr}_{\mathcal{E}}[\gamma(curv(i\not\partial_{\rm A}))] \in \mathcal{C}^\infty(M)$.

The form (43) of the Einstein-Hilbert action makes it most explicit how the metric $g_{\rm M}$ can be replaced by (a specific class of) Dirac type operators and hence how the Einstein-Hilbert functional determines a Clifford action γ on a twisted Grassmann bundle (31). Note that, despite of its appearance, (43) is actually independent of the connection on the twisting part $E \twoheadrightarrow M$ of (31). In other words, it is independent of the gauge potential A that (locally) determines the first order operator $\partial^{\rm E}$. The functional (43) thus yields a constraint only on how the vector bundle (31) can be actually regarded as a specific Clifford module. It thus determines γ as stated before. In fact, since $i\not\partial_{\rm A}$ is fully characterized by (37), it is straightforward to prove that these Dirac type operators provide the biggest class of Dirac type first order differential operators on a twisted Grassmann bundle such that the universal Dirac action (1) is proportional to the Einstein-Hilbert action and thus only depends on $g_{\rm M}$. Note that there is only a canonical choice for $\partial^{\rm E}$ if the Hermitian vector bundle $E \twoheadrightarrow M$ equals the trivial bundle $M \times V \twoheadrightarrow M$. Only in this case, there is a natural choice for $i\not\partial_{\rm A}$ given the Gauss-Bonnet like operator $d+\delta$. In the general case, however the latter operator is not gauge covariant. For this matter one has to chose some $\partial^{\rm E}$ to obtain an appropriate gauge covariant generalization $\mathrm{d}_{\rm A} + \delta_{\rm A}$ of $d+\delta$. Again, the functional (43) is independent of this arbitrary choice.

We are still left with the question whether it is possible to find a Dirac type operator $i\not D_{\rm A}$, say, on a certain twisted Grassmann bundle such that the Yang-Mills functional can be expressed in terms of the universal Dirac action (1).

The answer to this question turns out to be affirmative, actually, and has been discussed in some detail in [22] (c.f. also the appropriate references cited therein, in particular [2] in the case of a closed compact Riemannian manifold). In general, the Yang-Mills action may be written as

$$\begin{aligned}\mathcal{I}_{\rm YM}(g_{\rm M};A) &= \lambda'_{\rm YM}\left(\mathcal{I}_{\rm D}(i\not{D}_{\rm A}) - \mathcal{I}_{\rm D}(i\not{\partial}_{\rm A})\right)\\ &= \lambda'_{\rm YM}\int_M {\rm tr}_\gamma(curv(i\not{D}_{\rm A}) - curv(i\not{\partial}_{\rm A}))\,dvol_{\rm M}\,. \end{aligned}\tag{44}$$

where the corresponding Dirac type operator reads

$$i\not{D}_{\rm A} = i\not{\partial}_{\rm A} + \mathcal{I}\otimes\gamma(F_{\rm A})\,. \tag{45}$$

Here, $\mathcal{I} \equiv {\rm off}-{\rm diag}(-1,1)$ is an additional complex structure on the doubled twisted Grassmann bundle

$$2\mathcal{E}_{\Lambda,\,E} \equiv \mathcal{E}_{\Lambda,\,E}\oplus\mathcal{E}_{\Lambda,\,E}\Lambda_{\rm M}\otimes_{\mathbb{R}}(E\oplus E) \longrightarrow M\,. \tag{46}$$

The thus defined class of first order differential operators (45) are called Dirac operators of *"Pauli-type"*. The reason for this chosen terminology is that first order differential operators of the form

$$i\not{\partial}_{\rm A} + i\gamma(F_{\rm A}) \tag{47}$$

have been introduced in physics in order to describe the so-called "magnetic moment" of the proton long before it has been realized that the proton is a composite of more fundamental elementary particles (the "quarks"). In this context, the additional term $\gamma(F_{\rm A}) = \frac{i}{2}\gamma^\mu\gamma^\nu\otimes F_{\mu\nu}$, with $F\in\Omega^2(M)$ being the electromagnetic field strength, is named "Pauli-term" after the famous physicist W. Pauli. Note that the first order operator (47), however, is not a Dirac type first order operator. This is because the Pauli-term $\gamma(F_{\rm A})$ is an even operator in the sense that it commutes with the canonical $\mathbb{Z}_2-$grading provided by the Riemannian volume form: $\gamma_{\rm M} = i\gamma(dvol_{\rm M})$ (called "γ_5" in the physics literature). Indeed, a first order differential operator is said to be of Dirac type provided it is odd with respect to a given $\mathbb{Z}_2-$grading of the underlying Clifford module and the principal symbol of its square is given by the underlying metric. Only the latter feature is shared by the first order operator (47). In contrast, the first order operator (45) is both odd and its square is a "generalized Laplacian". It is thus of Dirac type.

Note that Dirac's original first order operator (or its gauge covariant generalization)

$$i\not{\partial}_{\rm A} - m \tag{48}$$

is also not of Dirac type for exactly the same reason as (47) is not of Dirac type. We shall come back to this in our third section where we discuss a specific class of Hermitian Clifford modules and the most general Dirac type operators thereof.

Concerning Dirac type operators of the form (45) the "square root" of the Yang-Mills Lagrangian becomes most obvious. It is not simply given by the traceless zero order operator $\gamma(F_{\rm A})$ itself but, instead, by Dirac operators of Pauli type. Basically, this is because of the additional grading one obtains from the doubling of (31). This additional grading also allows to express the "fermionic part" of the universal Dirac action (1) as

$$\langle \Psi,\, i\not{D}_{\rm A}\, \Psi\rangle_{2\mathcal{E}} = 2\langle \psi, i\not{\partial}_{\rm A}\, \psi\rangle_{\mathcal{E}}\,. \tag{49}$$

At least, this holds true for those sections $\Psi \in \mathfrak{Sec}(M, 2\mathcal{E}_{\Lambda,\,E})$ that are given by $\Psi = (\psi, \psi)$ and hence are determined by a section $\psi \in \mathfrak{Sec}(M, \mathcal{E}_{\Lambda,\,E})$. In other words, the "Pauli-term" does not contribute to the fermionic action but only to the bosonic action. This is a very desirable feature of this class of Pauli type Dirac operators, for it is well-known that the Pauli-term in the fermionic action yields a generalized Dirac equation that is not compatible with "quantization". We shall return to the Pauli type Dirac operators when considering a specific class of Hermitian Clifford modules and the corresponding most general Dirac type operators thereof. The underlying structure of this class of Clifford modules is basically motivated by our fourth equation: the *Majorana equation*:

$$i\not{\partial}\psi = m\psi^c \tag{50}$$

where ψ^c denotes the "charge conjugate" of ψ (c.f. below).

We call in mind that the Einstein-Hilbert functional may be expressed in terms of Dirac type operators of the form (32) with an arbitrary choice of $\partial^{\rm E}$. In contrast, when restricted to Pauli type Dirac operators $i\not{D}_{\rm A}$, the universal Dirac action (38) yields the combined Einstein-Hilbert-Yang-Mills functional. It reduces to the pure Yang-Mills functional only if $(M, g_{\rm M})$ is fixed to be (Ricci) flat. This is consistent with the Einstein equation, however, only with respect to the physical approximation that the gravitational field produced by the energy-momentum of the Yang-Mills field can be neglected to some extend. In general, however, (1) yields the coupled Einstein-Yang-Mills-Weyl equations as the corresponding Euler-Lagrange equations if (1) is restricted to Pauli type Dirac operators (45). In this case, the right-hand side of the Yang-Mills equation is similar to (10) and the energy-momentum current is a well-determined function of $(g_{\rm M}, F_{\rm A}, \psi)$.

We stress that the Pauli type Dirac operators are more general than those given by $i\not{\partial}_{\rm A}$. In particular, the relative curvature of $i\not{D}_{\rm A}$:

$$F_{\rm D} := curv(i\not{D}_{\rm A}) - R_g \tag{51}$$

is not $\gamma-$invariant, i.e.

$$F_{\rm D} \notin \Omega^2(M, {\rm End}_{\rm Cl}(2\mathcal{E}_{\Lambda,\,E}))\,. \tag{52}$$

For that matter, $\gamma(F_{\rm D}) \in \Omega^0(M, {\rm End}(\mathcal{E}))$ is not a traceless operator.

We close our motivation for the universal Dirac action (1) with the remark that the underlying invariance group of this functional is provided by the full diffeomorphism group $Diff(\mathcal{E}_{\Lambda,\,E})$ of (31).

This (infinite) gauge group decompose into the semi-direct product (c.f. [22]):

$$Diff(\mathcal{E}_{\Lambda,\,E})Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E}) \ltimes Diff(M) \tag{53}$$

with $Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E})$ consisting of all (bundle) automorphisms of (31) over the identity mapping on the base manifold M. Moreover, this group decomposes further into the direct sum of to sub-groups:

$$Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E})Aut_{\mathrm{EH}}(\mathcal{E}_{\Lambda,\,E}) \times Aut_{\mathrm{YM}}(\mathcal{E}_{\Lambda,\,E})\,. \tag{54}$$

Here, the "Yang-Mills" sub-group $Aut_{\mathrm{YM}}(\mathcal{E}_{\Lambda,\,E}) \subset Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E})$ consists of all automorphisms of (31) being isomorphic to the gauge transformations on the frame bundle that is induced by the vector bundle (31). It is thus a normal sub-group of $Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E})$ and

$$Aut_{\mathrm{EH}}(\mathcal{E}_{\Lambda,\,E}) : Aut_{\mathrm{M}}(\mathcal{E}_{\Lambda,\,E})/Aut_{\mathrm{YM}}(\mathcal{E}_{\Lambda,\,E})\,. \tag{55}$$

Locally, the "Einstein-Hilbert" sub-group $Aut_{\mathrm{EH}}(\mathcal{E}_{\Lambda,\,E})$ consists of all $SO(p,q)$ rotations of orthonormal frames of $TM \twoheadrightarrow M$ and $Aut_{\mathrm{YM}}(\mathcal{E}_{\Lambda,\,E})$ consists of all ordinary gauge transformations encountered in the usual geometrical description of Yang-Mills gauge theories in terms of principal G-bundles $G \hookrightarrow P \twoheadrightarrow M$. Thus, the universal Dirac action (1) contains all the physical symmetries which are usually imposed on physical field theories. To enlarge this symmetry to "super-symmetry" transformations, however, is still an open issue.

Having presented a detailed discussion of the motivation for the universal Dirac action (1) and how it is related to well-known field equations, we may proceed with a discussion in what sense the Dirac functional is more general than the ordinary Yang-Mills functional. In other words, in the following section we want to discuss the precise domain of dependence of the universal Dirac functional.

3 The Dirac action as a functional of "super fields"

In this section we discuss in more detail the domain of dependence of the (bosonic part of the) universal Dirac action:

$$\mathcal{I}_{\mathrm{D,bos}} : \int_M \mathrm{tr}_\gamma curv(\not{D})\, dvol_{\mathrm{M}}\,. \tag{56}$$

In the foregoing section we have shown how this functional covers both the Einstein-Hilbert and the Yang-Mills functional. In fact, the Dirac functional may be considered as a natural generalization of the Einstein-Hilbert functional of the form (43). Hence, when restricted to certain "sub-domains" on the "set of all Dirac type operators" (c.f. below), the universal functional (56) becomes a functional on (an appropriate subset of) $\mathfrak{S}ec(M,\mathcal{E}_{\mathrm{EH}})$ in the case of the Einstein-Hilbert action, or a functional on the affine manifold of all linear connections $\mathcal{A}(E)$ in the case of the Yang-Mills action. Accordingly, when restricted to Pauli type Dirac operators the (bosonic part of the) universal Dirac action becomes a functional on the smooth manifold $\mathfrak{Sec}(M, \mathcal{E}_{\rm EH}) \times \mathcal{A}(E)$. In general, (1) is considered as a functional on the smooth manifold

$$\mathcal{D}(\mathcal{E}_{\Lambda,\,E}) \times \mathfrak{Sec}(M, \mathcal{E}_{\Lambda,\,E}) \tag{57}$$

of all Dirac type first order operators on a twisted Grassmann bundle (31) and the module of smooth sections therein. Note that the "fermionic part" of (1)

$$\mathcal{I}_{\rm D,ferm} : \int_M \langle \psi, \not{D}\psi \rangle_{\mathcal{E}}\, dvol_{\rm M}\,, \tag{58}$$

is viewed simply as a quadratic form on $\mathfrak{Sec}(M, \mathcal{E}_{\Lambda,\,E})$. This quadratic form is fully determined by (symmetric) elements of $\mathcal{D}(\mathcal{E}_{\Lambda,\,E})$. For this reason, it suffices to focus on the affine manifold of all Dirac type operators on a given Grassmann bundle.

The aim of this section is to make this more precise and to show how the above two cases of the Einstein-Hilbert and the Yang-Mills functional are special cases of the more general functional (56). Basically, the reason is provided by the following (highly non-canonical) isomorphisms:

$$\mathcal{D}(\mathcal{E}_{\Lambda,\,E}) \simeq \Omega^0(M, {\rm End}(\mathcal{E}_{\Lambda,\,E})) \simeq \Omega^*(M, {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))\,, \tag{59}$$

which holds true for a fixed Clifford module structure on (31) (i.e. metric on M). The second isomorphism of (59) is implied by (33), where the abbreviation

$$\Omega^*(M, {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E})) \equiv \bigoplus_{p\in\mathbb{Z}} \Omega^p(M, {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E})) \tag{60}$$

has been used.

Consequently, any Dirac type operator on a Clifford module is determined by differential forms of all degrees. This is in strong contrast to connections on a vector bundle which are determined by one-forms, only.

To make this more precise, let again M be a smooth orientable manifold of even dimension $n = 2k \geq 2$. Also, let again $E \twoheadrightarrow M$ be a smooth Hermitian vector bundle over M and $\mathcal{E}_{\Lambda,\,E} \twoheadrightarrow M$ the corresponding twisted Grassmann bundle. We call the smooth fiber bundle

$$\mathcal{E}_{\rm D} : \mathcal{E}_{\rm EH} \times {\rm End}(\mathcal{E}_{\Lambda,\,E}) \longrightarrow M \tag{61}$$

the "Dirac bundle" associated to the twisted Grassmann bundle. So far (31) is considered as a vector bundle over M. There is no given Clifford structure at all, for M is not yet supposed to be endowed with a metric. We call in mind that a metric on M is in one-to-one correspondence with a section of the Dirac bundle given by

$$\begin{array}{rcl} \sigma_{\rm D} : M & \longrightarrow & \mathcal{E}_{\rm D} \\ x & \mapsto & (g_{\rm M}(x), 0) \end{array} \tag{62}$$

where $g_{\rm M} \in \mathfrak{Sec}(M, \mathcal{E}_{\rm EH})$.

We consider the following equivalence relation on the manifold of smooth sections $\mathfrak{Sec}(M,\mathcal{E}_{\rm D})$:

$$\sigma'_{\rm D} \equiv (g',\Phi') \sim \sigma_{\rm D} \equiv (g,\Phi) \in \mathfrak{Sec}(M,\mathcal{E}_{\rm D}) \tag{63}$$

iff $g' = g \in \mathfrak{Sec}(M,\mathcal{E}_{\rm EH})$ and there exists an $\alpha \in \Omega^1(M,{\rm End}(E)) \hookrightarrow \Omega^1(M,{\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,E}))$, such that $\Phi' = \Phi + \gamma(\alpha) \in \Omega^0(M,{\rm End}(\mathcal{E}_{\Lambda,E}))$. We put

$$\mathfrak{S}_{\rm D} := \mathfrak{Sec}(M,\mathcal{E}_{\rm D})/\sim\ . \tag{64}$$

There are various equivalent definitions available for Dirac type first order differential operators, depending on the appropriate focus (see, for example, [1], [5], [3], [4]). We present a different one which is most adopted to our purpose.

Definition 1. *Let $\mathcal{D}(\mathcal{E}_{\Lambda,E})$ be the set of all first order differential operators acting on $\mathfrak{Sec}(M,\mathcal{E}_{\Lambda,E})$, such that for $\not{D} \in \mathcal{D}(\mathcal{E}_{\Lambda,E})$ there exists a section $g_M \in \mathfrak{Sec}(M,\mathcal{E}_{EH})$ with*

$$\begin{array}{rcl} T^*M & \xrightarrow{\gamma} & {\rm End}(\mathcal{E}_{\Lambda,E}) \\ df & \mapsto & [\not{D}, f]\,. \end{array} \tag{65}$$

Here, the g_M-induced Clifford action γ is defined by (25).

A first order differential operator $\not{D} \in \mathcal{D}(\mathcal{E}_{\Lambda,E})$ is called a "Dirac type operator" provided it is odd with respect to the $\mathbb{Z}_2-$grading that is given by an involution $\tau_{\mathcal{E}} := \gamma_M \otimes \tau_E$.

The set of all Dirac type operators on $\mathcal{E}_{\Lambda,E}$ carries a natural action of the translational group

$$\mathfrak{T}_{\rm E} \equiv \Omega^1(M,{\rm End}^+(E)) \hookrightarrow \Omega^1(M,{\rm End}^+(\mathcal{E}_{\Lambda,E})) \tag{66}$$

that is given by

$$\begin{array}{rcl} \mathcal{D}(\mathcal{E}_{\Lambda,E}) \times \mathfrak{T}_{\rm E} & \xrightarrow{\mu} & \mathcal{D}(\mathcal{E}_{\Lambda,E}) \\ (\not{D},\alpha) & \mapsto & \not{D} + \gamma(\alpha)\,. \end{array} \tag{67}$$

Clearly, this action is free and the corresponding orbit space $\mathcal{D}(\mathcal{E}_{\Lambda,E})/\mu$ can be identified with $\mathfrak{S}_{\rm D}$. Furthermore, with respect to this identification

$$\begin{array}{rcl} \pi_{\rm D}:\ \mathcal{D}(\mathcal{E}_{\Lambda,E}) & \longrightarrow & \mathfrak{S}_{\rm D} \\ \not{D} & \mapsto & \mathfrak{s} \equiv [(g_{\rm M},\Phi)] \end{array} \tag{68}$$

is a principal fibering with structure group $\mathfrak{T}_{\rm E}$.

This principal fibering is actually trivial. However, every bijection

$$\begin{array}{rcl} \chi_{\rm A}:\ \mathcal{D}(\mathcal{E}_{\Lambda,E}) & \xrightarrow{\simeq} & \mathfrak{S}_{\rm D} \times \mathfrak{T}_{\rm E} \\ \not{D} & \mapsto & (\mathfrak{s},\alpha) \end{array} \tag{69}$$

strongly depends on the choice of $\partial^{\rm E}$. This holds true unless $E \twoheadrightarrow M$ is trivial.

Indeed, for every choice of $\partial^{\rm E}$ one may define

$$\mathcal{D}(\mathcal{E}_{\Lambda,\,E}) \ni \not{D} \equiv \chi_{\rm A}^{-1}(\mathfrak{s},\alpha) \;:=\; \not{\partial}_{\rm A} + \hat{\Phi}_{\rm A} + \gamma(\alpha) \\ \equiv\; \not{\partial}_{\rm A} + \Phi_{\rm A}\,. \tag{70}$$

Here, $\not{\partial}_{\rm A} \in \mathcal{D}(\mathcal{E}_{\Lambda,\,E})$ is given by (32) and

$$\hat{\Phi}_{\rm A} \in \mathfrak{S}ec(M, {\rm End}(\mathcal{E}_{\Lambda,\,E})) \simeq \mathfrak{S}ec(M, \Lambda_{\rm M} \otimes {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))\,, \tag{71}$$

which does not contain a one-form part. Note that $\hat{\Phi}_{\rm A}$ has to have odd total degree. It is uniquely defined as follows: every $\not{D} \in \mathcal{D}(\mathcal{E}_{\Lambda,\,E})$ can be decomposed (in a highly non-unique way) as $\not{D} = \not{\partial}_{\rm A} + \Phi_{\rm A}$ with $\Phi_{\rm A} \equiv \not{D} - \not{\partial}_{\rm A}$. Then, $\Phi_{\rm A} =: \hat{\Phi}_{\rm A} + \gamma(\alpha)$ and $\not{\partial}_{\rm A} + \hat{\Phi}_{\rm A} + \gamma(\alpha)$ is equivalent to $\not{\partial}_{\rm A} + \hat{\Phi}_{\rm A}$. It follows that $\pi_{\rm D}(\chi_{\rm A}^{-1}(\mathfrak{s},\alpha)) = {\rm pr}_1(\mathfrak{s},\alpha) = \mathfrak{s}$, if and only if $[\not{D}] \in \mathcal{D}(\mathcal{E})/\mu$ corresponds to $\mathfrak{s} \in \mathfrak{S}_{\rm D}$.

Proposition 1. *Let $\mathcal{E}_D \twoheadrightarrow M$ be the Dirac bundle associated with a twisted Grassmann bundle $\mathcal{E}_{\Lambda,\,E} \twoheadrightarrow M$. The functional (56) can be considered as a canonical functional on $\mathfrak{S}ec(M,\mathcal{E}_D)$:*

Proof: Since the value of the integral

$$\int_M {\rm tr}_\gamma curv(i\not{\partial}_{\rm A})\, dvol_{\rm M} \tag{72}$$

is independent of the choice of $\partial^{\rm E}$, it follows that (56) is constant along the fibers of (68). Hence, it descents to a well-defined functional on $\mathfrak{S}_{\rm D}$. For this matter $\mathcal{I}_{\rm D,bos}$ can be considered as a functional of $(g_{\rm M},\Phi)$ that constitutes a general section of the Dirac bundle (61). □

As a consequence

$$\mathcal{I}_{\rm D} = \mathcal{I}\,(g_{\rm M},\Phi,\psi) \tag{73}$$

with

$$\Phi \in \mathfrak{S}ec(M, \Lambda_{\rm M} \otimes {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E})) \bigoplus_{0\le l\le n} \mathfrak{S}ec(M, \Lambda^l T^*M \otimes {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))\,. \tag{74}$$

being a "super-field" of odd total degree that takes values in the $\gamma-$invariant endomorphisms on $\mathcal{E}_{\Lambda,\,E}\mathcal{E}^+_{\Lambda,\,E} \oplus \mathcal{E}^-_{\Lambda,\,E} \twoheadrightarrow M$.

Especially, for $\Phi = 0$, the action (1) reduces to the sum of the usual (massless) Dirac functional and the Einstein-Hilbert functional:

$$\int_M [\langle\psi,\, i\not{\partial}_{\rm A}\psi\rangle_{\mathcal{E}} + {\rm tr}_\gamma curv(i\not{\partial}_{\rm A})]\, dvol_{\rm M}\,. \tag{75}$$

In this case, the appropriate Euler-Lagrange equations are given by the combined Einstein-Weyl equations with the energy-momentum current τ being defined by $\psi \in \mathfrak{Sec}(M, \mathcal{E}_{\Lambda,\,E})$.

Likewise, to obtain the combined Einstein-Yang-Mills-Weyl equations one considers $\Phi = \mathcal{F}_{\rm A}$, with the requirement that $\mathcal{F}_{\rm A} \in \mathfrak{Sec}(M, \Lambda^2 T^*M \otimes {\rm End}_{\rm Cl}(2\mathcal{E}_{\Lambda,\,E}))$ being defined by the curvature $F_{\rm A} \in \Omega^2(M, {\rm End}(E))$ with respect to the chosen $\partial^{\rm E}$. In other words, one restrict the right-hand side of (1) to Pauli type Dirac operators on $2\mathcal{E}_{\Lambda,\,E} \twoheadrightarrow M$. In this case, the energy-momentum current τ is fully determined as a function of $(\psi, F_{\rm A})$, whereas the electromagnetic current is given by (10) (or an appropriate generalization thereof if $G \neq U(1)$). Of course, this reduces to pure Yang-Mills theory when one restricts to $\psi = 0$ and $g_{\rm M}$ (Ricci) flat.

Eventually, one can also recover the full action functional of the so-called Standard Model of elementary particles including the famous Higgs potential. For this one has to consider even more general super fields Φ for appropriate twisted Grassmann bundles. Interestingly, the structure of this bundle is determined by the topology of M and the choice of the "ground-state" of the still to find "Higgs boson" (c.f. [22] and the corresponding References cited therein).

The above mentioned examples may suffice to exhibit the generality of the Dirac action (1) and how it covers important classes of coupled partial differential equations as Euler-Lagrange equations of a natural generalization of the Einstein-Hilbert Lagrangian of Einstein's theory of gravity (43). Once one has the universal functional (1) one may ask for the corresponding form of the Euler-Lagrange equations. This, however, depends on the choice of the (twisting part of the) underlying twisted Grassmann bundle and is still a major challenge to exhibit in full generality. In the case, where the bundle is fixed and endowed with sufficient structure one may determine the most general Dirac type operator that is compatible with the endowed structure. Basically, this amounts to determine the most general super-field that is compatible with the given structure and then rewriting the universal Dirac action in terms of this super-field. As a specific example, this will be demonstrated in the next section in terms of a specific class of "real, Hermitian Clifford modules", called "particle-anti-particle modules".

Before, however, we want to briefly comment on "spin versus non-spin manifolds". So far, we concentrated on twisted Grassmann bundles and one may ask what does it give more than twisted spinor bundles. Also, one may ask how the latter fits with the geometrical frame of twisted Grassmann bundles.

First of all, if M is a spin-manifold (i.e. it has vanishing second Stiefel-Whitney classes) and $S \twoheadrightarrow M$ is the (complexified) spinor bundle with respect to a chosen spin-structure, then

$$\Lambda_{\rm M} \otimes \mathbb{C} \simeq S \otimes S^* \longrightarrow M \tag{76}$$

and hence

$$\mathcal{E}_{\Lambda,\,E} \simeq S \otimes W \longrightarrow M \tag{77}$$

where $W \equiv S^* \otimes E \twoheadrightarrow M$.

Moreover, if $\mathcal{S} \simeq Cl_{\mathrm{M}}\mathfrak{e} \equiv \{\mathfrak{a}\mathfrak{e} \in Cl_{\mathrm{M}} \mid \mathfrak{a} \in Cl_{\mathrm{M}}\}$ with $\mathfrak{e} \in \mathfrak{Sec}(M, Cl_{\mathrm{M}})$ being an appropriately global primitive idempotent and $S \simeq \mathcal{S} \otimes \mathbb{C}$, then

$$\begin{array}{rcl} S \otimes E & \hookrightarrow & \mathcal{E}_{\Lambda,\,E} \\ s \otimes y & \mapsto & s \otimes \mathfrak{e}^* \otimes y \end{array} \tag{78}$$

yields a canonical inclusion of the twisted spinor bundle

$$\mathcal{E}_{\mathrm{E}} := S \otimes E \longrightarrow M \tag{79}$$

into the twisted Grassmann bundle (31). Here, $\mathfrak{e}^*$ is the idempotent that yields the dual spinor module $\mathcal{S}^* := \mathfrak{e}\, Cl_{\mathrm{M}}$ and $S^* \simeq \mathcal{S}^* \otimes \mathbb{C}$. Note that we only consider complex modules.

In this way, the slightly more general situation of a twisted Grassmann bundles also covers the geometrical situation where M is supposed to be a spin-manifold. On the other hand, by a famous Theorem due to R. Geroch, a non-compact four-dimensional Lorentzian manifold possesses a spin-structure if and only if its frame bundle is trivial (c.f. [7]). Apparently, to propose that M is a spin-manifold is thus a very strong assumption about the topology of M. Note that locally every Clifford module looks like a twisted spinor bundle according to the fundamental decomposition (33).

Therefore, the geometrical setup of twisted Grassmann bundles is slightly more general than twisted spinor modules and much less restrictive (actually, twisted Grassmann bundles always exist). On the other hand, to consider arbitrary Clifford modules seems far too general. In particular, the metric g_{M} does not fix the Clifford module structure γ, in general, like (25) does in the case of a twisted Grassmann bundle. For that matter it becomes difficult to fix the domain of the universal Dirac action for general Clifford modules. Only in the case of twisted Grassmann bundles, the Einstein-Hilbert functional may interpret to provide restrictions also with respect to the module structure of the vector bundle (31).

We close this section by two remarks: First, one obtains for (M, g_{M}) denoting a closed compact and orientable Riemannian manifold of even dimension that there exists real constants α, β such that

$$\int_M \mathrm{tr}_{\mathcal{E}} V_{\mathrm{D}}\, dvol_{\mathrm{M}} = \alpha\, \mathcal{I}_{\mathrm{EH}}(i\partial\!\!\!/_{\mathrm{A}}) + \beta\, \mathcal{W}res\left(D\!\!\!\!/^{\,2-2k}\right) \tag{80}$$

independent of the chosen ∂^{E}. Here, "$\mathcal{W}res$" is the "Wodzicki residue", i.e. *the* trace functional on the algebra of classical pseudo-differential operators acting on $\mathfrak{Sec}(M, \mathcal{E}_{\Lambda,\,E})$ (see, for example, [20] and the given References therein; also see [2] and [21]).

Therefore, in the case of dim$M = 4$, the universal Dirac action is basically equal (up to a shift and the quadratic term in (4)) to the trace of the "propagator" (i.e. the Greens operator) of $D\!\!\!\!/^{\,2}$. This may demonstrate once again how natural the functional (1) actually is.

Second, the Dirac-like form (28) has been studied since from the beginning of the last century, c.f. [19], [13], [15], [14], [16], [10], [12], [11] and [17]. Apparently, this form of the Maxwell equations has a natural generalization:

$$\not{D} F_{\rm D} = 0 \tag{81}$$

where, again, $F_{\rm D} := curv(\not{D}) - R_g$ is the relative curvature with respect to $\not{D} \in \mathcal{D}(\mathcal{E}_{\Lambda,\,E})$. Accordingly, solutions of this generalized Maxwell equation like, for example, (anti-) self dual solutions may provide interesting restrictions to $\mathcal{D}(\mathcal{E}_{\Lambda,\,E})$ and hence to the Dirac action (1). Note that, when expressed in terms of the super field $\Phi \in \mathfrak{S}ec(M, \Lambda_{\rm M} \otimes {\rm End}_{\rm Cl}(\mathcal{E}_{\Lambda,\,E}))$ the generalized Maxwell equation (81) actually becomes a system of nonhomogeneous partial differential equations.

We finally mention that generalizations of the Dirac type operator $i\not{\partial}_{\rm A}$ also play a fundamental role in A. Connes's noncommutative geometry (c.f., for example, [6]) and in the case of the proof of the family index theorem, (c.f., for example, in [18], [4]).

4 Particle-anti-particle modules and Dirac operators of Pauli type

In this section we discuss another specific class of Clifford modules. These modules are mainly motivated by the structure that underpins the Majorana equation (50). These "particle-anti-particle" modules will also provide us with a better geometrical understanding of Pauli type Dirac operators. In particular, these modules will yield a geometrical motivation for the restriction of "diagonal sections" $\Psi = (\psi, \psi)$, such that the Pauli term appears in the bosonic part of the universal Dirac action (1) but drops out in fermionic part (58) of (1).

To get started let, again, $(M, g_{\rm M})$ be a given orientable, semi-Riemannian manifold of even dimension $n = 2k \geq 2$. Also, let $\tau_{\rm Cl} \equiv (Cl_{\rm M}, M)$ collect the data of the Clifford bundle $Cl_{\rm M} \twoheadrightarrow M$ associated with $(M, g_{\rm M})$.

Definition 2. *By a "real Hermitian Clifford module (bundle)" we understand a collection of data*

$$(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, J_{\mathcal{E}}, \gamma_{\mathcal{E}}) \tag{82}$$

where, respectively, $\mathcal{E}$ is the total space of a complex vector bundle $\xi_{\mathcal{E}} \equiv (\mathcal{E}, M, \pi_{\mathcal{E}})$ over M, $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ a fiber-wise Hermitian product turning $\xi_{\mathcal{E}}$ into a Hermitian vector bundle over M, $\tau_{\mathcal{E}} \in {\rm End}(\mathcal{E})$ is an involution giving rise to a $\mathbb{Z}_2-$grading of $\xi_{\mathcal{E}}$ and $J_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ denotes a real structure, i.e. an anti-linear involution on $\mathcal{E}$ that allows to identify $\xi_{\mathcal{E}}$ with its conjugate complex vector bundle $\xi_{\bar{\mathcal{E}}} := \bar{\xi}_{\mathcal{E}} \equiv (\bar{\mathcal{E}}, M, \bar{\pi}_{\mathcal{E}})$ over M. Finally,

$$\begin{array}{rccl} \gamma_{\mathcal{E}} : & T^*M & \longrightarrow & {\rm End}(\mathcal{E}) \\ & \nu & \mapsto & \gamma_{\mathcal{E}}(\nu) \end{array} \tag{83}$$

is a Clifford mapping such that all mappings are "quasi-Hermitian" (i.e. either Hermitian or skew-Hermitian) and $\tau_{\mathcal{E}}$ and $\gamma_{\mathcal{E}}$ are "quasi real" (i.e. either real or purely imaginary with respect to

$J_{\mathcal{E}})$:

$$\begin{aligned} J_{\mathcal{E}} \circ \tau_{\mathcal{E}} \circ J_{\mathcal{E}} &= \pm\tau_{\mathcal{E}} \\ J_{\mathcal{E}} \circ \gamma_{\mathcal{E}} \circ J_{\mathcal{E}} &= \pm\gamma_{\mathcal{E}} \,. \end{aligned} \tag{84}$$

Here, a real structure is called "quasi Hermitian" provided it fulfills

$$\langle J_{\mathcal{E}}(z), J_{\mathcal{E}}(w)\rangle_{\mathcal{E}} \pm \langle w, z\rangle_{\mathcal{E}} \tag{85}$$

for all $z, w \in \mathcal{E}$. Similar to complex linear mappings this is denoted by $J^{\rm t}_{\mathcal{E}} = \pm\, J_{\mathcal{E}}$, where, in general, " $^{\rm t}$" means Hermitian transpose with respect to $\langle\cdot,\cdot\rangle_{\mathcal{E}}$. If $J^{\rm t}_{\mathcal{E}} = +\, J_{\mathcal{E}}$, the real structure is also called an "anti-unitary involution".

In the following we are interested in a specific class of real Hermitian Clifford modules, called "particle modules".

Definition 3. *A real Hermitian Clifford module over $\tau_{\rm Cl}$ is called a "particle module" if*

1. *The involution is skew-Hermitian and purely imaginary;*
2. *The Clifford mapping is skew-Hermitian and real.*

The corresponding conjugate complex module is called an "anti-particle module" over $\tau_{\rm Cl}$.

We denote a particle module by

$$\xi_{\rm P} \equiv (\mathcal{P}, \langle\cdot,\cdot\rangle_{\rm P}, \tau_{\rm P}, J_{\rm P}, \gamma_{\rm P})\,. \tag{86}$$

A *particle-anti particle module* over M is a real Hermitian Clifford module (bundle) over $\tau_{\rm Cl}$

$$\xi_{\rm P\bar{P}} \equiv (\mathcal{P}\bar{\mathcal{P}}, \langle\cdot,\cdot\rangle_{\rm P\bar{P}}, \tau_{\rm P\bar{P}}, J_{\rm P\bar{P}}, \gamma_{\rm P\bar{P}}) \tag{87}$$

where, respectively

1. $\mathcal{P}\bar{\mathcal{P}} := \mathcal{P} \oplus_{\rm M} \bar{\mathcal{P}}$;
2. $\langle(z_1, w_1), (z_2, w_2)\rangle_{\rm P\bar{P}} := \frac{1}{2}\left(\langle z_1, z_2\rangle_{\rm P} + \langle w_1, w_2\rangle_{\rm P}\right)$;
3. $\tau_{\rm P\bar{P}}(z, w) : (\tau_{\rm P}(z), -\tau_{\rm P}(w))$;
4. $J_{\rm P\bar{P}}(z, w) : (J_{\rm P}(w), J_{\rm P}(z))$;
5. $\gamma_{\rm P\bar{P}}(z, w) : (\gamma_{\rm P}(z), \gamma_{\rm P}(w))$

for all $z, w, \ldots, w_2 \in \mathcal{P}$.

It follows that for all $\nu \in T^*M$:

1. $J^{\rm t}_{\rm P\bar{P}} = \pm\, J_{\rm P\bar{P}} \quad \Leftrightarrow \quad J^{\rm t}_{\rm P} = \pm\, J_{\rm P}\,;$
2. $\tau^{\rm t}_{\rm P\bar{P}} = \pm\, \tau_{\rm P\bar{P}} \quad \Leftrightarrow \quad \tau^{\rm t}_{\rm P} = \pm\, \tau_{\rm P}\,;$
3. $\gamma^{\rm t}_{\rm P\bar{P}}(\nu) = \pm\, \gamma_{\rm P\bar{P}}(\nu) \quad \Leftrightarrow \quad \gamma^{\rm t}_{\rm P}(\nu) = \pm\, \gamma_{\rm P}(\nu)\,;$
4. $J_{\rm P\bar{P}} \circ \tau_{\rm P\bar{P}} = \pm\, \tau_{\rm P\bar{P}} \circ J_{\rm P\bar{P}} \quad \Leftrightarrow \quad J_{\rm P} \circ \tau_{\rm P} = \mp\, \tau_{\rm P} \circ J_{\rm P}$
5. $J_{\rm P\bar{P}} \circ \gamma_{\rm P\bar{P}}(\nu) = \pm\, \gamma_{\rm P\bar{P}}(\nu) \circ J_{\rm P\bar{P}} \quad \Leftrightarrow \quad J_{\rm P} \circ \gamma_{\rm P}(\nu) = \pm\, \gamma_{\rm P}(\nu) \circ J_{\rm P}\,;$
6. $\tau_{\rm P\bar{P}} \circ \gamma_{\rm P\bar{P}}(\nu) = \pm\, \gamma_{\rm P\bar{P}}(\nu) \circ \tau_{\rm P\bar{P}} \quad \Leftrightarrow \quad \tau_{\rm P} \circ \gamma_{\rm P}(\nu) = \pm\, \gamma_{\rm P}(\nu) \circ \tau_{\rm P}\,.$

Theorem 1. *The most general real Dirac type operator on a particle-anti-particle module $\xi_{\rm P\bar{P}}$ reads*

$$\not{D}_{\rm P\bar{P}} \begin{pmatrix} \not{D}_P & J_P \circ \Phi_P \circ J_P \\ \Phi_P & J_P \circ \not{D}_P \circ J_P \end{pmatrix} \equiv \begin{pmatrix} \not{D}_P & \Phi^{\rm c}_P \\ \Phi_P & \not{D}^{\rm c}_P \end{pmatrix}, \tag{88}$$

with

$$\not{D}_P :\, \mathfrak{S}ec(M,\mathcal{P}) \longrightarrow \mathfrak{S}ec(M,\mathcal{P}) \tag{89}$$

being a general Dirac type operator on the underlying particle module ξ_P and $\Phi_P \in \mathfrak{S}ec(M, {\rm End}(\mathcal{P}))$ being a zero order operator that is even with respect to the $\mathbb{Z}_2-$grading on $\mathcal{P}$

Proof: To prove the statement we mention that an odd first order differential operator on a $\mathbb{Z}_2-$graded vector bundle $\xi_{\rm W} \equiv (\mathcal{W}, M, \pi_{\rm W})$ over a (semi-)Riemannian manifold $(M, g_{\rm M})$ is of Dirac type if and only if for all $f \in \mathcal{C}^\infty(M)$ the mapping

$$\begin{array}{rcl} \gamma_{\rm W} : T^*M & \longrightarrow & {\rm End}(\mathcal{W}) \\ df & \mapsto & [\not{D}, f] \equiv \not{D} \circ f - f \circ \not{D} \end{array} \tag{90}$$

yields a Clifford action on $\xi_{\rm W}$. Here, the ring $\mathcal{C}^\infty(M)$ acts multiplicatively on the (sheave) of sections $\mathfrak{S}ec(M, \mathcal{W})$.

Likewise, if $(\xi_{\rm W}, \gamma_{\rm W})$ denotes a Clifford module, then an odd first order differential operator

$$\not{D} : \mathfrak{S}ec(M,\mathcal{W}) \longrightarrow \mathfrak{S}ec(M,\mathcal{W}) \tag{91}$$

is of Dirac type (that is compatible with the given module structure) if and only if

$$[\not{D}, f] = \gamma_{\rm W}(df)\,. \tag{92}$$

Let $\xi_{\rm P\bar{P}}$ be a particle-anti-particle module and

$$\not{D} := \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \tag{93}$$

be a general first order differential operator acting on $\mathfrak{S}ec(M,\mathcal{P}\bar{\mathcal{P}})$:

$$D_k : \mathfrak{S}ec(M,\mathcal{P}) \longrightarrow \mathfrak{S}ec(M,\mathcal{P}) \tag{94}$$

for $k = 1,\ldots,4$.

The operator $\not{D}$ is odd with respect to $\tau_{\mathrm{P\bar{P}}}$ if and only if

$$\tau_{\mathrm{P\bar{P}}} \circ D_k - D_k \circ \tau_{\mathrm{P\bar{P}}} \tag{95}$$

for $k = 1,4$ and

$$\tau_{\mathrm{P\bar{P}}} \circ D_k + D_k \circ \tau_{\mathrm{P\bar{P}}} \tag{96}$$

for $k = 2,4$.

Then,

$$[\not{D}, f] = \gamma_{\mathrm{P\bar{P}}}(df) \tag{97}$$

for all $f \in \mathcal{C}^\infty(M)$ if and only if

$$[D_k, f] = \gamma_{\mathrm{P}}(df) \tag{98}$$

for $k = 1,4$ and

$$[D_k, f] = 0 \tag{99}$$

for $k = 2,3$.

Therefore, the first order differential operators $D_1 \equiv \not{D}_1$ and $D_4 \equiv \not{D}_2$ are of Dirac type on the underlying particle module ξ_{P}. In contrast, the operators $D_2 \equiv \Phi_2$ and $D_3 \equiv \Phi_1$ are of zero order.

Next, we consider the conditions on the Dirac type operator

$$\not{D} := \begin{pmatrix} \not{D}_1 & \Phi_2 \\ \Phi_1 & \not{D}_2 \end{pmatrix} \tag{100}$$

such that $\not{D}$ is real with respect to $J_{\mathrm{P\bar{P}}}$.

It follows that

$$J_{\mathrm{P\bar{P}}} \circ \not{D} \circ J_{\mathrm{P\bar{P}}} \not{D} \quad \Leftrightarrow \quad \begin{cases} \not{D}_2 &= J_{\mathrm{P}} \circ \not{D}_1 \circ J_{\mathrm{P}}\,, \\ \Phi_2 &= J_{\mathrm{P}} \circ \Phi_1 \circ J_{\mathrm{P}}\,. \end{cases} \tag{101}$$

This finally proves the statement. □

Note that neither $\not{D}_{\mathrm{P}}$, nor Φ_{P} are supposed to be real, in general.

Let

$$\mathcal{M}_{\mathrm{P\bar{P}}} : \{(z, z^{\mathrm{c}}) \in \mathcal{P}\bar{\mathcal{P}} \,|\, z, z^{\mathrm{c}} \equiv J_{\mathrm{P}}(z) \in \mathcal{P}\} \tag{102}$$

be the real subspace defined by $J_{\mathrm{P\bar{P}}}$ such that

$$\mathcal{P}\bar{\mathcal{P}} = \mathcal{M}_{\mathrm{P\bar{P}}} \otimes \mathbb{C}\,. \tag{103}$$

The corresponding real vector bundle is denoted by $\xi_{\mathcal{M}} \equiv (\mathcal{M}_{\mathrm{P\bar{P}}}, M, \pi_{\mathcal{M}})$ with the projection $\pi_{\mathcal{M}}$ being given by the restriction of $\pi_{\mathrm{P\bar{P}}}$ to $\mathcal{M}_{\mathrm{P\bar{P}}} \subset \mathcal{P}\bar{\mathcal{P}}$. Note that $\xi_{\mathcal{M}} \subset \xi_{\mathrm{P\bar{P}}}$ is a real τ_{Cl} submodule. Clearly, the latter itself contains a distinguished real sub-module given by $z \in \mathcal{P}$ fulfilling $z^{\mathrm{c}} = z$. That is, it is given by the real sub (bundle) space

$$\mathcal{M}_{\mathrm{P}} \oplus \mathcal{M}_{\mathrm{P}} : \{(z, z) \in \mathcal{P}\bar{\mathcal{P}} \,|\, J_{\mathrm{P}}(z) = z \in \mathcal{P}\} \subset \mathcal{M}_{\mathrm{P\bar{P}}}\,, \tag{104}$$

where $\mathcal{M}_{\mathrm{P}} := \{z \in \mathcal{P} \,|\, z = J_{\mathrm{P}}(z)\} \subset \mathcal{P}$, such that $\mathcal{P} = \mathcal{M}_{\mathrm{P}} \otimes \mathbb{C}$.

On a particle-anti-particle module, the first order differential operator (88) is the most general real Dirac type operator. Hence, one may restrict the universal Dirac action (1) to this type of Dirac operators:

$$\mathcal{I}_{\mathrm{D,real}} \;:\; \tfrac{1}{2}\int_M [\langle \Psi_{\mathrm{P\bar{P}}},\, \not{D}_{\mathrm{P\bar{P}}} \Psi_{\mathrm{P\bar{P}}}\rangle_{\mathrm{P\bar{P}}} + \mathrm{tr}_\gamma curv(\not{D}_{\mathrm{P\bar{P}}})]\; dvol_{\mathrm{M}} \tag{105}$$

with $\Psi_{\mathrm{P\bar{P}}} = (\Psi_{\mathrm{P}}, \Psi^{\mathrm{c}}_{\mathrm{P}}) \in \mathfrak{S}ec(M, \mathcal{M}_{\mathrm{P\bar{P}}})$ and $\not{D}_{\mathrm{P\bar{P}}}$ any real Dirac operator on the particle-anti-particle module $\xi_{\mathrm{P\bar{P}}}$.

Proposition 2. *When boundary terms are neglected, the Dirac action (105) decomposes as follows:*

$$\mathcal{I}_{D,real}\mathcal{I}_{D,ferm}(\not{D}_{\mathrm{P\bar{P}}}) + \mathcal{I}_{D,bos}(\not{D}_{\mathrm{P\bar{P}}}) \tag{106}$$

where

$$\begin{aligned} 2\mathcal{I}_{D,ferm}(\not{D}_{\mathrm{P\bar{P}}}) \;\equiv\; & \int_M [\langle \Psi_P, \not{D}_P \Psi_P\rangle_P + \langle \Psi^{\mathrm{c}}_P, \not{D}^{\mathrm{c}}_P \Psi^{\mathrm{c}}_P\rangle_P \\ & + \langle \Psi_P, \Phi^{\mathrm{c}}_P \Psi^{\mathrm{c}}_P\rangle_P + \langle \Psi^{\mathrm{c}}_P, \Phi_P \Psi_P\rangle_P]\, dvol_M\,; \end{aligned} \tag{107}$$

$$\begin{aligned} 2\mathcal{I}_{D,bos}(\not{D}_{\mathrm{P\bar{P}}}) \;\equiv\; & \int_M [\mathrm{tr}_\gamma curv(\not{D}_P) + \mathrm{tr}_\gamma curv(\not{D}^{c}_P) + 2\,\mathrm{tr}(\Phi^{\mathrm{c}}_P \circ \Phi_P) \\ & + 8\,(\mathrm{tr} \circ \mathrm{ev}_g)(\alpha^{\mathrm{c}}_P \circ \alpha_P) + 2\,(\mathrm{tr} \circ \mathrm{ev}_g)(\beta^{\mathrm{c}}_P \circ \beta_P)]\, dvol_M \end{aligned} \tag{108}$$

where $2\alpha_P(v) \,:\, \Phi_P \circ \gamma_P(v^\flat) \in \mathrm{End}(\mathcal{P})$ *and* $v^\flat(u) = g_M(v,u)$ *for all* $u, v \in TM$. *Accordingly,* $2\alpha^{\mathrm{c}}_P(v) \,:\, J_P \circ 2\alpha_P(v) \circ J_P^{-1}\Phi^{c}_P \circ \gamma_P(v^\flat) \in \mathrm{End}(\mathcal{P})$. *Furthermore,* $\beta_P \,:\, \mathrm{ext}_\Theta(\Phi_P - 2\,\not{\alpha}_P)$ *with* $\Theta \in \Omega^1(M, \mathrm{End}(\mathcal{P}))$ *being the canonical one-form that exists on every Clifford module (c.f. [22]) and* $\beta^{\mathrm{c}}_P : J_P \circ \beta_P \circ J_P^{-1}\mathrm{ext}_\Theta(\Phi^{\mathrm{c}}_P - 2\,\not{\alpha}^{c}_P)$.

In the sequel, we make use of the common "dagger" abbreviation: $\not{\alpha} \equiv \gamma(\alpha) \in \Omega^0(M, \mathrm{End}(\mathcal{P}))$ for any $\alpha \in \Omega^*(M, \mathrm{End}(\mathcal{P}))$. For example, for $\alpha = e^k \otimes \lambda_k$ one has $\not{\alpha} = \gamma^k \otimes \lambda_k$, etc.

Proof: The fermionic part is straightforward to prove. The bosonic part of the Dirac action is proved in several steps.

First, we prove the following

Lemma 1. *Let $\not{D}_1$ and $\not{D}_2$ be two Dirac type first order differential operators on an arbitrary Clifford module $(\xi_\mathcal{E}, \gamma_\mathcal{E}) \equiv (\mathcal{E}, M, \pi_\mathcal{E}, \gamma_\mathcal{E})$ with $\pi_\mathcal{E} : \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \to M$ being a $\mathbb{Z}_2-$graded (Hermitian) vector bundle over (M, g_M). The zero-order term V_H of the generalized Laplacian*

$$\begin{array}{rcl} H : \mathfrak{S}ec(M, \mathcal{E}) & \longrightarrow & \mathfrak{S}ec(M, \mathcal{E}) \\ \Psi & \mapsto & \not{D}_1(\not{D}_2 \Psi) \end{array} \tag{109}$$

has the explicit form:

$$V_H = V_D + \Phi \circ \not{\omega}_D + \mathrm{ev}_g(\alpha_H^2) + U \,. \tag{110}$$

Here, respectively, V_D and ω_D are the Dirac potential and Dirac form of $\not{D} \equiv \not{D}_2$ (c.f. [23]). Moreover, $\Phi := \not{D}_1 - \not{D}_2 \in \mathfrak{S}ec(M, \mathrm{End}^-(\mathcal{E}))$ and

$$U := \mathrm{ev}_g \left(\nabla_H^{T^*M \otimes \mathrm{End}(\mathcal{E})} \alpha_H \right) \tag{111}$$

with $\nabla_H^\mathcal{E}$ being the covariant derivative that defines the connection Laplacian of H :

$$\triangle_H : -\mathrm{ev}_g \left(\nabla_H^{T^*M \otimes \mathcal{E}} \circ \nabla_H^\mathcal{E} \right) \tag{112}$$

and $\alpha_H \in \Omega^1(M, \mathrm{End}_M(\mathcal{E}))$ is given by

$$\begin{array}{rcl} 2\,\alpha_H(\mathrm{grad}_g f) & := & [\Phi \circ \not{D}, f] \\ & = & \Phi \circ \gamma_\mathcal{E}(df) \end{array} \tag{113}$$

for all $f \in \mathcal{C}^\infty(M)$.

Here and henceforth we make use of the following notation: "ev_g" means "evaluation/contraction" with respect to g_M. For instance, $\mathrm{ev}_g(\alpha^2) \stackrel{\mathrm{loc.}}{=} \mathrm{ev}_g(e^\mu \otimes e^\nu \otimes \alpha_\mu \circ \alpha_\nu) : g_\mathrm{M}(e^\mu, e^\nu)\, \alpha_\mu \circ \alpha_\nu \in \mathrm{End}(\mathcal{E})$ for all $\alpha \in \Omega^1(M, \mathrm{End}(\mathcal{E}))$ etc.

Proof: With $\not{D}_1 = \not{D}_2 + \Phi \equiv \not{D} + \Phi$ it becomes sufficient to consider Laplace type operators of the form

$$H = \not{D}^2 + \Phi \circ \not{D} \,. \tag{114}$$

Every generalized Laplacian H decomposes as (see, for instance, in [3])

$$H = -\triangle_{\rm H} + V_{\rm H} \tag{115}$$

with $\nabla^{\mathcal{E}}_{\rm H}$ being given by

$$2\,{\rm ev}(f_0\,{\rm grad}f_1, \nabla^{\mathcal{E}}_{\rm H}\Psi) := f_0\left([H, f_1] + \triangle_g f_1\right)\Psi \tag{116}$$

for all $f_0, f_1 \in \mathcal{C}^\infty(M)$ and $\Psi \in \mathfrak{S}ec(M, \mathcal{E})$. Here, $\triangle_g$ denotes the usual Laplace-Beltrami operator restricted to zero-forms on M.

It follows that

$$\nabla^{\mathcal{E}}_{\rm H}\nabla^{\mathcal{E}}_{\rm D} + \alpha_{\rm H} \tag{117}$$

with $\nabla^{\mathcal{E}}_{\rm D}$ being the covariant derivative that defines the Bochner-Laplacian of $\not{D}^2$.

As a consequence, the connection Laplacian of H may be expressed in terms of the Bochner-Laplacian[1] of $\not{D}^2$:

$$\triangle_{\rm H} = \triangle_{\rm D} + 2\,{\rm ev}_g\left(\alpha_{\rm H} \circ \nabla^{\mathcal{E}}_{\rm D}\right) + {\rm ev}_g(\alpha^2_{\rm H}) + U\,. \tag{118}$$

The statement then follows by comparison of the general Lichnerowicz decomposition (115), taking into account that $\not{D}^2 = -\triangle_{\rm D} + V_{\rm D}$ and

$$2\,{\rm ev}_g(\alpha_{\rm H} \circ \nabla^{\mathcal{E}}_{\rm D}) = \Phi \circ \not{\nabla}^{\mathcal{E}}_{\rm D}\,. \tag{119}$$

□

Note that

$${\rm tr}_{\mathcal{E}} U = {\rm div}_g\,\xi_{\rm H} \tag{120}$$

with

$$\xi_{\rm H} := ({\rm tr}_{\mathcal{E}}\alpha_{\rm H})^\sharp \in \mathfrak{S}ec(M, TM)\,. \tag{121}$$

Hence,

$${\rm tr}_{\mathcal{E}} V_{\rm H}\, dvol_{\rm M} \left[{\rm tr}_{\mathcal{E}} V_{\rm D} + {\rm tr}_{\mathcal{E}}(\Phi \circ \not{\phi}_{\rm D}) + ({\rm tr}_{\mathcal{E}} \circ {\rm ev}_g)(\alpha^2_{\rm H})\right]\, dvol_{\rm M} + \pounds_{\xi_{\rm H}} dvol_{\rm M} \tag{122}$$

which demonstrates that

$$[*{\rm tr}_{\mathcal{E}} V_{\rm H}][*{\rm tr}_{\mathcal{E}}(V_{\rm D} + \Phi \circ \not{\phi}_{\rm D} + {\rm ev}_g(\alpha^2_{\rm H}))] \in {\rm H}^{\rm n}_{\rm deR}(M) \tag{123}$$

with "$*$" being the Hodge map induced by $g_{\rm M}$ and the orientation defined by $dvol_{\rm M}$.

[1] I would like to thank M. Schneider for appropriate comments.

Clearly, for $\not{D}_1 = \not{D}_2 = \not{D}$ one has

$$V_{\rm H} = V_{\rm D}\,. \tag{124}$$

Next, we present a Bochner-Lichnerowicz-Weizenböck type formula for a slightly more general Laplace type second order differential operator H'.

Corollary 2. *Let again $(\xi_{\mathcal{E}}, \gamma_{\mathcal{E}})$ be a Clifford module over (M, g_M). Also, let $D_k = \not{D}_k + \Phi_k$ $(k = 1, 2)$ be two first order differential operators acting on $\mathfrak{Sec}(M, \mathcal{E})$. The zero-order term $V_{\rm H'}$ of the generalized Laplacian*

$$\begin{array}{rcl} H' : \mathfrak{Sec}(M,\mathcal{E}) & \longrightarrow & \mathfrak{Sec}(M,\mathcal{E}) \\ \Psi & \mapsto & D_1(D_2\Psi) \end{array} \tag{125}$$

reads:

$$V_{\rm H'} = V_H + V \tag{126}$$

where V_H is given by (110) with the replacement

$$\Phi \;:=\; D_1 - D_2 + 2\Phi_2 \tag{127}$$

and

$$V := (\Phi - \Phi_2)\circ\Phi_2 + [\not{D}_2, \Phi_2]\,. \tag{128}$$

Proof: We put $D_1 = \not{D}_2 + \Phi_0 + \Phi_1 \equiv \not{D} + \Phi_{01}$ and rewrite H' as

$$\begin{array}{rcl} H' & = & \not{D}^2 + \Phi\circ\not{D} + V \\ & = & H + V\,. \end{array} \tag{129}$$

Hence, the connection Laplacian $\triangle_{\rm H'}$ of H' is the same as the connection Laplacian $\triangle_{\rm H}$ of H. One may thus apply the former Lemma 1 to prove the statement. □

As a consequence, one obtains explicitly (neglecting boundary terms):

$$\mathrm{tr}_{\mathcal{E}} V_{\rm H'} \mathrm{tr}_{\mathcal{E}} V_{\rm D} + \mathrm{tr}_{\mathcal{E}}(\Phi\circ\Phi_2 - \Phi_2^2) + \mathrm{tr}_{\mathcal{E}}[\not{D}_2, \Phi_2] + \mathrm{tr}_{\mathcal{E}}(\Phi\circ\not{\omega}_{\rm D}) + (\mathrm{tr}_{\mathcal{E}}\circ \mathrm{ev}_g)(\alpha_{\rm H}^2)\,. \tag{130}$$

We are now in the position to prove the bosonic part of Proposition 2. In fact, this will be an immediate consequence of the following more general

Proposition 3. *Let $(\xi_{\mathcal{E}}, \gamma_{\mathcal{E}})$ be a Clifford module over (M, g_M). Also, let $(\xi_{2\mathcal{E}}, \gamma_{2\mathcal{E}})$ be the Clifford module that is defined by the corresponding Whitney sum:*

$$\xi_{2\mathcal{E}} : \xi_{\mathcal{E}} \oplus \xi_{\mathcal{E}}\,, \quad \tau_{2\mathcal{E}} : \tau_{\mathcal{E}} \oplus (-\tau_{\mathcal{E}})\,, \quad \gamma_{2\mathcal{E}} : \gamma_{\mathcal{E}} \oplus \gamma_{\mathcal{E}} \tag{131}$$

with $\tau_{\mathcal{E}}$ being the grading involution on $\xi_{\mathcal{E}}$.

The zero order term $V_{\mathcal{D}} \in \mathfrak{Sec}(M, \mathrm{End}(2\mathcal{E}))$ associated with the (square of the) most general Dirac type first order differential operator

$$\mathcal{D} \equiv \begin{pmatrix} \not{D}_1 & \Phi_2 \\ \Phi_1 & \not{D}_2 \end{pmatrix} : \begin{matrix} \mathfrak{Sec}(M,\mathcal{E}) \\ \oplus \\ \mathfrak{Sec}(M,\mathcal{E}) \end{matrix} \longrightarrow \begin{matrix} \mathfrak{Sec}(M,\mathcal{E}) \\ \oplus \\ \mathfrak{Sec}(M,\mathcal{E}) \end{matrix} \tag{132}$$

reads:

$$V_{\mathcal{D}} =$$

$$\begin{pmatrix} V_1 + \Phi_2 \circ \Phi_1 + 4\,\mathrm{ev}_g(\alpha_2 \circ \alpha_1) & [\not{D}_1, \Phi_2] + \Phi_2 \circ (\not{D}_1 - \not{D}_2) + 2\,\Phi_2 \circ \not{\omega}_2 \\ [\not{D}_2, \Phi_1] + \Phi_1 \circ (\not{D}_2 - \not{D}_1) + 2\,\Phi_1 \circ \not{\omega}_1 & V_2 + \Phi_1 \circ \Phi_2 + 4\,\mathrm{ev}_g(\alpha_1 \circ \alpha_2) \end{pmatrix} \tag{133}$$

where, respectively, V_k and ω_k denote the Dirac potential and the Dirac form of $\not{D}_k$ $(k = 1, 2)$ and α_k is defined in terms of $\not{D}_k^2$, similar to α_H of Lemma (1).

Proof: We may write

$$\mathcal{D} = \not{D} + \Phi \tag{134}$$

with

$$\not{D} : \begin{pmatrix} \not{D}_1 & 0 \\ 0 & \not{D}_2 \end{pmatrix}, \qquad \Phi : \begin{pmatrix} 0 & \Phi_2 \\ \Phi_1 & 0 \end{pmatrix}. \tag{135}$$

Then, we simply make use of the preceding Corollary 2 and apply the corresponding Bochner-Lichnerowicz-Weizenböck type formula to $H' := \mathcal{D}^2$. Note that the Bochner-Laplacian of $\mathcal{D}$ is given by

$$\nabla^{2\mathcal{E}}_{\mathcal{D}} \nabla^{2\mathcal{E}}_{\mathrm{D}} + \beta_{\mathcal{D}} \tag{136}$$

with $\beta_{\mathcal{D}} \in \Omega^1(M, \mathrm{End}(2\mathcal{E}))$ being

$$\beta_{\mathcal{D}} : \begin{pmatrix} 0 & 2\alpha_2 \\ 2\alpha_1 & 0 \end{pmatrix}. \tag{137}$$

Here, again

$$\begin{aligned} 2\,\alpha_k(\mathrm{grad}_g f) &:= [\Phi_k \circ \not{D}_k\,, f] \\ &= \Phi_k \circ \gamma_{\mathcal{E}}(df) \end{aligned} \tag{138}$$

for all $f \in \mathcal{C}^\infty(M)$ and $k = 1, 2$.

Then, similar to the results presented before

$$V_{\mathcal{D}} = V_{\mathrm{D}} + U + 2\,\Phi \circ \not{\omega}_{\mathrm{D}} + \mathrm{ev}_g(\beta^2_{\mathcal{D}}) \tag{139}$$

where

$$\not{\omega}_{\mathrm{D}} : \gamma_{2\mathcal{E}}(\omega_{\mathrm{D}}) \equiv \begin{pmatrix} \not{\omega}_1 & 0 \\ 0 & \not{\omega}_2 \end{pmatrix}, \qquad U := \mathrm{ev}_g\left(\nabla^{T^*M\otimes\mathrm{End}(2\mathcal{E})}_{\mathcal{D}}\beta_{\mathcal{D}}\right). \tag{140}$$

□

Therefore,

$$\mathrm{tr}_{2\mathcal{E}}V_{\mathcal{D}}\mathrm{tr}_{\mathcal{E}}V_1 + \mathrm{tr}_{\mathcal{E}}V_2 + 2\,\mathrm{tr}_{\mathcal{E}}(\Phi_1 \circ \Phi_2) + 8\,(\mathrm{tr}_{\mathcal{E}} \circ \mathrm{ev}_g)(\alpha_1 \circ \alpha_2) + div_g\xi_{\mathcal{D}} \tag{141}$$

with

$$\xi_{\mathcal{D}} : (\mathrm{tr}_{2\mathcal{E}}\beta_{\mathcal{D}})^\sharp\,. \tag{142}$$

The bosonic part of the Proposition (2) is then implied by (again, omitting all boundary terms):

$$\begin{aligned} \mathrm{tr}_\gamma(curv(\not{\mathcal{D}})) &= \mathrm{tr}_{2\mathcal{E}}V_{\mathcal{D}} + (\mathrm{tr}_{2\mathcal{E}}\circ \mathrm{ev}_g)(\omega^2_{\mathcal{D}}) \\ &= \mathrm{tr}_\gamma(curv(\not{D}_1)) + \mathrm{tr}_\gamma(curv(\not{D}_2)) + \\ &\quad + 2\,\mathrm{tr}_{\mathcal{E}}(\Phi_1 \circ \Phi_2) + 8\,(\mathrm{tr}_{\mathcal{E}} \circ \mathrm{ev}_g)(\alpha_1 \circ \alpha_2) + \\ &\quad + 2\,(\mathrm{tr}_{\mathcal{E}} \circ \mathrm{ev}_g)(\sigma_1 \circ \sigma_2)\,, \end{aligned} \tag{143}$$

where $\sigma_k := \mathrm{ext}_\Theta(\Phi_k - 2\,\not{\alpha}_k) \in \Omega^1(M, \mathrm{End}(\mathcal{E}))$ and $2\alpha_k(v)\Phi_k \circ \gamma_{\mathcal{E}}(v^\flat)$ for all $v \in TM$ and $k = 1, 2$.

This finally ends the proof of Proposition (2). □

The functionals (107–108) may look complicated at first glance. However, they yield a straightforward generalization of the usual action of the Standard Model of particle physics as discussed in [22], which allows to also include Majorana mass terms. The latter feature will be discussed in detail elsewhere.

Proposition 4. *Let J_P be anti-unitary and $\Psi_{\mathrm{P\bar{P}}}(\Psi_P, \Psi_P) \in \mathfrak{Sec}(M, \mathcal{M}_P \oplus \mathcal{M}_P)$. Also, let $\not{D}_P$ be real with respect to J_P and both $\not{D}_P$ and the real part of Φ_P be formally self-adjoint. Then, the Dirac action*

$$\mathcal{I}_{D,real}\mathcal{I}_{D,ferm}(\not{D}_{\mathrm{P\bar{P}}}) + \mathcal{I}_{D,bos}(\not{D}_{\mathrm{P\bar{P}}}) \tag{144}$$

reads:

$$\begin{aligned}\mathcal{I}_{D,ferm}(\not{D}_{\mathrm{P}\bar{\mathrm{P}}}) &= \int_M [\langle\Psi_P,\,(\not{D}_P + \mathcal{Y}_P)\Psi_P\rangle_P]\,dvol_M\,,\\ \mathcal{I}_{D,bos}(\not{D}_{\mathrm{P}\bar{\mathrm{P}}}) &= \int_M \big[\mathrm{tr}_\gamma curv(\not{D}_P) + (\mathrm{tr}\circ\mathrm{ev}_{g'})(Y_P^2) - (\mathrm{tr}\circ\mathrm{ev}_{g'})(F_P^2)\big]\,dvol_M \end{aligned}\qquad(145)$$

where $Y_P,\ F_P \in \Omega^*(M, \mathrm{End}_{Cl}(\mathcal{P}))$.

Proof: This is a simple application of Proposition 2 taking into account that the (endomorphism valued) one-forms $\alpha_{\mathrm{P}},\ \beta_{\mathrm{P}} \in \Omega^1(M, \mathrm{End}(\mathcal{P}))$ are linearly determined by the zero order operator $\Phi_{\mathrm{P}} \in \mathrm{Sec}(M, \mathrm{End}(\mathcal{P}))$. Furthermore, due to the fundamental decomposition (33) every zero order operator locally reads:

$$\Phi_{\mathrm{P}}\gamma^{\mathrm{I}} \otimes \phi_{\mathrm{I}} \qquad(146)$$

where $\mathrm{I} = (i_1, i_2, \ldots, i_l)$ is a multi-index ($1 \le i_k \le n$ for $k = 0, 1, \ldots, n$), $\gamma^{\mathrm{I}} \equiv \gamma^{i_1}\gamma^{i_2}\cdots\gamma^{i_l}$ and ϕ_{I} are local sections of $\mathrm{Sec}(M, \mathrm{End}_{\mathrm{Cl}}(\mathcal{P}))$ which are totally antisymmetric with respect to the multi index I. To avoid double counting the summation is thus take only for the ordered indices: $i_1 < i_2 < \ldots < i_l$, $l = 0, 1, , \ldots, n$. In other words, the zero order section $\Phi_{\mathrm{P}} \in \Omega(M, \mathrm{End}(\mathcal{P}))$ is in one-to-one correspondence with a general section $\phi_{\mathrm{P}} \in \Omega^*(M, \mathrm{End}_{\mathrm{Cl}}(\mathcal{P}))$. Then, $\phi_{\mathrm{P}} = e^{\mathrm{I}} \otimes \phi_{\mathrm{I}}$ with $\{e^{\mathrm{I}} \equiv e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_l} \,|\, l = 0, 1, \ldots, n\}$ being a local basis of $\Lambda_{\mathrm{M}} \twoheadrightarrow M$.

Hence, the bosonic part of (105) reduces to

$$2\mathcal{I}_{\mathrm{D,bos}}(\not{D}_{\mathrm{P}\bar{\mathrm{P}}}) \equiv \int_M [\mathrm{tr}_\gamma curv(\not{D}_{\mathrm{P}}) + \mathrm{tr}_\gamma curv(\not{D}^{\mathrm{c}}_{\mathrm{P}}) + 2(\mathrm{tr}\circ\mathrm{ev}_{g'})(\phi^{\mathrm{c}}_{\mathrm{P}} \circ \phi_{\mathrm{P}})]\,dvol_{\mathrm{M}} \qquad(147)$$

where the evaluation map on the right hand side refers to the re-scaled (fiber) metric $g'_{\Lambda\mathrm{M}}$ on the Grassmann bundle $\Lambda_{\mathrm{M}} \twoheadrightarrow M$ that is defined by

$$\begin{aligned} g'^{\mathrm{IJ}} \equiv \lambda'\, g_{\Lambda\mathrm{M}}(e^{\mathrm{I}}, e^{\mathrm{J}}) \ &:=\ \mathrm{tr}\gamma^{\mathrm{I}}\gamma^{\mathrm{J}} + \tfrac{1}{4}\, g_{ij}\,\mathrm{tr}\gamma^{\mathrm{I}}\gamma^i\gamma^{\mathrm{J}}\gamma^j + \\ &\quad + \tfrac{1}{n^2}\, g_{ij}\,\mathrm{tr}\big(\gamma^i\gamma^{\mathrm{I}} - g_{ab}\,\gamma^i\gamma^a\gamma^{\mathrm{I}}\gamma^b\big)\big(\gamma^j\gamma^{\mathrm{J}} - g_{cd}\,\gamma^j\gamma^c\gamma^{\mathrm{J}}\gamma^d\big)\,. \end{aligned}\qquad(148)$$

Indeed, one explicitly has

$$\begin{aligned} \Phi_{\mathrm{P}} \circ \Phi^{\mathrm{c}}_{\mathrm{P}} &= \gamma^{\mathrm{I}}\gamma^{\mathrm{J}} \otimes \phi_{\mathrm{I}} \circ \phi^{\mathrm{c}}_{\mathrm{J}}\,,\\ \mathrm{ev}_g(\alpha_{\mathrm{P}} \circ \alpha^{\mathrm{c}}_{\mathrm{P}}) &= \tfrac{1}{4}\, g_{ij}\,\gamma^{\mathrm{I}}\gamma^i\gamma^{\mathrm{J}}\gamma^j \otimes \phi_{\mathrm{I}} \circ \phi^{\mathrm{c}}_{\mathrm{J}}\,,\\ \mathrm{ev}_g(\beta_{\mathrm{P}} \circ \beta^{\mathrm{c}}_{\mathrm{P}}) &= \tfrac{1}{n^2}\, g_{ij}\,\big(\gamma^i\gamma^{\mathrm{I}} - g_{ab}\,\gamma^i\gamma^a\gamma^{\mathrm{I}}\gamma^b\big)\big(\gamma^j\gamma^{\mathrm{J}} - g_{cd}\,\gamma^j\gamma^c\gamma^{\mathrm{J}}\gamma^d\big)\otimes\phi_{\mathrm{I}}\circ\phi^{\mathrm{c}}_{\mathrm{J}} \end{aligned}\qquad(149)$$

where $g_{ij} \equiv g_{\mathrm{M}}(e_i, e_j)$ etc., $\mathrm{J} = (j_1, j_2, \ldots, j_k)$ is again a multi-index and Einstein's summation convention is applied. Let us call in mind that we do not distinguish between the metric on the tangent and the co-tangent space of M.

As a consequence,

$$\mathrm{tr}(\Phi^{\mathrm{c}}_{\mathrm{P}} \circ \Phi_{\mathrm{P}}) + 4\,(\mathrm{tr} \circ \mathrm{ev}_g)(\alpha^{\mathrm{c}}_{\mathrm{P}} \circ \alpha_{\mathrm{P}}) + (\mathrm{tr} \circ \mathrm{ev}_g)(\beta^{\mathrm{c}}_{\mathrm{P}} \circ \beta_{\mathrm{P}})(\mathrm{tr} \circ \mathrm{ev}_{g'})(\phi^{\mathrm{c}}_{\mathrm{P}} \circ \phi_{\mathrm{P}})\,. \tag{150}$$

Furthermore, every real Dirac type first order differential operator on a particle-anti-particle module may be rewritten as

$$\not{D}_{\mathrm{P}\bar{\mathrm{P}}} \begin{pmatrix} \not{D}_{\mathrm{P}} & \mathcal{Y}_{\mathrm{P}} - \mathcal{F}_{\mathrm{P}} \\ \mathcal{Y}_{\mathrm{P}} + \mathcal{F}_{\mathrm{P}} & \not{D}^{\mathrm{c}}_{\mathrm{P}} \end{pmatrix}. \tag{151}$$

Here,

$$\begin{aligned} \mathcal{Y}_{\mathrm{P}} &:= \tfrac{1}{2}(\Phi_{\mathrm{P}} + \Phi^{\mathrm{c}}_{\mathrm{P}}) \equiv \mathrm{Re}_{\mathrm{J}}\Phi_{\mathrm{P}}\,, & (152)\\ \mathcal{F}_{\mathrm{P}} &:= \tfrac{1}{2}(\Phi_{\mathrm{P}} - \Phi^{\mathrm{c}}_{\mathrm{P}}) \equiv i\mathrm{Im}_{\mathrm{J}}\Phi_{\mathrm{P}}\,, & (153)\end{aligned}$$

such that $\mathcal{Y}_{\mathrm{P}}$ is the real and $-i\mathcal{F}_{\mathrm{P}}$ is the imaginary part of the zero order operator Φ_{P} with respect to the real structure J_{P}.

According to the general case, we put

$$\begin{aligned} \mathcal{Y}_{\mathrm{P}} &= \gamma^{\mathrm{I}} \otimes Y_{\mathrm{I}}\,,\\ \mathcal{F}_{\mathrm{P}} &= \gamma^{\mathrm{J}} \otimes F_{\mathrm{J}}\,. \end{aligned} \tag{154}$$

The statement then follows from

$$\mathrm{tr}(\Phi^{\mathrm{c}}_{\mathrm{P}} \circ \Phi_{\mathrm{P}})\mathrm{tr}(\mathcal{Y}^2_{\mathrm{P}}) - \mathrm{tr}(\mathcal{F}^2_{\mathrm{P}})\,, \tag{155}$$

which is analogous to the case of complex numbers (remember that $\mathcal{F}^{\mathrm{c}}_{\mathrm{P}} - \mathcal{F}_{\mathrm{P}}$). □

We note that $\not{D}_{\mathrm{P}\bar{\mathrm{P}}}$ leaves the real submodule $\mathfrak{S}ec(M, \mathcal{M}_{\mathrm{P}} \oplus \mathcal{M}_{\mathrm{P}}) \subset \mathfrak{S}ec(M, \mathcal{M}_{\mathrm{P}\bar{\mathrm{P}}})$ invariant if and only if $\not{D}^{\mathrm{c}}_{\mathrm{P}} = \not{D}_{\mathrm{P}}$ and $\mathcal{F}_{\mathrm{P}} = 0$.

Clearly, for $\mathcal{Y}_{\mathrm{P}} = 0$ and $\mathcal{F}_{\mathrm{P}}$ the curvature of $\not{D}_{\mathrm{P}} := i\not{\partial}_{\mathrm{A}}$ we get back the Pauli type Dirac operators as specific real Dirac type operators on the real Hermitian Clifford module $\xi_{\mathrm{P}\bar{\mathrm{P}}}$. Moreover, the "diagonal sections" are motivated by the distinguished real submodule

$$\mathfrak{S}ec(M, \mathcal{M}_{\mathrm{P}} \oplus \mathcal{M}_{\mathrm{P}}) \subset \mathfrak{S}ec(M, \mathcal{P}\bar{\mathcal{P}}) \tag{156}$$

of $\xi_{\mathrm{P}\bar{\mathrm{P}}}$. Note that it is the doubling of $\mathcal{M}_{\mathrm{P}}$ (which we may identify with our former twisted Grassmann bundle $\mathcal{E}_{\Lambda,\,E}$) which allows to add the Pauli-term to $i\not{\partial}_{\mathrm{A}}$ such that the resulting first order operator is still of Dirac type. Also note that the additional complex structure encountered in the definition of (45) corresponds to the assumption that the zero order part of (88) is purely imaginary. For the same matter it has to drop out in the fermionic part of the universal Dirac action since it would yield a non-real contribution.

5 Outlook

We presented a detailed motivation of "Dirac type gauge theories" which are gauge theories that are based on the universal Dirac action (1). In particular, we have exhibit how the Dirac action covers well-known differential equations, like the Maxwell and the Einstein equation. Indeed, the Dirac action turns out to be a natural generalization of the Einstein-Hilbert functional. To also obtain the Yang-Mills functional, one has to introduce a specific class of Dirac type operators and we discussed their geometrical origin in terms of real Hermitian Clifford modules. We also discussed the domain of the Dirac action from a geometrical point of view. We thereby proved several Lichnerowicz type formulae for decomposable Laplace type operators which generalize the corresponding result presented in [23].

It is well-known that there is a one-to-one correspondence between Dirac type operators on a Clifford module and Clifford super-connections (see, for instance, in [3]). For this reason, the domain of dependence of the Dirac action may not come as a surprise, especially because of the isomorphisms (59). However, the latter hold true only when the module structure is fixed from the outset. This, of course, does not permit interpreting the Einstein-Hilbert functional as a constraint on the module structure. Moreover, our discussion clearly demonstrates that there exists a natural functional on the Dirac bundle provided by the Dirac action.

The presented results clearly exhibit in what sense Dirac type operators and Clifford modules provide a more general geometrical setting to describe gauge theories than connections and principal bundles. Indeed, Dirac type gauge theories allow to describe different types of gauge theories, like Yang-Mills theory, Einstein's theory of gravity and spontaneously broken Yang-Mills gauge theories, in a geometrically unified setting based on the same universal Dirac functional. This is independent of whether the base manifold ("space-time") M is supposed to be spin or not.

In order to gain more insight, however, one has to deal with the "moduli space of Dirac operators"

$$\mathfrak{M}(\mathcal{E}) \equiv \mathcal{D}(\mathcal{E})/Diff(\mathcal{E}) \tag{157}$$

on which the Dirac functional descents. Of course, this set is probably far too wild and thus has to be restricted to appropriate subsets like the solutions of

$$*F_{\rm D} = \pm F_{\rm D}\,, \tag{158}$$

similar to the moduli space of (anti-) self-dual solutions of the ordinary Yang-Mills equation: $*F_{\rm A} \pm F_{\rm A}$. For this, however, the domain of the Dirac functional has to be discussed more seriously, in particular from an analytical point of view.

In contrast to the ordinary Yang-Mills equations one obtains still another reasonable constrains to Dirac type operators, similar to the (anti-) self-duality condition as, for instance, the "unimodularity" condition:

$$\pounds_{\xi_{\rm D}} dvol_{\rm M} = 0\,. \tag{159}$$

Finally, one may pose the question to what extent is there a relation between the stationary points of the Dirac action (1) and the generalized Maxwell equation

$$\not{D} F_{\mathrm{D}} = 0 \,. \tag{160}$$

Again, in full generality this seems a hopeless task. However, it might be reasonable to discuss this question using appropriate simple geometrical settings. This will be done in a forthcoming work.

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Discrete Clifford analysis: an overview

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ABSTRACT

We give an account of our current research results in the development of a higher dimensional discrete function theory in a Clifford algebra context. On the simplest of all graphs, the rectangular $\mathbb{Z}^m$ grid, the concept of a discrete monogenic function is introduced. To this end new Clifford bases, involving so–called forward and backward basis vectors and introduced by means of their underlying metric, are controlling the support of the involved operators. As our discrete Dirac operator is seen to square up to a mixed discrete Laplacian, the resulting function theory may be interpreted as a refinement of discrete harmonic analysis. After a proper definition of some topological concepts, function theoretic results amongst which Cauchy's theorem and a Cauchy integral formula are obtained. Finally a first attempt is made at creating a general model for the Clifford bases used, involving geometrically interpretable curvature vectors.

RESUMEN

Nosotros damos un relato de los resultados de investigación actual en el desarrollo de la teoría de funciones discretas de dimensión grande en un álgebra de Clifford. Sobre el mas simple de todos los gráficos, la red de rectangulos $\mathbb{Z}^m$, el concepto de función monogénica discreta es presentado. Con esta finalidad nuevas bases de Clifford, envolviendo las bases de vectores llamadas forward and backward, son introducidas mediante su métrica fundamental, estas controlan el soporte de los operadores envueltos. Como nuestro operador de Dirac discreto puede ser visto como un operador

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Laplaciano discreto mixto, la teoría de funciones resultante puede ser interpretada como refinamiento de análisis armónico discreto. Después de definir algunos conceptos topológicos, resultados de teoría de funciones entre los cuales el Teorema de Cauchy y la fórmula de Cauchy integral son obtenidos. Finalmente, una primera tentativa es hacer uso de un modelo general de bases de Clifford envolviendo vectores de curvatura geométricamente interpretables.

Key words and phrases: *discrete Clifford analysis, discrete function theory, discrete Cauchy formula.*

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1 Introduction to the Clifford analysis setting

Clifford analysis (see e.g. [3, 4, 14]) is a higher dimensional function theory centred around the notion of monogenic functions, i.e. null solutions of the rotation invariant vector valued Dirac operator $\partial_{\underline{x}}$, defined below. It is a popular viewpoint to consider this function theory both as a higher dimensional analogue of the theory of holomorphic functions in the complex plane and as a refinement of classical harmonic analysis. In order to clarify these statements, let us introduce the underlying framework.

To this end, let $\mathbb{R}^{0,m}$ be endowed with a non–degenerate quadratic form of signature $(0, m)$, let $(e_1, \ldots, e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$ and let $\mathbb{R}_{0,m}$ be the real Clifford algebra constructed over $\mathbb{R}^{0,m}$, see e.g. [22]. The non–commutative multiplication in $\mathbb{R}_{0,m}$ is governed by

$$e_j e_k + e_k e_j = -2\delta_{jk}, \qquad j, k = 1, \ldots, m \tag{1}$$

A basis for $\mathbb{R}_{0,m}$ is obtained by considering for each set $A = \{j_1, \ldots, j_h\} \subset \{1, \ldots, m\}$ the element $e_A = e_{j_1} \ldots e_{j_h}$, with $1 \leq j_1 < j_2 < \ldots < j_h \leq m$. For the empty set $\emptyset$ one puts $e_\emptyset = 1$, the identity element. Any Clifford number a in $\mathbb{R}_{0,m}$ may thus be written as $a = \sum_A e_A a_A$, $a_A \in \mathbb{R}$.

When allowing for complex constants, the same set of generators $(e_1, \ldots, e_m)$, still satisfying the anti–commutation rules (1), also produces the complex Clifford algebra $\mathbb{C}_m$, as well as all real Clifford algebras $\mathbb{R}_{p,q}$ of any signature $(p + q = m)$.

The Euclidean space $\mathbb{R}^{0,m}$ is embedded in $\mathbb{R}_{0,m}$ by identifying $(x_1, \ldots, x_m)$ with the Clifford vector

$$\underline{x} = \sum_{j=1}^{m} e_j x_j$$

The multiplication of two vectors $\underline{x}$ and $\underline{y}$ is given by $\underline{x}\,\underline{y} = \underline{x} \bullet \underline{y} + \underline{x} \wedge \underline{y}$ with

$$\begin{aligned}\underline{x} \bullet \underline{y} &= -\sum_{j=1}^{m} x_j y_j = \frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\underline{x}) \\ \underline{x} \wedge \underline{y} &= \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\underline{x})\end{aligned}$$

being the scalar valued dot product (equalling the Euclidean inner product up to a minus sign) and the bivector valued wedge product, respectively. Note that the square of a vector $\underline{x}$ is scalar valued and equals the norm squared up to a minus sign: $\underline{x}^2 = - < \underline{x}, \underline{x} > = -|\underline{x}|^2$.

Conjugation in $\mathbb{R}_{0,m}$ is defined as the anti-involution for which $\bar{e}_j = -e_j$, $j = 1, \ldots, m$. In particular for a vector $\underline{x}$ we have $\bar{\underline{x}} = -\underline{x}$.

The Fourier dual of the vector $\underline{x}$ is the vector valued first order differential operator

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$$

called Dirac operator. It is precisely this Dirac operator which underlies the notion of monogenicity of a function, a notion which may be considered as the higher dimensional counterpart of holomorphy in the complex plane. A function f defined and differentiable in an open region Ω of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$ is called left–monogenic in Ω if $\partial_{\underline{x}}[f] = 0$. In what follows, we will use the concept of inner spherical monogenics; these are homogeneous polynomials $P_k(\underline{x})$ of degree k ($k \in \mathbb{N}$), which are moreover monogenic, i.e. for which it holds that $\partial_{\underline{x}}[P_k](\underline{x}) = 0$. Since the Dirac operator factorizes the Laplacian, $\Delta = -\partial_{\underline{x}}^2$, monogenicity may also be regarded as a refinement of harmonicity; in this sense, spherical monogenics can be seen as refinements of spherical harmonics. The fundamental group leaving the Dirac operator $\partial_{\underline{x}}$ invariant is the special orthogonal group SO(m), doubly covered by the Spin(m) group of the Clifford algebra $\mathbb{R}_{0,m}$. For this reason, the Dirac operator is called a rotation invariant operator. In the present context, we will refer to this setting as the continuous case, as opposed to the discrete setting treated in this paper.

Recently, several authors have shown interest in finding an appropriate framework for the development of discrete counterparts of the basic notions and concepts of Clifford analysis, see a.o. [15, 16, 9, 10, 12]. Some, yet not all, of these contributions are explicitly oriented towards the numerical treatment of problems from potential theory and boundary value problems, rather than towards discrete function theoretic results, see also [17, 18]. In this paper, however, we will abandon the path of possible applications in order to focus on the fundamental features of a concrete model for a Clifford algebra framework in which discrete Dirac operators and the corresponding discrete function theories can be developed, see also [5, 6]. Seen the above mentioned connection between continuous Clifford analysis and complex analysis in the plane, special attention should be paid to the important property of the discrete Dirac operator factorizing a discrete Laplacian. This was also the case in the study of holomorphic functions on $\mathbb{Z}^2$, see e.g. [13, 19, 8] and, more recently [20, 21].

Discrete mathematics always involve graphs; here, we will only consider the simplest of all graphs in Euclidean space, namely the one corresponding to the rectangular $\mathbb{Z}^m$ grid.

2 Definition of a discrete Dirac operator

As announced above, we will consider the natural graph corresponding to the equidistant grid $\mathbb{Z}^m$; thus a Clifford vector $\underline{x}$ as introduced above will now only show integer co–ordinates. For the pointwise discretization of the partial derivatives $\frac{\partial}{\partial x_j}$ we then introduce the traditional one–sided forward and backward differences, respectively given by

$$\Delta_j^+[f](\underline{x}) = f(\ldots, x_j + 1, \ldots) - f(\ldots, x_j, \ldots) = f(\underline{x} + e_j) - f(\underline{x}), \quad j = 1, \ldots, m$$

$$\Delta_j^-[f](\underline{x}) = f(\ldots, x_j, \ldots) - f(\ldots, x_j - 1, \ldots) = f(\underline{x}) - f(\underline{x} - e_j), \quad j = 1, \ldots, m$$

We then first introduce a discrete Laplacian by its usual definition for an arbitrary connected graph.

Definition 1. *Let f be a function defined on the vertices of a connected graph and let $\underline{x}$ be such an arbitrary vertex. Then the action of the discrete Laplace operator on f at $\underline{x}$ is defined by*

$$\Delta f(\underline{x}) = \sum_{\underline{y} \sim \underline{x}} \big(f(\underline{y}) - f(\underline{x})\big) = \sum_{\underline{y} \sim \underline{x}} f(\underline{y}) - \big(\#\mathcal{N}_{\underline{x}}\big)\ f(\underline{x})$$

where the notation $\underline{y} \sim \underline{x}$ means that there is an edge in the graph under consideration which links the vertex $\underline{y}$ to $\underline{x}$, and where $\mathcal{N}_{\underline{x}}$ stands for the neighbourhood of $\underline{x}$ with respect to the graph, i.e. the set of all points $\underline{y} \sim \underline{x}$.

In the present case, with respect to the $\mathbb{Z}^m$ neighbourhood of $\underline{x}$, the above definition explicitly reads

$$\Delta^*[f](\underline{x}) = \sum_{j=1}^{m} \left[\Delta_j^+[f](\underline{x}) - \Delta_j^-[f](\underline{x})\right] = \sum_{j=1}^{m} [f(\underline{x} + e_j) + f(\underline{x} - e_j)] - 2mf(\underline{x}) \qquad (2)$$

where we have denoted the corresponding discrete Laplacian by Δ^*; it is usually called the *star Laplacian* and involves the values of the considered function at the midpoints of the faces of the unit cube centred at $\underline{x}$. Clearly, with respect to the same grid, but changing the graph, other discrete Laplacians may be defined, involving e.g. the function values at the vertices of the cube (the cross Laplacian), or at the midpoints of the "edges".

For now, we restrict ourselves to the star Laplacian (2); note that it can also be written as

$$\Delta^*[f](\underline{x}) = \sum_{j=1}^{m} \Delta_j^+ \Delta_j^- [f](\underline{x}) = \sum_{j=1}^{m} \Delta_j^- \Delta_j^+ [f](\underline{x})$$

When passing to the Dirac operator, we cannot simply combine each discretized partial derivative, be it forward or backward, with the corresponding basis vector e_j, $j = 1, \ldots, m$, since such attempts do not serve our aim at developing a discrete function theory in which the notion of discrete monogenicity implies discrete harmonicity, as has been shown in [5]. Instead, an alternative approach is followed, in which the basis vectors will carry an orientation, just like the forward and backward differences do. To this end, we need to embed the Clifford algebra $\mathbb{R}_{0,m}$ into a bigger one, with an underlying vector space of the double dimension, e.g. $\mathbb{C}_{2m}$, where we consider $2m$ vectors e_j^+ and e_j^-, $j = 1, \ldots, m$, satisfying the following anti–commutator relations:

$$e_j^+ e_k^+ + e_k^+ e_j^+ = -2g_{jk}^+, \quad e_j^- e_k^- + e_k^- e_j^- = -2g_{jk}^-, \quad e_j^+ e_k^- + e_k^- e_j^+ = -2M_{jk}$$

where the symmetric tensors (g_{jk}^+), (g_{jk}^-) and the general tensor (M_{jk}) determine the corresponding metric, see also [12]. Three subsequent assumptions on this metric will now significantly reduce the degrees of freedom in the choice of the metric scalars.

Assumption 1. *The forward and the backward basis vector in each particular cartesian direction add up to the traditional basis vector in that direction, i.e.* $e_j^+ + e_j^- = e_j$, $j = 1, \ldots, m$.

Assumption 2. *There are no preferential cartesian directions, or: all cartesian directions play the same role in the metric. This assumption will be referred to as the principle of dimensional democracy and may be seen as a kind of rotational invariance.*

Assumption 3. *The positive and negative orientations of any cartesian direction play an equivalent role. This assumption may be interpreted as a kind of reflection invariance.*

On the basis of the second and third assumptions, one may put $g_{11}^+ = g_{22}^+ = \ldots = g_{mm}^+ = g_{11}^- = g_{22}^- = \ldots = g_{mm}^- = \lambda$, where $g_{jj}^{\pm} = -(e_j^{\pm})^2$, $j = 1, \ldots, m$, and $M_{11} = M_{22} = \ldots = M_{mm} = \mu$, where $2M_{jj} = -(e_j^+ e_j^- + e_j^- e_j^+)$, $j = 1, \ldots, m$. Furthermore, also $g_{jk}^{\pm}$ and M_{jk}, for $j \neq k$, should be independent of their subscripts, whence we put $g_{jk}^{\pm} = g$ and $M_{jk} = M_{kj} = M$, $j, k = 1, \ldots, m$, $j \neq k$. The first assumption, combined with the traditional Clifford multiplication rules, then leads to the additional conditions $\lambda + \mu = \frac{1}{2}$ and $g + M = 0$. Summarizing, the forward and backward basis vectors e_j^+ and e_j^-, $j = 1, \ldots, m$, will submit to the following multiplication rules:

- $e_j^+ e_k^+ + e_k^+ e_j^+ = e_j^- e_k^- + e_k^- e_j^- = -2g$, $j \neq k$
- $e_j^+ e_k^- + e_k^- e_j^+ = 2g$, $j \neq k$
- $(e_j^+)^2 = (e_j^-)^2 = -\lambda$, $j = 1, \ldots, m$
- $e_j^+ e_j^- + e_j^- e_j^+ = 2\lambda - 1$, $j = 1, \ldots, m$

We are now led to the definition of our discrete Dirac operator.

Definition 2. *The discrete Dirac operator ∂ is the first order, Clifford vector valued difference operator given by*

$$\partial = \partial^+ + \partial^-$$

where the forward and backward discrete Dirac operators ∂^+ and ∂^- are respectively given by

$$\partial^+ = \sum_{j=1}^{m} e_j^+ \Delta_j^+ \quad \textit{and} \quad \partial^- = \sum_{j=1}^{m} e_j^- \Delta_j^-$$

We obtain, using the above multiplication rules, that

$$\partial^2 = -\lambda \sum_{j=1}^{m} (\Delta_j^+ \Delta_j^+ + \Delta_j^- \Delta_j^-) + (2\lambda - 1) \sum_{j=1}^{m} \Delta_j^+ \Delta_j^- + g \sum_{j \neq k} (2\Delta_j^+ \Delta_k^- - \Delta_j^- \Delta_k^- - \Delta_j^+ \Delta_k^+)$$

If we require the support of ∂^2 to remain at least in the unit cube centred at $\underline{x}$, the isotropy of the forward and backward basis vectors needs to be imposed, i.e. we have to put $\lambda = (e_j^+)^2 = (e_j^-)^2 = 0$ as in [12], whence in our case it follows in addition that $\mu = \frac{1}{2}$, or $e_j^+ e_j^- + e_j^- e_j^+ = -1$, $j = 1, \ldots, m$. One thus finally arrives at

- $e_j^+ e_k^+ + e_k^+ e_j^+ = e_j^- e_k^- + e_k^- e_j^- = -2g$, $j \neq k$
- $e_j^+ e_k^- + e_k^- e_j^+ = 2g$, $j \neq k$
- $(e_j^+)^2 = (e_j^-)^2 = 0$, $j = 1, \ldots, m$
- $e_j^+ e_j^- + e_j^- e_j^+ = -1$, $j = 1, \ldots, m$

see also [5]. These relations completely determine the metric of the underlying $2m$–dimensional space in terms of one free scalar parameter g, the metric tensor being given by

$$m_{jk} = \begin{cases} e_j^+ \bullet e_k^+, & j, k = 1, \ldots, m \\ e_j^+ \bullet e_k^-, & j = 1, \ldots, m, \; k = m+1, \ldots, 2m \\ e_j^- \bullet e_k^+, & j = m+1, \ldots, 2m, \; k = 1, \ldots, m \\ e_j^- \bullet e_k^-, & j, k = m+1, \ldots, 2m \end{cases}$$

or explicitly:

$$M = \left(\begin{array}{cccc|cccc} 0 & -g & \cdots & -g & -\frac{1}{2} & g & \cdots & g \\ -g & 0 & \ddots & \vdots & g & -\frac{1}{2} & \ddots & \vdots \\ \vdots & \ddots & 0 & -g & \vdots & \ddots & -\frac{1}{2} & g \\ -g & \cdots & -g & 0 & g & \cdots & g & -\frac{1}{2} \\ \hline -\frac{1}{2} & g & \cdots & g & 0 & -g & \cdots & -g \\ g & -\frac{1}{2} & \ddots & \vdots & -g & 0 & \ddots & \vdots \\ \vdots & \ddots & -\frac{1}{2} & g & \vdots & \ddots & 0 & -g \\ g & \cdots & g & -\frac{1}{2} & -g & \cdots & -g & 0 \end{array}\right)$$

Its determinant reads

$$\det M = (-1)^m \frac{(1+4g)^{m-1}(1-4(m-1)g)}{4^m}$$

whence it should hold that $g \neq -\frac{1}{4}$ and $g \neq \frac{1}{4(m-1)}$, since these specific values would induce a collapse of dimension; for a further discussion of this phenomenon we refer to Section 7. Under the above conditions, ∂^2 takes the form

$$\begin{aligned} \partial^2 &= -\sum_{j=1}^{m} \Delta_j^+ \Delta_j^- + g \sum_{j \neq k} (\Delta_j^+ \Delta_k^- + \Delta_k^+ \Delta_j^- - \Delta_j^- \Delta_k^- - \Delta_j^+ \Delta_k^+) \\ &= (4(m-1)g - 1)\Delta^* - 2g \sum_{j<k} \widetilde{\Delta}_{jk} \end{aligned} \tag{3}$$

where Δ^* is the star Laplacian (2), and

$$\widetilde{\Delta}_{jk} = f(\underline{x} + e_j + e_k) + f(\underline{x} + e_j - e_k) + f(\underline{x} - e_j + e_k) + f(\underline{x} - e_j - e_k) - 4f(\underline{x}), \quad j < k$$

each $\widetilde{\Delta}_{jk}$ being interpretable as a cross Laplacian on the corresponding (e_j, e_k) plane, see also [12]. Note however that the grid points involved in these additional terms do not respect the neighbourhood $\mathcal{N}_{\underline{x}}$ of the vertex $\underline{x}$ in the originally chosen $\mathbb{Z}^m$ graph; we will consider in the next section the particular case where this term disappears. Anyhow, observe that, if (3) is to be interpreted as a similar result to the continuous factorization $\partial_{\underline{x}}^2 = -\Delta$, then we should in fact restrict the metric scalar g to the range $[0, \frac{1}{4(m-1)}[$.

3 Special case: the star Laplacian factorized

In the special case of the above approach where $g = 0$, the defining relations for the forward and backward basis vectors reduce to

- $e_j^+ + e_j^- = e_j$, $j = 1, \ldots, m$
- $\{e_j^+, e_k^+\} = \{e_j^-, e_k^-\} = \{e_j^+, e_k^-\} = 0$, $j, k = 1, \ldots, m$, $j \neq k$

- $(e_j^+)^2 = (e_j^-)^2 = 0$, $j = 1, \ldots, m$
- $\{e_j^+, e_j^-\} = -1$, $j = 1, \ldots, m$

(with the usual notation $\{.,.\}$ for the anti–commutator). This particular choice for the metric scalar causes the second term in (3) to drop, whence we are left with a factorization of the star Laplacian, i.e. $\partial^2 = -\Delta^*$, the support of the involved operators now staying in the $\mathbb{Z}^m$ neighbourhood of $\underline{x}$. As has been remarked in [5, 11], there is a well–known model for these particular forward and backward vectors, namely the so–called Witt basis of the Clifford algebra $\mathbb{C}_{2m}$. In order to understand this model properly, provide $\mathbb{C}_{2m}$ with the structure of a Hermitean space by introducing a so–called complex structure J on the underlying orthogonal space $\mathbb{R}^{0,2m}$, i.e. $J \in \mathrm{SO}(2m)$ with $J^2 = -\mathbf{1}$. For details on the construction, we refer to [1, 2]; for our purpose the following observations are sufficient. Start from the given orthonormal basis $(e_1, \ldots, e_m)$ of $\mathbb{R}^{0,m}$ and complement it with additional vectors $(e_{m+1}, \ldots, e_{2m})$ yielding an orthonormal basis of $\mathbb{R}^{0,2m}$, i.e. $e_j e_k + e_k e_j = -2\delta_{jk}$, $j, k = 1, \ldots, 2m$. Without loss of generality, the complex structure J may always be chosen such that it maps the m–dimensional subspaces spanned by $(e_1, \ldots, e_m)$ and by $(e_{m+1}, \ldots, e_{2m})$ onto each other. A commonly used choice is $J[e_j] = -e_{m+j}$ and $J[e_{m+j}] = e_j$, $j = 1, \ldots, m$, but other choices are possible as well. The Witt basis $(\mathfrak{f}_j, \mathfrak{f}_j^c)_{j=1}^m$ for the complex Clifford algebra $\mathbb{C}_{2m}$ is then obtained through the action of the projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ on the basis elements e_j:

$$\begin{aligned} \mathfrak{f}_j &= \frac{1}{2}(e_j + i\,J[e_j]) = \frac{1}{2}(e_j - i\,e_{m+j}), \qquad j = 1, \ldots, m \\ \mathfrak{f}_j^c &= \frac{1}{2}(e_j - i\,J[e_j]) = \frac{1}{2}(e_j + i\,e_{m+j}), \qquad j = 1, \ldots, m \end{aligned}$$

It holds that $\mathfrak{f}_j + \mathfrak{f}_j^c = e_j$, $j = 1, \ldots, m$ and moreover the Witt basis elements satisfy the Grassmann identities $\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j = \mathfrak{f}_j^c \mathfrak{f}_k^c + \mathfrak{f}_k^c \mathfrak{f}_j^c = 0$, $j, k = 1, \ldots, m$, which also implies their isotropy ($(\mathfrak{f}_j)^2 = (\mathfrak{f}_j^c)^2 = 0$, $j = 1, \ldots, m$), and the duality identities $\mathfrak{f}_j \mathfrak{f}_k^c + \mathfrak{f}_k^c \mathfrak{f}_j = -\delta_{jk}$, $j, k = 1, \ldots, m$. These properties exactly coincide with the above conditions on the vectors e_j^+ and e_j^-, so that we may put $e_j^+ = \mathfrak{f}_j$ and $e_j^- = \mathfrak{f}_j^c$, $j = 1, \ldots, m$ and we are left with the Witt discrete Dirac operator $\partial = \partial^+ + \partial^-$, with $\partial^+ = \sum_{j=1}^m \mathfrak{f}_j\, \Delta_j^+$ and $\partial^- = \sum_{j=1}^m \mathfrak{f}_j^c\, \Delta_j^+$. This setting was already mentioned in [21], however without any function theoretic aims.

4 Discrete monogenic functions

In order to define discrete monogenicity, one first needs some discrete topology. So, consider a bounded set $B \subset \mathbb{Z}^m$ and its characteristic function

$$\psi_B(\underline{x}) = \begin{cases} 1 & \text{if } \underline{x} \in B \\ 0 & \text{if } \underline{x} \notin B \end{cases}$$

as well as the discrete operator

$$\check{\partial} = \sum_{j=1}^m e_j^+ \Delta_j^- + \sum_{j=1}^m e_j^- \Delta_j^+$$

The vector valued function

$$\psi_B \check{\partial} = \sum_{j=1}^{m} e_j^+ \Delta_j^- [\psi_B] + \sum_{j=1}^{m} e_j^- \Delta_j^+ [\psi_B]$$

is called the oriented boundary of B. Observe that $\mathrm{supp}(\psi_B \check{\partial})$ contains points which do not belong to B. In fact, it consists of all vertices the $\mathbb{Z}^m$ neighbourhood of which contains points of both B and $\mathrm{co}(B) \equiv \mathbb{Z}^m \setminus B$. In addition to this definition of the boundary, one may then also define the interior of B (respectively the exterior of B) to be the set of all points of B (respectively of $\mathrm{co}(B)$) which do not belong to $\mathrm{supp}(\psi_B \check{\partial})$. Each bounded set $B \subset \mathbb{Z}^m$ thus gives rise to a partition of $\mathbb{Z}^m$ into its interior, its exterior and the support of its oriented boundary.

The above concepts now allow to give a definition of a discrete monogenic function.

Definition 3. *Let B be a bounded set in $\mathbb{Z}^m$ and let the Clifford algebra valued function f be defined on $B \cup \mathrm{supp}(\psi_B \check{\partial})$. The f is called discrete (left) monogenic in B if and only if it holds that $\partial[f](\underline{x}) = 0$ for all $\underline{x} \in B$.*

Defined in this way, discrete monogenicity constitutes a proper generalization to higher dimension of discrete holomorphy in the Isaacs or the Ferrand sense, see [13, 19]. Moreover, it may be seen as a refinement of discrete harmonicity, since the right hand side of (3) can be interpreted as a generalized discrete Laplacian, also called mixed Laplacian, see [12], which even coincides with the star Laplacian when $g = 0$.

5 Some function theoretic results

We consider Clifford algebra valued functions defined on $\mathbb{Z}^m$.

Then, first of all, a discrete version of Leibniz's rule is obtained by direct calculation. Observe that, as compared to its continuous counterpart, it contains an extra term, which fortunately will turn out to become small when considering finer grids.

Lemma 1 (Leibniz's rule)**.** *Let f and g be Clifford algebra valued functions defined on $\mathbb{Z}^m$. Then*

(i) $\Delta_j^{\pm}[fg] = (\Delta_j^{\pm} f)\, g + f\, (\Delta_j^{\pm} g) \pm (\Delta_j^{\pm} f)\, (\Delta_j^{\pm} g)$;

(ii) if f is scalar–valued, then

$$[fg]\, \check{\partial} = g\, (f \check{\partial}) + f\, (g \check{\partial}) + \sum_{j=1}^{m} \left((\Delta_j^+ f)(\Delta_j^+ g) e_j^- - (\Delta_j^- f)(\Delta_j^- g) e_j^+ \right).$$

Next, the integral of a discrete function f is quite naturally defined as

$$\int f = \sum_{\underline{x}\in\mathbb{Z}^m} f(\underline{x})$$

where, in order to ensure integrability, integrands are required to have compact supports. The following results were then directly obtained, see [6].

Lemma 2 (partial integration)**.** *Let f and g be Clifford algebra valued functions defined on $\mathbb{Z}^m$, where at least one of both has compact support, then*

$$\int f\Delta_j^{\pm}[g] = -\int \Delta_j^{\mp}[f]\, g$$

Lemma 3 (Stokes' theorem)**.** *Let f and g be Clifford algebra valued functions defined on $\mathbb{Z}^m$, where at least one of both has compact support, then*

$$\int f\,(\partial g) = -\int (f\check{\partial})\, g \quad \textit{and} \quad \int f\,(\check{\partial} g) = -\int (f\partial)\, g$$

Observe that the domains of integration on both sides of the formulae in the above lemmata need not to be the same.

On account of Stokes' theorem, one now easily arrives at a first fundamental result.

Theorem 1 (Cauchy's theorem)**.** *Let f be a Clifford algebra valued function defined on $\mathbb{Z}^m$, which is discrete left monogenic in the bounded set B, then*

$$\int (\psi_B\,\check{\partial})\, f = 0$$

Corollary 1. *If B is a bounded set in $\mathbb{Z}^m$, then*

$$\int \psi_B\,\check{\partial} = 0$$

Clearly, for the further development of this function theory, a Cauchy integral formula is essential. So, assume that E is the fundamental solution of operator $\check{\partial}$, i.e.

$$E(\underline{x})\,\check{\partial} = \delta(\underline{x}) = \left\{ \begin{array}{ll} 0, & \underline{x}\neq\underline{0} \\ 1 & \underline{x}=\underline{0} \end{array} \right\} = \prod_{j=1}^{m} \delta_{0\,x_j} \tag{4}$$

and

$$E(\underline{x}-\underline{y})\,\check{\partial} = \delta(\underline{x}-\underline{y}) = \left\{ \begin{array}{ll} 0, & \underline{x}\neq\underline{y} \\ 1 & \underline{x}=\underline{y} \end{array} \right\} = \prod_{j=1}^{m} \delta_{x_j\,y_j} \tag{5}$$

For further use, we then define

$$GT(\underline{x},\underline{y}) = \sum_{j=1}^{m} \left(\Delta_j^{+}[\psi_B(\underline{x})]\Delta_j^{+}[E(\underline{x}-\underline{y})]\, e_j^{-} - \Delta_j^{-}[\psi_B(\underline{x})]\Delta_j^{-}[E(\underline{x}-\underline{y})]\, e_j^{+} \right) \tag{6}$$

The following results were then obtained in [6].

Theorem 2 (Cauchy–Pompeiu formula)**.** *Let B be a bounded set in $\mathbb{Z}^m$ and let f be a Clifford algebra valued function defined on $B \cup supp(\psi_B\, \check{\partial})$, then for all points $\underline{y} \in B$ it holds that*

$$-f(\underline{y}) = \int \psi_B(\underline{x})\, E(\underline{x} - \underline{y})\partial f(\underline{x}) + \int E(\underline{x} - \underline{y})\, (\psi_B \check{\partial})\, f(\underline{x}) + \int GT(\underline{x}, \underline{y})\, f(\underline{x})$$

while for all points $\underline{y} \in co(B)$:

$$0 = \int \psi_B(\underline{x})\, E(\underline{x} - \underline{y})\partial f(\underline{x}) + \int E(\underline{x} - \underline{y})\, (\psi_B \check{\partial})\, f(\underline{x}) + \int GT(\underline{x}, \underline{y})\, f(\underline{x})$$

where $GT(\underline{x}, \underline{y})$ is given by (6).

The first and the second term at the right hand side in the above formulae are 'traditional' terms, representing a volume integral over the bounded set B and a surface integral over the oriented boundary of B, respectively. On the contrary, the third term is an additional one, arising due to the grid (and more precisely: it originates from the additional term already arising in Leibniz's rule). We call this term the 'grid tension' term, which explains the notation $GT(\underline{x}, \underline{y})$, introduced above.

Theorem 3 (Cauchy's integral formula)**.** *Let B be a bounded set in $\mathbb{Z}^m$ and let the function f be discrete monogenic on B, then for all points $\underline{y} \in B$ it holds that*

$$-f(\underline{y}) = \int E(\underline{x} - \underline{y})\, (\psi_B \check{\partial})\, f(\underline{x}) + \int GT(\underline{x}, \underline{y})\, f(\underline{x})$$

while for all points $\underline{y} \in co(B)$:

$$0 = \int E(\underline{x} - \underline{y})\, (\psi_B \check{\partial})\, f(\underline{x}) + \int GT(\underline{x}, \underline{y})\, f(\underline{x})$$

where $GT(\underline{x}, \underline{y})$ is given by (6).

Obviously, in the above results, an essential role is played by the so–called fundamental solution $E(\underline{x})$, defined by (4)–(5). In order to obtain $E(\underline{x})$ explicitly, we will pass to frequency space by means of the discrete–time Fourier transform, defined for a discrete Clifford algebra valued function $f(\underline{x})$ with compact support as follows:

$$\mathcal{F}[f(\underline{x})](\underline{\xi}) = \int f(\underline{x})\, \exp(-i\langle \underline{\xi}, \underline{x}\rangle) = \sum_{\underline{x} \in \mathbb{Z}^m} \exp(-i\langle \underline{\xi}, \underline{x}\rangle)\, f(\underline{x}), \qquad \underline{\xi} \in \mathbb{Z}^m \tag{7}$$

and yielding a periodic function of $\underline{\xi}$ with period $(2\pi)^m$. Elementary properties of this discrete–time Fourier transform are listed in the following lemma.

Lemma 4. *Let $f(\underline{x})$ be a Clifford algebra valued function defined on $\mathbb{Z}^m$ with compact support and let its discrete–time Fourier transform be given by (7), then it holds that*

- $\mathcal{F}[f(\underline{x} \pm e_j)](\underline{\xi}) = \exp(\pm i\xi_j)\, \mathcal{F}[f(\underline{x})](\underline{\xi})$;

- $\mathcal{F}[\Delta_j^{\pm} f(\underline{x})](\underline{\xi}) = \mp(1 - \exp(\pm i\xi_j))\,\mathcal{F}[f(\underline{x})](\underline{\xi})$;
- $\mathcal{F}[f(\underline{x})\,\check{\partial}](\underline{\xi}) = \mathcal{F}[f(\underline{x})](\underline{\xi})\,G(\underline{\xi})$, *where*

$$G(\underline{\xi}) = \sum_{j=1}^{m} \left[(1 - \exp(-i\xi_j))\, e_j^{+} + (\exp(i\xi_j) - 1)\, e_j^{-}\right] \tag{8}$$

- $\mathcal{F}[\delta(\underline{x})](\underline{\xi}) = 1$.

On account of these calculus rules, it was then obtained in [6] that

$$\hat{E}(\underline{\xi}) \equiv \mathcal{F}[E(\underline{x})](\underline{\xi}) = \frac{G(\underline{\xi})}{(G(\underline{\xi}))^2}, \qquad \text{wherever } G(\underline{\xi}) \neq 0 \tag{9}$$

with $G(\underline{\xi})$ being given by (8).

In Section 7, $G(\underline{\xi})$ and $\hat{E}(\underline{\xi})$ are obtained even more explicitly, when passing to a concrete model for the Clifford forward and backward bases.

6 Discrete monogenic polynomials

Here our aim is to establish a notion of discrete spherical monogenic, i.e. the discrete counterpart of a monogenic homogeneous polynomial. To this end, one should observe that, for polynomials, it is not necessary to distinguish between the continuous and the discrete world. Indeed, if a polynomial is defined in the continuous variable $\underline{x} \in \mathbb{R}^m$, then it is trivially defined on $\mathbb{Z}^m$. Conversely, for each polynomial $P(\underline{x})$, there exists a number N such that, if $P(\underline{x})$ is defined on a subset $A \subset \mathbb{Z}^m$, with $|A| = N$, then $P(\underline{x})$ is well–defined in the whole of $\mathbb{R}^m$. So we are able to use at the same time derivatives and differences of polynomials.

For further use, we list a few auxiliary results in this respect, see also [7].

Lemma 5. *The operators $\Delta_j^{\pm} - \partial_{x_j}$, $j = 1, \ldots, m$, turn a homogeneous polynomial of degree k into a polynomial of degree $(k-2)$.*

Corollary 2. *The operator $\partial - \partial_{\underline{x}}$ turns a homogeneous polynomial of degree k into a polynomial of degree $(k-2)$.*

Corollary 3. *A homogeneous polynomial of degree k is left monogenic if and only if the discrete Dirac operator ∂ turns it into a polynomial of degree $(k-2)$.*

Proposition 1. *Let $L_k(\underline{x})$ be a polynomial of degree k, and let $P_k(\underline{x})$ be its homogeneous part of degree k, i.e. let*

$$L_k(\underline{x}) = P_k(\underline{x}) + R_{k-1}(\underline{x})$$

the meaning of $R_{k-1}(\underline{x})$ *being obvious. If* $L_k(\underline{x})$ *is discrete monogenic, i.e.* $\partial[L_k](\underline{x}) = 0$*, then* $P_k(\underline{x})$ *is an inner spherical monogenic, i.e.* $\partial_{\underline{x}}[P_k](\underline{x}) = 0$.

Corollary 4. *A homogeneous discrete monogenic polynomial is automatically an inner spherical monogenic.*

Although, fortunately, the converse is not true, the above corollary nevertheless indicates that it makes no sense to define an inner spherical discrete monogenic to be a discrete monogenic homogeneous polynomial. The question thus raises if an inner spherical monogenic can be completed, possibly uniquely, to a discrete monogenic polynomial of the same degree. The answer is given in the proposition below.

Proposition 2. *Let* $P_k(\underline{x})$ *be an inner spherical monogenic of degree* k*. Then there exists a unique polynomial* R_{k-2} *of degree* $k-2$*, such that*

$$Q_k(\underline{x}) = P_k(\underline{x}) - \underline{x}\, R_{k-2}(\underline{x})$$

is a discrete monogenic polynomial of degree k*.*

This induces the following fundamental result.

Theorem 4. *A discrete monogenic polynomial* $L_k(\underline{x})$ *of degree* k *may be uniquely decomposed as*

$$L_k(\underline{x}) = Q_k(\underline{x}) + L_{k-1}(\underline{x})$$

where $Q_k(\underline{x})$ *is a discrete monogenic polynomial of degree* k *showing the specific form*

$$Q_k(\underline{x}) = P_k(\underline{x}) - \underline{x}\, R_{k-2}(\underline{x})$$

$P_k(\underline{x})$ *being an inner spherical monogenic, and where* $L_{k-1}(\underline{x})$ *is a discrete monogenic polynomial of degree* $(k-1)$*.*

The above observations now give rise to the following definition.

Definition 4. *A discrete monogenic polynomial* $Q_k(\underline{x})$ *of degree* k*, showing the specific form*

$$Q_k(\underline{x}) = P_k(\underline{x}) - \underline{x}\, R_{k-2}(\underline{x})$$

$P_k(\underline{x})$ *being an inner spherical monogenic, is called an inner spherical discrete monogenic of degree* k*.*

By subsequent application of the above theorem, we may now conclude the following.

Corollary 5. *For each discrete monogenic polynomial* $L_k(\underline{x})$ *of degree* k*, there exists a unique set of inner spherical discrete monogenics* $(Q_j(\underline{x}))_{j=0}^{k}$*, such that*

$$L_k(\underline{x}) = Q_k(\underline{x}) + Q_{k-1}(\underline{x}) + \ldots + Q_1(\underline{x}) + Q_0(\underline{x})$$

7 A model for the forward and backward basis vectors

In Section 2 we have introduced our discrete Dirac operator with respect to the $\mathbb{Z}^m$ graph, a crucial role in its definition being played by the so–called forward and backward Clifford basis vectors e_j^+ and e_j^-, $j = 1, \ldots, m$, for which we have already provided a concrete model in the special case treated in Section 3. In this section, a feasible model is given in the general case where the metric scalar g does not equal zero, see also [5].

To this end so–called curvature vectors B_j, $j = 1, \ldots, m$ are introduced, by means of which one puts

$$e_j^+ = \frac{1}{2}(e_j + B_j) \quad \text{and} \quad e_j^- = \frac{1}{2}(e_j - B_j), \qquad j = 1, \ldots, m$$

meanwhile ensuring that $e_j^+ + e_j^- = e_j$, $j = 1, \ldots, m$. As these forward and backward Clifford vectors should satisfy the relations derived in Section 2, it should hold that

$$\begin{cases} B_j^2 = +1 \\ \{e_j, B_j\} = 2(e_j \bullet B_j) = 0 \end{cases} \qquad j = 1, \ldots, m \tag{10}$$

and that

$$\begin{cases} \{e_k, B_j\} = 2(e_k \bullet B_j) = 0 \\ \{B_k, B_j\} = 2(B_k \bullet B_j) = -8g \end{cases} \qquad j, k = 1, \ldots, m, \; j \neq k \tag{11}$$

Note that the second condition in (10) and the first one in (11) together express the orthogonality of the space spanned by the curvature vectors and the original m-dimensional space with basis $(e_1, \ldots, e_m)$. As a consequence, the curvature vectors may be written explicitly as

$$B_j = \sum_{\ell=1}^{m} b_j^{(\ell)} (ie_{m+\ell}) = \sum_{\ell=1}^{m} b_j^{(\ell)} \epsilon_\ell, \qquad j = 1, \ldots, m$$

where $\epsilon_\ell^2 = (ie_{m+\ell})^2 = +1$, $\ell = 1, \ldots, m$ and $\sum_{\ell=1}^{m} (b_j^{(\ell)})^2 = 1$, $j = 1, \ldots, m$. Note that here the Clifford dot product of any two curvature vectors equals their Euclidean inner product, these inner products all being equal to the same scalar $-4g$. $(B_1, \ldots, B_m)$ may thus be interpreted as a set of vectors on the unit sphere S^{m-1} of $\mathbb{R}^m$, containing two by two the same fixed angle α, with $\cos(\alpha) = -4g$. To this end the metric scalar g needs to be restricted to the interval $]-\frac{1}{4}, \frac{1}{4}]$, creating then a kind of 'umbrella' of vectors, which will open and close according to varying g. In particular, if $g = 0$ then $\alpha = \frac{\pi}{2}$, in agreement with the Witt case of Section 3.

The above relations (10)–(11) are summarized in the metric tensor $\widetilde{M}$:

$$\widetilde{M} = \left(\begin{array}{ccccc|ccccc} -1 & 0 & \cdots & \cdots & 0 & & & & & \\ 0 & -1 & 0 & \ddots & \vdots & & & & & \\ \vdots & 0 & -1 & 0 & \vdots & & & 0 & & \\ \vdots & \ddots & 0 & -1 & 0 & & & & & \\ 0 & \cdots & \cdots & 0 & -1 & & & & & \\ \hline & & & & & +1 & -4g & \cdots & \cdots & -4g \\ & & & & & -4g & +1 & -4g & \ddots & \vdots \\ & & 0 & & & \vdots & -4g & +1 & -4g & \vdots \\ & & & & & \vdots & \ddots & -4g & +1 & -4g \\ & & & & & -4g & \cdots & \cdots & -4g & +1 \end{array}\right)$$

its entries being equal to the Clifford dot products of the vectors $(e_1, \ldots, e_m, B_1, \ldots, B_m)$, in this specific order. Its determinant equalling $(-1)^m(1+4g)^{m-1}(1-4(m-1)g)$, we are again confronted with the non–admissible values $-\frac{1}{4}$ and $\frac{1}{4(m-1)}$ for the metric scalar g, already obtained in Section 2. Indeed, in those cases we no longer dispose of a basis for a $2m$–dimensional space: instead, for $g = \frac{1}{4(m-1)}$ we have that rank $(\widetilde{M}) = 2m-1$, while for $g = -\frac{1}{4}$ we have rank $(\widetilde{M}) = m+1$. We will further comment on this from a geometrical point of view. To this end, first take $g = -\frac{1}{4}$. Here $B_k \bullet B_j = \langle B_k, B_j \rangle = +1$, $j, k = 1, \ldots, m$, whence their contained angle α is zero. So, the 'umbrella' completely closes, all curvature vectors coincide and the dimension of the space spanned by them becomes 1, in accordance with the rank of the metric tensor. In the case where $g = \frac{1}{4(m-1)}$, the rank of $\widetilde{M}$ shows that the space spanned by the curvature vectors should be $(m-1)$–dimensional, i.e. they should be on the intersection of the unit sphere S^{m-1} with a hyperplane in m–dimensional space. In [5], the contained angle of the vectors in this situation has been explicitly determined for dimensions $m = 3$ and $m = 4$, showing that it indeed corresponds to the given value of g.

Remark 1. *It is worth noting that, in this concrete model for the forward and backward Clifford bases, one has*

$$G(\underline{\xi}) = \sum_{j=1}^{m} [(1 - \cos \xi_j)\, B_j + i\, \sin \xi_j\, e_j]$$

and

$$(G(\underline{\xi}))^2 = 4 \sum_{j=1}^{m} \sin^2 \frac{\xi_j}{2} - 32g \sum_{j<k} \sin^2 \frac{\xi_j}{2} \sin^2 \frac{\xi_k}{2}$$

whence the fundamental solution $\hat{E}(\underline{\xi})$ *in frequency space, (9), explicitly reads*

$$\hat{E}(\underline{\xi}) = \frac{1}{4} \frac{\displaystyle\sum_{j=1}^{m} \left[(1 - \cos \xi_j)\, B_j + i\, \sin \xi_j\, e_j\right]}{\displaystyle\sum_{j=1}^{m} \sin^2 \frac{\xi_j}{2} - 8g \sum_{j<k} \sin^2 \frac{\xi_j}{2} \sin^2 \frac{\xi_k}{2}}$$

This explicit expression also allows to investigate when the denominator of $\hat{E}(\underline{\xi})$ *will be zero (i.e., when* $G(\underline{\xi}) = 0$*), see [6] for the treatment of low dimensional cases.*

It is our intention to extend this first model in a forthcoming paper, taking into account generalized curvature tensors controlling the support of all involved operators.

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On mapping properties of monogenic functions

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ABSTRACT

Main goal of this paper is to study the description of monogenic functions by their geometric mapping properties. At first monogenic functions are studied as general quasi-conformal mappings. Moreover, dilatations and distortions of these mappings are estimated in terms of the hypercomplex derivative. Then pointwise estimates from below and from above are given by using a generalized Bohr's theorem and a Borel-Carathéodory theorem for monogenic functions. Finally it will be shown that monogenic functions can be defined as mappings which map infinitesimal balls to special ellipsoids.

RESUMEN

El principal objetivo de este artículo es estudiar la descripción de funciones monogénicas a través de las propiedades geométricas de sus mapeos. Primero son estudiadas funciones monogénicas como aplicaciones casi-conformes generales. Además, dilataciones y distorciones de estas aplicaciones son estimadas en términos de la derivada hipercompleja. Entonces estimativas puntuales por abajo y por arriba son dadas usando un teorema de Bohr generalizado y un Teorema de Borel-Carathéodory para funciones monogénicas. Finalmente es demostrado que funciones monogénicas pueden ser definidas como mapeos que aplican bolas infinitesimales en elipsoides especiales.

Key words and phrases: *monogenic functions, quasi-conformal mappings, geometric mapping properties.*

Math. Subj. Class.: *30G35.*

1 Introduction

Quaternionic analysis provides us with a spatial analogue of the one-dimensional complex function theory in the plane. Generalizing ideas from the complex case to the higher dimensional real Euclidean space, quaternionic analysis is applied to construct solutions of some classes of partial differential equations for vector-valued functions in three or four dimensions (see, e.g., [17, 18, 31, 32]), involving the Dirac operator or a generalized Cauchy-Riemann operator, respectively.

We are mainly interested in the study of spatial generalizations of holomorphic or anti-holomorphic functions. The crucial fact of such functions (transformations) is that they describe essentially mappings of the unit disk onto or into the unit disk transforming partial differential equations to other differential equations and moreover, they embrace Cauchy's inequalities, maximum modulus principle, Bohr's Theorem, Schwarz Lemma, Hadamard Theorem and others. Since they provide with the best description of the pointwise behavior from a given function, at first one has to ask whether or not those results can be generalized to "holomorphic" (resp. anti-holomorphic) functions in higher dimensions (sections 3 and 4). In addition, to deal with these results it is also necessary to study the mapping properties of such functions more detailed. It is already known, as the ideas of [7] and [5] show, that these functions preserve geometrical properties like length, distance and angles, while mapping domains to the ball. Therefore they are applied as well for the transformation of differential equations. Our main task is to characterize such mappings which map technically relevant domains to mathematically simple domains and to find out if such class of functions leave the differential equations and/or some geometrical properties invariant.

At this point it is of interest to know that in contrast to the situation in the plane, the set of conformal mappings is restricted only to the set of Möbius transformations. But the theory of generalized holomorphic functions (by historical reasons they are also called monogenic functions, cf. [8]) does not cover the set of Möbius tranformations in $\mathbb{R}^{n+1}$, and since the Möbius transformations are not monogenic, one can only expect that monogenic functions represent certain quasi-conformal mappings. On the other hand, the class of all quasi-conformal mappings is much bigger than the class of monogenic functions. The question arises if monogenic functions correspond to a special subclass of quasi-conformal mappings.

In the paper [28], the concept of monogenic-conformal mappings realized by functions in $\mathbb{R}^{n+1}$ and with values in the Clifford algebra $Cl_{0,n}$ was already considered. Together with the geometric interpretation of the hypercomplex derivative (see [17]), dilatations and distortions of these mappings can be estimated. Compared with the related work, the advantage of our approach lies in the possibility to study the description of monogenic functions by their geometric mapping properties. The local mapping properties of a monogenic function or of a real analytic function are mainly determined by the behaviour of the linear part of their Taylor expansions. According to this, in [23] the geometric behaviour of the linear part of a monogenic function was considered.

As a consequence, it is shown that monogenic functions can be defined as mappings which map infinitesimal balls to special ellipsoids and vice versa. Here we extend this result considering the all series expansion of a function.

The text is based on recent publications of the authors [19, 20, 21, 22, 23] and contains new material as well.

2 Preliminaries

Let $\mathbb{H} := \{\mathbf{a} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3,\ \ a_i \in \mathbb{R},\ i = 0,1,2,3\}$ be the algebra of the real quaternions, where the imaginary units $\mathbf{e}_i$ $(i = 1,2,3)$ are subject to the multiplication rules

$$\begin{aligned} &\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1, \\ &\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3 = -\mathbf{e}_2\mathbf{e}_1,\ \mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1 = -\mathbf{e}_3\mathbf{e}_2,\ \mathbf{e}_3\mathbf{e}_1 = \mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_3. \end{aligned}$$

Through this paper we shall denote by $\mathbf{Sc}(\mathbf{a}) := a_0$ the scalar part of $\mathbf{a}$ and by $\mathbf{Vec}(\mathbf{a}) := a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ its vector part. Analogously to the complex case, the (quaternion-)conjugate element of $\mathbf{a}$ is the quaternion

$$\overline{\mathbf{a}} := \mathbf{Sc}(\mathbf{a}) - \mathbf{Vec}(\mathbf{a}) = a_0 - a_1\mathbf{e}_1 - a_2\mathbf{e}_2 - a_3\mathbf{e}_3.$$

Also, we shall use the Euclidean norm $|\mathbf{a}|^2 = \mathbf{a}\overline{\mathbf{a}} = \left(a_0^2 + a_1^2 + a_2^2 + a_3^2\right)^{1/2}$. The real vector space $\mathbb{R}^3$ is to be embedded in the subset $\mathcal{A} := span_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{H}$ via the identification of each element $\mathbf{x} = (x_0, \underline{x}) = (x_0, x_1, x_2) \in \mathbb{R}^3$ with the paravector (also called reduced quaternion)

$$\mathbf{x} := x_0 + \underline{x} = x_0 + x_1\mathbf{e}_1 + x_2\mathbf{e}_2 \in \mathcal{A}.$$

As a consequence, no distinction will be made between $\mathbf{x}$ as a point in $\mathbb{R}^3$ or its correspondent reduced quaternion. Also, we emphasize that $\mathcal{A}$ is only a real vector space but not a subalgebra of $\mathbb{H}$. For more details on the real algebra of quaternions we refer e.g. to [8], [31], [18], [15].

Let now Ω be an open subset of $\mathbb{R}^3$ with piecewise smooth boundary. A quaternion-valued function or, briefly, an $\mathbb{H}$-valued function is a mapping $f : \Omega \longrightarrow \mathbb{H}$ such that

$$f(\mathbf{x}) = \sum_{i=0}^{3} f_i(\mathbf{x})\mathbf{e}_i,$$

where $\mathbf{e}_0 = 1$ and the coordinate-functions f_i $(i = 0,1,2,3)$ are real-valued in Ω. Properties such as continuity, differentiability or integrability are ascribed coordinate-wisely.

For continuously real-differentiable functions $f : \Omega \longrightarrow \mathbb{H}$, the operator

$$D = \partial_{x_0} + \mathbf{e}_1\partial_{x_1} + \mathbf{e}_2\partial_{x_2} \tag{1}$$

is called generalized Cauchy-Riemann operator. The conjugate generalized Cauchy-Riemann operator is defined by

$$\overline{D} = \partial_{x_0} - \mathbf{e}_1\partial_{x_1} - \mathbf{e}_2\partial_{x_2}. \tag{2}$$

A function $f : \Omega \longrightarrow \mathbb{H}$ is called *left* (resp. *right*) *monogenic* in Ω if

$$Df = 0 \ \text{ in } \ \Omega \ \ (\text{resp.}, fD = 0 \ \text{ in } \ \Omega).$$

Remark 1. *In general, left (resp. right) monogenic functions are not right (resp. left) monogenic. From now on, we refer only to left monogenic functions. For simplicity, we will call them monogenic. However, all results achieved to left monogenic functions can easily be adapted to right monogenic functions.*

The generalized Cauchy-Riemann operator (1) and its conjugate (2) factorize the Laplace operator in $\mathbb{R}^3$. In fact, it holds

$$\Delta_3 f = D\overline{D}f = \overline{D}Df,$$

which implies that any monogenic function is also a harmonic function. Analogously as in the complex one-dimensional case $\frac{1}{2}\bar{D}$ defines a derivative of monogenic functions. This was shown in [16], where $\frac{1}{2}\bar{D}f$ was called *hypercomplex derivative* of f.

A monogenic function $f : \Omega \longrightarrow \mathbb{H}$ with an identically vanishing hypercomplex derivative (i.e. a function from $\ker D \cap \ker \bar{D}$) is called hyperholomorphic constant (see again [16]). It is immediately clear that such function depends only on x_1 and x_2.

Additionally, we introduce the following notations: $B_R := B_R(0)$ will denote the ball of radius R in $\mathbb{R}^3$ centered at the origin, $S_R = \partial B_R$ its boundary and $d\sigma_R$ (resp. dV_R) the Lebesgue measure on S_R (resp. B_R). For simplicity, in the case $R = 1$ we omit R in the notations. We will also denote by $L_2(S_R; \mathbb{X}; \mathbb{R})$ (resp. $L_2(B_R; \mathbb{X}; \mathbb{R})$) the $\mathbb{R}$-linear Hilbert space of square integrable functions on S_R (resp. B_R) with values in $\mathbb{X}$ ($\mathbb{X} = \mathbb{R}$ or $\mathcal{A}$). In the case $\mathbb{X} = \mathbb{R}$ we abbreviate $L_2(S_R; \mathbb{R}; \mathbb{R})$ (resp. $L_2(B_R; \mathbb{R}; \mathbb{R})$) briefly by $L_2(S_R)$ (resp. $L_2(B_R)$). Also, the real-valued inner product in $L_2(S_R; \mathcal{A}; \mathbb{R})$ (resp. $L_2(B_R; \mathcal{A}; \mathbb{R})$) is given by

$$\langle f, g \rangle_{L_2(S_R;\mathcal{A};\mathbb{R})} = \int_{S_R} \mathbf{Sc}(\overline{f}g)d\sigma_R, \tag{3}$$

respectively,

$$\langle f, g \rangle_{L_2(B_R;\mathcal{A};\mathbb{R})} = \int_{B_R} \mathbf{Sc}(\overline{f}g)dV_R, \tag{4}$$

for any $f, g \in L_2(S_R; \mathcal{A}; \mathbb{R})$ (resp. $L_2(B_R; \mathcal{A}; \mathbb{R})$). Each homogeneous harmonic polynomial P_n of degree n can be written in spherical coordinates as

$$P_n(x) = R^n P_n(\omega), \ \omega \in S_R, \tag{5}$$

its restriction, $P_n(\omega)$, to the boundary of the ball B_R is called *spherical harmonic*[1] of degree n. From (5), it is clear that a homogeneous polynomial is determined by its restriction to S_R. Denoting by $\mathcal{H}_n(S_R)$ the space of real-valued spherical harmonics of degree n on S_R, it is well-known (see [2] and [29]) that

$$\dim \mathcal{H}_n(S_R) = 2n + 1.$$

It is also known (see [2] and [29]) if $n \neq m$, the spaces $\mathcal{H}_n(S_R)$ and $\mathcal{H}_m(S_R)$ are orthogonal in $L_2(S_R)$.

Let us denote the homogeneous monogenic polynomials of degree n by H_n. In an analogous way to the spherical harmonics, the restriction of H_n to the boundary of the ball B_R is called *spherical monogenic*[2] of degree n. Now let $M^+(\Omega; \mathcal{A}; n)$ be the space of $\mathcal{A}$-valued homogeneous monogenic polynomials of degree n in $\Omega \subset \mathbb{R}^3$. In [27], it is shown that the space $M^+(\Omega; \mathcal{A}; n)$ has dimension $2n + 3$. Later, this result was generalized for arbitrary higher dimensions by R. Delanghe in [12].

Consider, for each $n \in \mathbb{N}_0$, a basis $\{H_n^\nu : \nu 1, ..., \dim M^+(\Omega; \mathcal{A}; n)\}$ of $M^+(\Omega; \mathcal{A}; n)$. Since the coordinates of H_n^ν are harmonic, for arbitrary $n, k = 0, 1, ...$, we have

$$\|H_n^\nu\|^2_{L_2(B_R;\mathcal{A};\mathbb{R})} = \frac{R^{2n+3}}{2n+3} \|H_n^\nu\|^2_{L_2(S;\mathcal{A};\mathbb{R})}. \tag{6}$$

3 Homogeneous Monogenic Polynomials

In [9] and [10], a special $\mathbb{R}$-linear complete orthonormal system of $\mathcal{A}$-valued homogeneous monogenic polynomials defined in the unit ball of $\mathbb{R}^3$ is explicitly constructed. The main idea of this construction is based on the already referred factorization of the Laplace operator. The authors took a system of real-valued homogeneous harmonic polynomials and applied the $\overline{D}$ operator in order to obtain a system of $\mathcal{A}$-valued homogeneous monogenic polynomials.

For an easier description, we introduce spherical coordinates

$$x_0 = r\cos\theta,\ x_1 = r\sin\theta\cos\varphi,\ x_2 = r\sin\theta\sin\varphi,$$

where $0 < r < \infty$, $0 < \theta \leq \pi$, $0 < \varphi \leq 2\pi$. Each point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{R}^3 \setminus \{0\}$ admits a unique representation $\mathbf{x} = r\boldsymbol{\omega}$, where $r|\mathbf{x}|$ and $|\boldsymbol{\omega}| = 1$. Therefore, $\omega_i = \frac{x_i}{r}$ for $i = 0, 1, 2$.

As described, the homogeneous monogenic polynomials (solid spherical monogenics)

$$\{X_n^{0,\dagger},\ X_n^{m,\dagger},\ Y_n^{m,\dagger} : m = 1, ..., n+1\}, \tag{7}$$

[1]This restriction is also called surface spherical harmonic by some authors (see [8]).
[2]Such restriction is also called surface inner spherical monogenic (see [8]).

formed by the extensions to the ball of $\{X_n^0, X_n^m, Y_n^m : m = 1, ..., n+1\}$ are obtained by applying the operator $\frac{1}{2}\overline{D}$ to the system of homogeneous harmonic polynomials

$$U_{n+1}^{0,\dagger}, U_{n+1}^{m,\dagger}, V_{n+1}^{m,\dagger}, \quad m = 1, \ldots, n+1,$$

with the notations

$$\begin{aligned} U_{n+1}^{0,\dagger} &:= r^{n+1} U_{n+1}^0 = r^{n+1} P_{n+1}(\cos\theta) \\ U_{n+1}^{m,\dagger} &:= r^{n+1} U_{n+1}^m = r^{n+1} P_{n+1}^m(\cos\theta)\cos m\varphi \\ V_{n+1}^{m,\dagger} &:= r^{n+1} V_{n+1}^m = r^{n+1} P_{n+1}^m(\cos\theta)\sin m\varphi, \ m = 1, ..., n+1. \end{aligned}$$

Hereby P_{n+1} stands for the Legendre polynomial of degree $n+1$ and the functions P_{n+1}^m are the *associated Legendre functions*[3]. The set $\{U_{n+1}^0, U_{n+1}^m, V_{n+1}^m : m = 1, \ldots, n+1\}$ denotes the standard orthogonal basis of spherical harmonics of degree $n+1$ in $\mathbb{R}^3$ (considered, e.g., in [34]) with respect to the inner product

$$\langle f, g \rangle_{L_2(S)} = \int_S f g \, d\sigma.$$

Moreover, their norms are given by

$$\begin{aligned} \| U_{n+1}^0 \|_{L_2(S)} &= 2\sqrt{\frac{\pi}{2n+3}} \\ \| U_{n+1}^m \|_{L_2(S)} &= \| V_{n+1}^m \|_{L_2(S)} = \sqrt{\frac{2\pi}{2n+3}\frac{(n+1+m)!}{(n+1-m)!}}. \end{aligned}$$

We begin by considering the following norm estimates already obtained in [9] which will be used later on.

Proposition 1. *For a given fixed $n \in \mathbb{N}_0$, the spherical monogenics X_n^0, X_n^m and Y_n^m $(m = 1, \ldots, n+1)$ are orthogonal to each other with respect to the inner product* (3) *and their norms are given by*

$$\begin{aligned} \| X_n^0 \|_{L_2(S;\mathcal{A};\mathbb{R})} &= \sqrt{\pi(n+1)} \\ \| X_n^m \|_{L_2(S;\mathcal{A};\mathbb{R})} &= \| Y_n^m \|_{L_2(S;\mathcal{A};\mathbb{R})} = \sqrt{\frac{\pi}{2}(n+1)\frac{(n+1+m)!}{(n+1-m)!}}. \end{aligned}$$

Remark 2. *A similar result can be obtained for the homogeneous monogenic polynomials* (7) *if one takes into account relation* (6).

[3]These functions were introduced in 1877 by Ferrers. For that reason, some authors (c.f. [34]) call them *Ferrers functions*.

In the second and third sections we will look more closely to the pointwise behavior of a given function. For that reason in what follows we present pointwise estimates of our basis polynomials (7), already obtained by the authors in [21].

Proposition 2. *Let $n \in \mathbb{N}_0$. For the homogeneous monogenic polynomials* (7) *the following estimates hold:*

$$\begin{aligned} |X_n^{0,\dagger}(\mathbf{x})| &\leq \frac{1}{2\sqrt{\pi}}(n+1)(2r)^n \parallel X_n^0 \parallel_{L_2(S;\mathcal{A};\mathbb{R})} \\ |X_n^{m,\dagger}(\mathbf{x})| &\leq \frac{1}{2\sqrt{\pi}}(n+1)(2r)^n \parallel X_n^m \parallel_{L_2(S;\mathcal{A};\mathbb{R})} \\ |Y_n^{m,\dagger}(\mathbf{x})| &\leq \frac{1}{2\sqrt{\pi}}(n+1)(2r)^n \parallel Y_n^m \parallel_{L_2(S;\mathcal{A};\mathbb{R})}, \end{aligned}$$

with $m = 1, \ldots, n+1$.

An interesting point to note here is that the real part of these polynomials are again related with the set $\{U_n^0, U_n^m, V_n^m : m = 1, \ldots, n\}$. These relations are given in the next theorem:

Theorem 1. *Given a fixed $n \in \mathbb{N}_0$, we have the following relations:*

$$\begin{aligned} \mathbf{Sc}(X_n^0) &= \frac{(n+1)}{2}\, U_n^0(\theta,\varphi) \\ \mathbf{Sc}(X_n^m) &= \frac{(n+m+1)}{2}\, U_n^m(\theta,\varphi) \\ \mathbf{Sc}(Y_n^m) &= \frac{(n+m+1)}{2}\, V_n^m(\theta,\varphi), \end{aligned}$$

for $m = 1, \ldots, n$.

Proof. We just prove the relation for the spherical harmonics $\mathbf{Sc}(X_n^l)$ $(l = 0, \ldots, n)$. The proof for $\mathbf{Sc}(Y_n^m)$ $(m = 1, \ldots, n)$ is similar. Taking results from [9], the real part of the spherical monogenics X_n^l $(l = 0, ..., n)$ is given by

$$\mathbf{Sc}(X_n^l) = A^{l,n}(\theta)\cos(l\varphi)$$

with

$$A^{l,n}(\theta) = \frac{1}{2}\left(\sin^2\theta \frac{d}{dt}\left[P_{n+1}^l(t)\right]_{t=\cos\theta} + (n+1)\cos\theta P_{n+1}^l(\cos\theta)\right).$$

It is well known that the Legendre polynomials, together with the associated Legendre functions, satisfy in particular the recurrence formula

$$(1-t^2)(P_{n+1}^l(t))' = (n+l+1)P_n^l(t) - (n+1)tP_{n+1}^l(t), \tag{8}$$

for $l = 0, \ldots, n+1$. Now, making the change of variable $\cos\theta = t$ in $A^{l,n}(\theta)$ and using the previous recurrence formula, it follows immediately that

$$\begin{aligned} A^{l,n}(\arccos t) &= \frac{1}{2}\left[(1-t^2)(P^l_{n+1}(t))' + (n+1)tP^l_{n+1}(t)\right] \\ &= \frac{(n+l+1)}{2} P^l_n(t). \end{aligned}$$

Making again a change of variable $t = \cos\theta$ our statement is proved. □

As a consequence, it turns out the result:

Theorem 2. *For a fixed $n \in \mathbb{N}_0$, the spherical harmonics*

$$\left\{\mathbf{Sc}(X^0_n), \mathbf{Sc}(X^m_n), \mathbf{Sc}(Y^m_n) : \; m = 1, \ldots, n\right\}$$

are orthogonal in $L_2(S)$.

With such relations together with the norms of the spherical harmonics, we are ready to establish, as well, the L_2-norms of $\mathbf{Sc}(X^0_n)$, $\mathbf{Sc}(X^m_n)$ and $\mathbf{Sc}(Y^m_n)$ $(m = 1, ..., n)$.

Proposition 3. *For a fixed $n \in \mathbb{N}_0$, the norms of the spherical harmonics $\mathbf{Sc}(X^0_n)$, $\mathbf{Sc}(X^m_n)$ and $\mathbf{Sc}(Y^m_n)$ are given by*

$$\|\mathbf{Sc}(X^0_n)\|_{L_2(S)} = (n+1)\sqrt{\frac{\pi}{2n+1}}$$

and

$$\|\mathbf{Sc}(X^m_n)\|_{L_2(S)} = \|\mathbf{Sc}(Y^m_n)\|_{L_2(S)}(n+1+m)\sqrt{\frac{\pi}{2}\frac{1}{(2n+1)}\frac{(n+m)!}{(n-m)!}},$$

for $m = 1, \ldots, n$.

Remark 3. *Using a different measure, we have established Theorem 2 and the previous proposition already in [21].*

For future use we need also the next results:

Proposition 4. *Given a fixed $n \in \mathbb{N}_0$, the spherical harmonics*

$$\left\{\mathbf{Sc}(X^{n+1}_n \mathbf{e}_i), \mathbf{Sc}(Y^{n+1}_n \mathbf{e}_i) : i = 1, 2\right\}$$

are orthogonal to each other with respect to the inner product (3) *and their norms are given by*

$$\|\mathbf{Sc}(X^{n+1}_n \mathbf{e}_i)\|_{L_2(S)} = \|\mathbf{Sc}(Y^{n+1}_n \mathbf{e}_i)\|_{L_2(S)} = \frac{1}{2}\sqrt{\pi(n+1)(2n+2)!}.$$

The case $i = 1$ was already studied by the authors in [20]. For $i = 2$ the proof is similar.

Remark 4. *The orthogonality is ensured if one takes into account the following representations* ([9], *Proposition* 3.4.3)

$$\begin{aligned} X_n^{n+1} &= -C^{n+1,n}\cos(n\varphi)\mathbf{e}_1 + C^{n+1,n}\sin(n\varphi)\mathbf{e}_2 \\ Y_n^{n+1} &= -C^{n+1,n}\sin(n\varphi)\mathbf{e}_1 - C^{n+1,n}\cos(n\varphi)\mathbf{e}_2. \end{aligned} \quad (9)$$

Since some of our further results are not only restricted to the unit ball, from now on we represent by $X_n^{0,\dagger,*_R}, X_n^{m,\dagger,*_R}, Y_n^{m,\dagger,*_R}$ $(m = 1,...,n+1)$ the normalized basis functions $X_n^{0,\dagger}, X_n^{m,\dagger}, Y_n^{m,\dagger}$ in $L_2(B_R;\mathcal{A};\mathbb{R})$. Based on these functions, in [9] and [10] the following orthonormal basis is constructed, therein restricted to the unit ball.

Theorem 3. *For each n, the set of $2n+3$ homogeneous monogenic polynomials*

$$\left\{X_n^{0,\dagger,*_R}, X_n^{m,\dagger,*_R}, Y_n^{m,\dagger,*_R},\ m1,...,n+1\right\} \quad (10)$$

forms an orthonormal basis in $M^+(B_R;\mathcal{A};n)$ with respect to the inner product (4).

Remark 5. *The estimates stated in Proposition 2 are still valid for this new system of polynomials* (10)*. In particular, taking into account relation* (6) *it follows:*

$$|X_n^{0,\dagger,*_R}(\mathbf{x})| = \sqrt{\frac{2n+3}{R^{2n+3}}}\frac{|X_n^{0,\dagger}(\mathbf{x})|}{\|X_n^{0,\dagger}\|_{L_2(S;\mathcal{A};\mathbb{R})}}$$

and moreover, from Proposition 2

$$|X_n^{0,\dagger}(\mathbf{x})| = \left|R^n X_n^{0,\dagger}\left(\frac{\mathbf{x}}{R}\right)\right| \le R^n\frac{1}{2\sqrt{\pi}}(n+1)2^n\left|\frac{\mathbf{x}}{R}\right|^n \parallel X_n^0 \parallel_{L_2(S;\mathcal{A};\mathbb{R})}$$

for $0 < |\mathbf{x}| = r < R$.

Theorem 3 makes it possible to define the Fourier expansion of a square integrable $\mathcal{A}$-valued monogenic function in $L_2(B_R)$. Moreover, each monogenic function can be decomposed in an orthogonal sum of a monogenic "main part" (g) of the function and a hyperholomorphic constant (h). More precisely, it holds:

Lemma 1. *A monogenic L_2-function $f:\Omega\subset\mathbb{R}^3 \longrightarrow \mathcal{A}$ can be decomposed into*

$$f = f(\mathbf{0}) + g + h, \quad (11)$$

where the functions g and h have Fourier series

$$\begin{aligned} g(\mathbf{x}) &= \sum_{n=1}^{\infty}\left(X_n^{0,\dagger,*_R}(\mathbf{x})\alpha_n^0 + \sum_{m=1}^{n}\left[X_n^{m,\dagger,*_R}(\mathbf{x})\alpha_n^m + Y_n^{m,\dagger,*_R}(\mathbf{x})\beta_n^m\right]\right) \\ h(\mathbf{x}) &= \sum_{n=1}^{\infty}\left[X_n^{n+1,\dagger,*_R}(\mathbf{x})\alpha_n^{n+1} + Y_n^{n+1,\dagger,*_R}(\mathbf{x})\beta_n^{n+1}\right]. \end{aligned}$$

The associated Fourier coefficients $\alpha_n^0, \alpha_n^m, \beta_n^m$ $(m = 1, ..., n+1)$ *are real-valued.*

4 Bohr's Theorem for monogenic functions

During the last years the standard Bohr's phenomena attracted a lot of attention. In 1914, H. Bohr discovered that there exists a radius $r \in (0,1)$ such that if a power series of a holomorphic function converges in the unit disk and its sum has a modulus less than 1, then for $|z| < r$ the sum of the absolute values of its terms is again less than 1. The significance of the theorem is that such radius does not depend on the function.[4] To be more precise, the classical Bohr's Theorem says that:

Theorem 4. [6] *Let f be a bounded analytic function in the open unit disk, with Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ convergent in the unit disk and with modulus less than 1. Then $\sum_{n=0}^{\infty} |a_n| r^n < 1$ for $0 \leq r < \frac{1}{3}$.*

This result, known as Bohr's inequality, is true for $0 < r < \frac{1}{3}$ and the constant $\frac{1}{3}$ cannot be improved, that is, the inequality fails for any $r \geq \frac{1}{3}$. Originally, this theorem was proved for $0 \leq r < \frac{1}{6}$, but soon improved to the sharp result by M. Riesz, I. Schur, and N. Wiener independently. In Bohr's paper [6] his own proof was published as well as a proof by Wiener based on function theory methods. Later, S. Sidon gave a different proof [35], which was subsequently rediscovered by M. Tomić [36].

Recently, multi-dimensional analogues and other generalizations of Bohr's theorem are treated by several mathematicians such as Aizenberg [1], Beneteau, Dahlner and Khavinson [3], Boas and Khavinson [4], Dineen and Timoney [14], Paulsen, Popescu and Singh [30], and many others. In several of these papers, the proof of Bohr's inequality or of Bohr-type inequalities, respectively, in the theory of holomorphic functions of one or n variables is based on the orthogonality of the powers of the complex variable(s). To use similar ideas in the quaternionic case, it seems to be natural to work with the Fourier expansion of monogenic functions. It is not so simple as in the complex case to switch between the Taylor expansion and the Fourier series of a function. The reason for this is that the Taylor expansion with respect to the Fueter variables does not give us orthogonal summands.

It should be also remarked that in some papers (see, e.g., [30]) the idea is to work with the Fourier coefficients of the boundary values of a holomorphic function. We prefer here to consider (analogously to the original formulation of Bohr's theorem) only monogenic functions in the ball. It follows directly from the supposed boundedness of the monogenic functions that they are also square integrable in the ball and therefore we can work with Fourier series there. The existence of integrable boundary values needs additional assumptions.

[4] For the physicists, the notion "Bohr radius" is associated to Niels Bohr, the founder of the quantum theory and winner of the Nobel Prize in Physics in 1922.

In the remainder of this section, we collect generalizations and different modifications of this theorem (see [19, 21]) and we show that the result can be extended to the all class of monogenic functions with $|f(\mathbf{x})| < 1$ in B. In [19], the first version of a quaternionic Bohr type theorem was obtained, therein restricted to the case of functions with $f(\mathbf{0}) = \mathbf{0}$.

Theorem 5. [19] *Let f be a square integrable $\mathcal{A}$-valued monogenic function with $f(\mathbf{0}) = \mathbf{0}$ and $|f(\mathbf{x})| < 1$ in B and let*

$$\sum_{n=1}^{\infty} \left[X_n^{0,\dagger,*}\alpha_n^0 + \sum_{m=1}^{n+1} \left(X_n^{m,\dagger,*}\alpha_n^m + Y_n^{m,\dagger,*}\beta_n^m \right) \right]$$

be its Fourier expansion. Then

$$\sum_{n=1}^{\infty} \left| X_n^{0,\dagger,*}\alpha_n^0 + \sum_{m=1}^{n+1} \left(X_n^{m,\dagger,*}\alpha_n^m + Y_n^{m,\dagger,*}\beta_n^m \right) \right| < 1$$

holds in the ball $\{\mathbf{x} : |\mathbf{x}| < 0.047\}$.

This result is adapted very well to the complex situation. The absolute value is taken from all summands of the same degree n. In the complex case this is also a first important step. All the considered functions with $f(0) = 0$ are orthogonal to the constants. This is used later on to estimate all Fourier coefficients of a general holomorphic function with $|f(z)| \leq 1$ by the first Fourier coefficient (see, e.g., [30]).

However, it is important to remark that in the quaternionic context, the set of "constants" is much bigger. Hereby constants are also monogenic functions which have an identically vanishing hypercomplex derivative. Then it is immediately clear that the constant function and all monogenic functions which depend only on x_1 and x_2 are the so called *hyperholomorphic constants*. Moreover, if we, as in this paper, consider only $\mathcal{A}$-valued functions then a non-trivial hyperholomorphic constant must have values in $\text{span}_{\mathbb{R}}\{\mathbf{e_1}, \mathbf{e_2}\}$. With these observations it seems to be natural to study at first the class of functions which are orthogonal to the non-trivial hyperholomorphic constants in $L_2(B; \mathcal{A}; \mathbb{R})$ with $|f(\mathbf{x})| < 1$ in B (see [21]). This approach is also supported by the fact that in Lemma 1 it is shown that each monogenic function can be decomposed in an orthogonal sum of a monogenic "main part" of the function and a hyperholomorphic constant. Remind that an orthonormal basis of the subspace of hyperholomorphic constants is given by the set $\{X_n^{n+1,\dagger}, Y_n^{n+1,\dagger}\}_{n=0}^{\infty}$.

The Fourier representation in the hypothesis of the next Theorem describes the general form of these main parts. The non-trivial hyperholomorphic constants in the decomposition do not influence the real part of the function at the origin because their image lies in $\text{span}_{\mathbb{R}}\{\mathbf{e_1}, \mathbf{e_2}\}$.

Theorem 6. [21] Let f be an $\mathcal{A}$-valued monogenic function such that $f(\mathbf{x}) - f(\mathbf{0})$ is orthogonal to the hyperholomorphic constants with respect to the inner product (4) with $|f(\mathbf{x})| < 1$ in B and let

$$\sum_{n=0}^{\infty}\left[X_n^{0,\dagger,*}\alpha_n^0+\sum_{m=1}^{n}\left(X_n^{m,\dagger,*}\alpha_n^m+Y_n^{m,\dagger,*}\beta_n^m\right)\right]$$

be its Fourier expansion. Then

$$\sum_{n=0}^{\infty}\left[|X_n^{0,\dagger,*}||\alpha_n^0|+\sum_{m=1}^{n}\left(|X_n^{m,\dagger,*}||\alpha_n^m|+|Y_n^{m,\dagger,*}||\beta_n^m|\right)\right]<1$$

holds in the ball of radius r, with $0\leq r<0.004$.

Proof. We give only the main ideas of the proof. For more details see [21]. At first it is important to note that in the previous series the sum which contains the variable m runs now only from 1 to n. This fact expresses the supposed orthogonality to the hyperholomorphic constants $X_n^{n+1,\dagger}$ and $Y_n^{n+1,\dagger}$. Since the basis polynomials are homogeneous, the value of f at the origin is

$$\begin{aligned} f(\mathbf{0}) &= \sum_{n=0}^{\infty}\left(X_n^{0,\dagger,*}(\mathbf{0})\alpha_n^0+\sum_{m=1}^{n}\left[X_n^{m,\dagger,*}(\mathbf{0})\alpha_n^m+Y_n^{m,\dagger,*}(\mathbf{0})\beta_n^m\right]\right)\\ &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0\in\mathbb{R}. \end{aligned}$$

Without loss of generality we assume that $f(\mathbf{0})$ is positive (otherwise we work with $-f$). Since the associated Fourier coefficients are real-valued, the real part of f is given by

$$\mathbf{Sc}(f)=\sum_{n=0}^{\infty}\left\{\mathbf{Sc}(X_n^{0,\dagger,*})\alpha_n^0+\sum_{m=1}^{n}\left[\mathbf{Sc}(X_n^{m,\dagger,*})\alpha_n^m+\mathbf{Sc}(Y_n^{m,\dagger,*})\beta_n^m\right]\right\}.$$

Basically, the main idea of the proof is to find relations between the general Fourier coefficients and the coefficient of the zeroth term, i.e., α_0^0. Multiplying both sides of the equation

$$\mathbf{Sc}(1-f)=1-\mathbf{Sc}(f) \tag{12}$$

by the solid spherical harmonics $\{\mathbf{Sc}(X_k^{0,\dagger,*}),\mathbf{Sc}(X_k^{p,\dagger,*}),\mathbf{Sc}(Y_k^{p,\dagger,*}):p=1,...,k\}$, integrating over the ball and applying the modulus we get the following relations:

$$\begin{aligned} |\alpha_k^0| &\leq \max_B|X_k^{0,\dagger,*}|\frac{2\sqrt{\frac{\pi}{3}}}{\|\mathbf{Sc}(X_k^{0,\dagger,*})\|^2_{L_2(B)}}\left(2\sqrt{\frac{\pi}{3}}-\alpha_0^0\right)\\ |\alpha_k^p| &\leq \max_B|X_k^{p,\dagger,*}|\frac{2\sqrt{\frac{\pi}{3}}}{\|\mathbf{Sc}(X_k^{p,\dagger,*})\|^2_{L_2(B)}}\left(2\sqrt{\frac{\pi}{3}}-\alpha_0^0\right)\\ |\beta_k^p| &\leq \max_B|Y_k^{p,\dagger,*}|\frac{2\sqrt{\frac{\pi}{3}}}{\|\mathbf{Sc}(Y_k^{p,\dagger,*})\|^2_{L_2(B)}}\left(2\sqrt{\frac{\pi}{3}}-\alpha_0^0\right),\ p=1,...,k. \end{aligned}$$

With some calculations, applying Propositions 1-3 we arrive at

$$\begin{aligned}
&\frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0 \;+\; \sum_{n=1}^{\infty}\left[|X_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n}\left(|X_n^{m,\dagger,*}||\alpha_n^m| + |Y_n^{m,\dagger,*}||\beta_n^m|\right)\right]\\
\le\;& \frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0 \;+\; \frac{1}{\sqrt{3\pi}}\left(2\sqrt{\frac{\pi}{3}} - \alpha_0^0\right)\sum_{n=1}^{\infty}(4r)^n(n+1)^4(2n+3).
\end{aligned}$$

The principal significance is that

$$\frac{1}{2}\sqrt{\frac{3}{\pi}}\alpha_0^0 \;+\; \sum_{n=1}^{\infty}\left[|X_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n}\left(|X_n^{m,\dagger,*}||\alpha_n^m| + |Y_n^{m,\dagger,*}||\beta_n^m|\right)\right] < 1$$

if $\frac{2}{3}\sum_{n=1}^{\infty}(4r)^n(n+1)^4(2n+3) < 1$. We see that the last series converges for $r < \frac{1}{4}$, and therefore, the inequality is satisfied for $0 \le r < 0.004$. □

As we have seen, the set $\{X_n^{n+1,\dagger}, Y_n^{n+1,\dagger}\}$ belonging to h play a special role. In order to extend the previous result, next we present some important properties of this function and/or of its coordinates. For simplicity we restrict ourselves to the unit ball.

Lemma 2. *The hyperholomorphic constant h can be written as*

$$h = h_1\mathbf{e}_1 + h_2\mathbf{e}_2,$$

where its coordinates have Fourier series

$$\begin{aligned}
h_1(\underline{x}) &= \sum_{n=1}^{\infty}\left([X_n^{n+1,\dagger,*}(\underline{x})]_1\alpha_n^{n+1} + [Y_n^{n+1,\dagger,*}(\underline{x})]_1\beta_n^{n+1}\right)\\
h_2(\underline{x}) &= \sum_{n=1}^{\infty}\left([X_n^{n+1,\dagger,*}(\underline{x})]_2\alpha_n^{n+1} + [Y_n^{n+1,\dagger,*}(\underline{x})]_2\beta_n^{n+1}\right).
\end{aligned}$$

Moreover, the following properties hold:

Proposition 5. *The harmonic functions h_1 and h_2 are orthogonal with respect to the inner product* (3).

Proof. Because relation (6) we just need to ensure the orthogonality in $L_2(S)$. For technical reasons, we rewrite the function h as follows

$$h(\underline{x}) = \sum_{n=1}^{\infty}\sqrt{2n+3}\,r^n\left(X_n^{n+1,*}(\theta,\varphi)\alpha_n^{n+1} + Y_n^{n+1,*}(\theta,\varphi)\beta_n^{n+1}\right).$$

By definition of the real-valued inner product in $L_2(S)$, using representation (9) and the previous expression we have

$$< h_1, h_2 >_{L_2(S)} = \sum_{n=1}^{\infty} \sum_{n'=1}^{\infty} \frac{\sqrt{2n+3}\, r^n}{\|X_n^{n+1}\|_{L_2(S;\mathcal{A};\mathbb{R})}} \frac{\sqrt{2n'+3}\, r^{n'}}{\|X_{n'}^{n'+1}\|_{L_2(S;\mathcal{A};\mathbb{R})}} \int_S A_n(\theta,\varphi) B_{n'}(\theta,\varphi) d\sigma$$

where

$$\begin{aligned} A_n(\theta,\varphi) &= -C^{n+1,n}(\theta)\left(\cos(n\varphi)\alpha_n^{n+1} + \sin(n\varphi)\beta_n^{n+1}\right) = A_n(\theta)A_n(\varphi) \\ B_{n'}(\theta,\varphi) &= C^{n'+1,n'}(\theta)\left(\sin(n'\varphi)\alpha_{n'}^{n'+1} - \cos(n'\varphi)\beta_{n'}^{n'+1}\right) = B_{n'}(\theta)B_{n'}(\varphi). \end{aligned}$$

If the degrees of the summands in the series of h_1 and h_2 are different ($n \neq n'$) then the orthogonality is ensured. Therefore it is only necessary to prove that the previous integral vanishes for $n = n'$. And since

$$\int_S A_n(\theta,\varphi) B_n(\theta,\varphi) d\sigma = \int_0^{\pi} A_n(\theta) B_n(\theta) \sin\theta d\theta \int_0^{2\pi} A_n(\varphi) B_n(\varphi) d\varphi,$$

it is enough to prove that the second integral on the right-hand side is zero. Moreover,

$$\begin{aligned} \int_0^{2\pi} A_n(\varphi) B_n(\varphi) d\varphi &= \left[(\alpha_n^{n+1})^2 - (\beta_n^{n+1})^2\right] \int_0^{2\pi} \cos(n\varphi)\sin(n\varphi) d\varphi \\ &+ \alpha_n^{n+1}\beta_n^{n+1} \left(\int_0^{2\pi} \sin^2(n\varphi) d\varphi - \int_0^{2\pi} \cos^2(n\varphi) d\varphi \right) \\ &= 0. \end{aligned}$$

□

Proposition 6. *The harmonic functions h_1 and h_2 verify the following relation:*

$$\sup_B |h_1(\underline{x})| = \sup_B |h_2(\underline{x})|.$$

The proof follows immediately from representation (9).

We are thus led to the following generalization of Theorem 6.

Theorem 7. *Let f be an $\mathcal{A}$-valued monogenic function with $|f(\mathbf{x})| < 1$ in B and let*

$$\sum_{n=0}^{\infty} \left[X_n^{0,\dagger,*}\alpha_n^0 + \sum_{m=1}^{n+1} \left(X_n^{m,\dagger,*}\alpha_n^m + Y_n^{m,\dagger,*}\beta_n^m \right) \right]$$

be its Fourier expansion. Then

$$\sum_{n=0}^{\infty} \left[|X_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n+1} \left(|X_n^{m,\dagger,*}||\alpha_n^m| + |Y_n^{m,\dagger,*}||\beta_n^m| \right) \right] < 1$$

holds in the ball of radius r, with $0 \leq r < 0.004$.

Proof. Considering f written as in (11) (Lemma 1)

$$f(\mathbf{x}) = f(\mathbf{0}) + g(\mathbf{x}) + h(\underline{x}),$$

with $f(\mathbf{0}) = g(\mathbf{0}) + h(\underline{0})$. The study of the function g was already considered in Theorem 6. We showed that

$$\begin{aligned} |g(\mathbf{x})| &\leq \sum_{n=1}^{\infty} \left[|X_n^{0,\dagger,*}||\alpha_n^0| + \sum_{m=1}^{n} \left(|X_n^{m,\dagger,*}||\alpha_n^m| + |Y_n^{m,\dagger,*}||\beta_n^m| \right) \right] \\ &\leq \frac{1}{\sqrt{3\pi}} \left(2\sqrt{\frac{\pi}{3}} - |\alpha_0^0| \right) \sum_{n=1}^{\infty} (4r)^n (n+1)^4 (2n+3). \end{aligned}$$

We consider now the function h written as Fourier series

$$h(\underline{x}) = \sum_{n=1}^{\infty} \left(X_n^{n+1,\dagger,*}(\underline{x})\alpha_n^{n+1} + Y_n^{n+1,\dagger,*}(\underline{x})\beta_n^{n+1} \right).$$

The proof for the function h follows the same idea as the previous one. According to its Fourier expansion, in this case we should find relations with α_0^1 and/or β_0^1. However, it is important to note that the function h has no real part, and therefore, it is not possible to apply straight the previous idea. Because h lies in $\text{span}_{\mathbb{R}}\{\mathbf{e_1}, \mathbf{e_2}\}$, the interesting point is that multiplying h at right by $\mathbf{e}_1$ (resp., by $\mathbf{e}_2$) the real part is different of zero, standing then for $-h_1$ (resp., $-h_2$). Moreover, since there are two different Fourier coefficients in the zeroth term, it is natural to ask: which coefficient should be compared, α_0^1 or β_0^1? Multiplying the function h at right by $\mathbf{e}_1$ we obtain

$$\begin{aligned} \tilde{h}(\underline{0}) + \tilde{h}(\underline{x}) &:= h\mathbf{e}_1(\underline{0}) + h\mathbf{e}_1(\underline{x}) \\ &= (X_0^{1,\dagger,*}\mathbf{e}_1)\alpha_0^1 + (Y_0^{1,\dagger,*}\mathbf{e}_1)\beta_0^1 \\ &+ \sum_{n=1}^{\infty} \left[(X_n^{n+1,\dagger,*}\mathbf{e}_1)(\underline{x})\alpha_n^{n+1} + (Y_n^{n+1,\dagger,*}\mathbf{e}_1)(\underline{x})\beta_n^{n+1} \right] \end{aligned}$$

where

$$(X_0^{1,\dagger,*}\mathbf{e}_1)\alpha_0^1 + (Y_0^{1,\dagger,*}\mathbf{e}_1)\beta_0^1 = \frac{1}{2\sqrt{\pi}}\alpha_0^1 + \frac{1}{2\sqrt{\pi}}\beta_0^1\mathbf{e}_3.$$

For this case, taking into account the previous assumption, it is obviously that we should consider for zeroth term the coefficient α_0^1. In a similar way, considering a new function $\tilde{\tilde{h}} := h\mathbf{e}_2$, β_0^1 should be surely considered.

Having disposed of this preliminary step let us return to the proof. It follows easily multiplying h at right by $\mathbf{e}_1$

$$\begin{aligned} \mathbf{Sc}(\tilde{h}) &= \mathbf{Sc}(h\mathbf{e}_1) \\ &= \sum_{n=1}^{\infty} \left[\mathbf{Sc}(X_n^{n+1,\dagger,*}\mathbf{e}_1)\alpha_n^{n+1} + \mathbf{Sc}(Y_n^{n+1,\dagger,*}\mathbf{e}_1)\beta_n^{n+1} \right]. \end{aligned}$$

Taking into account Proposition 6, for $0 < \delta_2 \leq 1$ we multiply both sides of the equation

$$\mathbf{Sc}\left(\frac{\delta_2}{2} - \tilde{h}\right) = \frac{\delta_2}{2} - \mathbf{Sc}(\tilde{h})$$

by the orthonormal solid spherical harmonics $\mathbf{Sc}(X_k^{k+1,\dagger,*}\mathbf{e}_1)$ (resp. $\mathbf{Sc}(Y_k^{k+1,\dagger,*}\mathbf{e}_1)$), integrating over the ball and taking the modulus it follows

$$\begin{aligned} |\alpha_k^{k+1}| &\leq \max_B |X_k^{k+1,\dagger,*}| \frac{2\sqrt{\frac{\pi}{3}}}{\|\mathbf{Sc}(X_k^{k+1,\dagger,*})\|^2_{L_2(B)}} \left(\sqrt{\frac{\pi}{3}}\delta_2 - |\alpha_0^1|\right) \\ |\beta_k^{k+1}| &\leq \max_B |X_k^{k+1,\dagger,*}| \frac{2\sqrt{\frac{\pi}{3}}}{\|\mathbf{Sc}(X_k^{k+1,\dagger,*})\|^2_{L_2(B)}} \left(\sqrt{\frac{\pi}{3}}\delta_2 - |\alpha_0^1|\right) \end{aligned}$$

Now with some calculations, using Propositions 1-3 and applying the Maximum modulus principle, the previous inequalities can be rewritten as follows

$$\begin{aligned} |\alpha_k^{k+1}| &\leq \frac{2}{\sqrt{3}} 2^k \sqrt{2k+3}(k+1)\left(\sqrt{\frac{\pi}{3}}\delta_2 - |\alpha_0^1|\right) \\ |\beta_k^{k+1}| &\leq \frac{2}{\sqrt{3}} 2^k \sqrt{2k+3}(k+1)\left(\sqrt{\frac{\pi}{3}}\delta_2 - |\alpha_0^1|\right). \end{aligned}$$

In a similar way, from the study of the function $\tilde{\tilde{h}}$ we obtain

$$\begin{aligned} |\alpha_k^{k+1}| &\leq \frac{2}{\sqrt{3}} 2^k \sqrt{2k+3}(k+1)\left(\sqrt{\frac{\pi}{3}}\delta_2 - |\beta_0^1|\right) \\ |\beta_k^{k+1}| &\leq \frac{2}{\sqrt{3}} 2^k \sqrt{2k+3}(k+1)\left(\sqrt{\frac{\pi}{3}}\delta_2 - |\beta_0^1|\right). \end{aligned}$$

Using the previous inequalities we end with

$$\begin{aligned} |h(\underline{x})| &\leq \sum_{n=1}^{\infty} \left(|X_n^{n+1,\dagger,*}||\alpha_n^{n+1}| + |Y_n^{n+1,\dagger,*}||\beta_n^{n+1}|\right) \\ &\leq \frac{1}{\sqrt{3\pi}}\left(2\sqrt{\frac{\pi}{3}}\delta_2 - |\alpha_0^1| - |\beta_0^1|\right)\sum_{n=1}^{\infty}(4r)^n(n+1)^2(2n+3). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} |f(\mathbf{x})| &\leq \frac{1}{2}\sqrt{\frac{3}{\pi}}|\alpha_0^0 - \alpha_0^1\mathbf{e}_1 - \beta_0^1\mathbf{e}_2| \\ &+ \frac{1}{\sqrt{3\pi}}\left(2\sqrt{\frac{\pi}{3}}(1+\delta_2) - |\alpha_0^0| - |\alpha_0^1| - |\beta_0^1|\right)\sum_{n=1}^{\infty}(4r)^n(n+1)^4(2n+3), \end{aligned}$$

and since $|f(\mathbf{x})| < 1$ it is clear that

$$\frac{1}{\sqrt{3\pi}}\left(\frac{2\sqrt{\frac{\pi}{3}}(1+\delta_2) - |\alpha_0^0| - |\alpha_0^1| - |\beta_0^1|}{2\sqrt{\frac{\pi}{3}} - |\alpha_0^0| - |\alpha_0^1| - |\beta_0^1|}\right)\sum_{n=1}^{\infty}(4r)^n(n+1)^4(2n+3) < 1.$$

Of crucial importance is the fact that the coefficient

$$\frac{2\sqrt{\frac{\pi}{3}} - |\alpha_0^0| - |\alpha_0^1| - |\beta_0^1|}{2\sqrt{\frac{\pi}{3}}(1+\delta_2) - |\alpha_0^0| - |\alpha_0^1| - |\beta_0^1|}$$

is bounded from above by 1. A simple calculation shows that the last series converges for $r < \frac{1}{4}$, and therefore, the inequality is satisfied for $0 \leq r < 0.004$. □

This shows that such a radius exists in the three-dimensional Euclidean ball. It has to be studied how the estimate for the Bohr radius can be improved.

5 Real part Theorems for monogenic functions

In the remainder of this section, we refer to the results as "real part theorems" in honor to the first assertion of such a kind, the classical (improved) Hadamard's real part theorem (1892). His work on functions of a complex variable was one of the first to examine the general theory of analytic functions. Since then, the acceptance of his work is worldwide. Looking back to all of these years one can say that time has shown that his topic has a wide range of applications. Some important indicators for such a development is that based on it, many recent results with strong applications are still coming out. Moreover, they provide with the best description of the pointwise behavior of analytic functions from a given space. A lot of results and extended list of references concerning these and other fundamental inequalities, as well as their applications, can be found in the book by Kresin and Maz'ya (see [33]).

In the complex case, Hadamard's real part theorem contains only the modulus of the function in the left-hand side of an inequality and bounds the growth of a function by the growth of its real part. More precisely, for $r < R$ the inequality

$$|f(z) - f(0)| \leq \frac{Cr}{R-r} \sup_{|\xi|\leq R} |\mathbf{Re}\,(f(\xi) - f(0))\,|, \tag{13}$$

holds for analytic functions on a closed disk of radius R centered at the origin. Such an inequality appeared first in 1892 (see [24]) with $C = 4$. Later, Borel and Carathéodory found the sharp constant $C = 2$. As a matter of this fact, a more general estimate for $|f(z)|$ with $f(0) \neq 0$ was noticed by Carathéodory (see Landau [25, 26])

$$|f(z)| \leq \frac{2r}{R-r} \sup_{|\xi|\leq R} |\mathbf{Re} f(\xi)| + \frac{R+r}{R-r}|f(0)|. \tag{14}$$

Having in mind the type of inequalities we want to prove, we will restrict ourselves to the three-dimensional case. The frequent use of quaternionic analysis in the study of three-dimensional problems motivate us to consider functions defined in $\mathbb{R}^3$ and with values in the reduced quaternions. It is also known that already in the four-dimensional case (quaternion-valued functions) there are a lot of non-trivial monogenic functions with vanishing scalar part. For such functions we cannot get the result as desired.

In [20, 22] it is shown that it is possible to generalize Borel-Carathéodory and Hadamard's real part theorems to monogenic functions, therein restricted to the unit ball in the Euclidean space $\mathbb{R}^3$. In this section we generalize these results for an arbitrary ball of radius R, analogously to the complex case.

Remark 6. *Several proofs of Bohr's inequality are based on estimating all Fourier coefficients by the first one. We will observe that the proof of both theorems follows from this idea. Here we consider relations between the Fourier coefficients of the function and the Fourier coefficients of its real part.*

The referred relations come with the next lemma:

Lemma 3. *Let f be a square integrable $\mathcal{A}$-valued monogenic function and $n \in \mathbb{N}_0$. Then the Fourier coefficients of f are given by*

$$
\begin{aligned}
\alpha_n^0 &= \frac{\|X_n^{0,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{0,\dagger})\|^2_{L_2(B_R)}} \int_{B_R} \mathbf{Sc}(f)\mathbf{Sc}(X_n^{0,\dagger})dV_R \\
\alpha_n^m &= \frac{\|X_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{m,\dagger})\|^2_{L_2(B_R)}} \int_{B_R} \mathbf{Sc}(f)\mathbf{Sc}(X_n^{m,\dagger})dV_R \\
\beta_n^m &= \frac{\|Y_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(Y_n^{m,\dagger})\|^2_{L_2(B_R)}} \int_{B_R} \mathbf{Sc}(f)\mathbf{Sc}(Y_n^{m,\dagger})dV_R, \quad m = 1, ..., n \\
\alpha_n^{n+1} &= \frac{\|X_n^{n+1,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{n+1,\dagger}\mathbf{e}_1)\|^2_{L_2(B_R)}} \int_{B_R} \mathbf{Sc}(h\mathbf{e}_1)\mathbf{Sc}(X_n^{n+1,\dagger}\mathbf{e}_1)dV_R \\
\beta_n^{n+1} &= \frac{\|Y_n^{n+1,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(Y_n^{n+1,\dagger}\mathbf{e}_1)\|^2_{L_2(B_R)}} \int_{B_R} \mathbf{Sc}(h\mathbf{e}_1)\mathbf{Sc}(Y_n^{n+1,\dagger}\mathbf{e}_1)dV_R.
\end{aligned}
$$

Originally the Fourier coefficients are defined by the inner product of the function f and elements of the space $M^+(\mathbb{R}^3; \mathcal{A}, n)$. Now as we can see these coefficients, up to a factor, are also associated with the scalar part of f.

Proof. We give only some ideas of the proof. For more details see [20]. According to Lemma 1, f

can be written as Fourier series respecting the decomposition (11)

$$f(\mathbf{x}) = f(\mathbf{0}) \quad + \quad \underbrace{\sum_{n=1}^{\infty}\left(X_n^{0,\dagger,*_R}(\mathbf{x})\alpha_n^0 + \sum_{m=1}^{n}\left[X_n^{m,\dagger,*_R}(\mathbf{x})\alpha_n^m + Y_n^{m,\dagger,*_R}(\mathbf{x})\beta_n^m\right]\right)}_{=g}$$

$$+ \quad \underbrace{\sum_{n=1}^{\infty}\left[X_n^{n+1,\dagger,*_R}(\underline{x})\alpha_n^{n+1} + Y_n^{n+1,\dagger,*_R}(\underline{x})\beta_n^{n+1}\right]}_{=h}.$$

We will present the proof only for the coefficients α_n^0 of g. The remaining coefficients α_n^m and β_n^m $(m = 1, ..., n)$ are obtained in a similar way. As described, we aim to compare each Fourier coefficient α_n^0 with $\mathbf{Sc}(f)$. We have seen several times before that

$$\mathbf{Sc}(f) = \sum_{n=0}^{\infty}\left(\mathbf{Sc}(X_n^{0,\dagger,*_R})\alpha_n^0 + \sum_{m=1}^{n}\left[\mathbf{Sc}(X_n^{m,\dagger,*_R})\alpha_n^m + \mathbf{Sc}(Y_n^{m,\dagger,*_R})\beta_n^m\right]\right).$$

Multiplying both sides of the previous expression by the solid spherical harmonics $\{\mathbf{Sc}(X_k^{0,\dagger,*_R}), \mathbf{Sc}(X_k^{p,\dagger,*_R}), \mathbf{Sc}$ $p = 1, ..., k\}$ $(k \geq 1)$ and integrating over the ball we get the desired relations.

For the study of the coefficients α_n^{n+1} and β_n^{n+1} we multiply the equation

$$\mathbf{Sc}(h\mathbf{e}_1) = \sum_{n=0}^{\infty}\left[\mathbf{Sc}(X_n^{n+1,\dagger,*_R}\mathbf{e}_1)\alpha_n^{n+1} + \mathbf{Sc}(Y_n^{n+1,\dagger,*_R}\mathbf{e}_1)\beta_n^{n+1}\right]$$

by the orthogonal solid spherical harmonics $\mathbf{Sc}(X_k^{k+1,\dagger,*_R}\mathbf{e}_1)$ (resp.
$\mathbf{Sc}(Y_k^{k+1,\dagger,*_R}\mathbf{e}_1)$) $(k \geq 1)$ and integrating over the ball carries our results. □

Lemma 4. *Let f be a square integrable $\mathcal{A}$-valued monogenic function. For each $n \in \mathbb{N}_0$, the Fourier coefficients satisfy the inequalities*

$$\begin{aligned}
|\alpha_n^0| &\leq 2\sqrt{\frac{\pi}{3}}\sqrt{R^3}\frac{\|X_n^{0,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{0,\dagger})\|_{L_2(B_R)}}\sup_{|\xi|\leq R}|\mathbf{Sc}\,(f(\xi))\,| \\
|\alpha_n^m| &\leq 2\sqrt{\frac{\pi}{3}}\sqrt{R^3}\frac{\|X_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{m,\dagger})\|_{L_2(B_R)}}\sup_{|\xi|\leq R}|\mathbf{Sc}\,(f(\xi))\,| \\
|\beta_n^m| &\leq 2\sqrt{\frac{\pi}{3}}\sqrt{R^3}\frac{\|Y_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(Y_n^{m,\dagger})\|_{L_2(B_R)}}\sup_{|\xi|\leq R}|\mathbf{Sc}\,(f(\xi))\,|,\ \ m = 1, ..., n \\
|\alpha_n^{n+1}| &\leq 2\sqrt{\frac{\pi}{3}}\sqrt{R^3}\frac{\|X_n^{n+1,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{n+1,\dagger}\mathbf{e}_1)\|_{L_2(B_R)}}\sup_{|\xi|\leq R}|\mathbf{Sc}\,(h\mathbf{e}_1(\xi))\,| \\
|\beta_n^{n+1}| &\leq 2\sqrt{\frac{\pi}{3}}\sqrt{R^3}\frac{\|Y_n^{n+1,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(Y_n^{n+1,\dagger}\mathbf{e}_1)\|_{L_2(B_R)}}\sup_{|\xi|\leq R}|\mathbf{Sc}\,(h\mathbf{e}_1(\xi))\,|.
\end{aligned}$$

The previous inequalities are basic results to prove the following theorem:

Theorem 8 (Real-Part Theorem)**.** *Let f be a square integrable $\mathcal{A}$-valued monogenic function in B_R. Then, for $0 \leq r < \frac{R}{2}$ we have the inequality*

$$\begin{aligned}|f|_r &\leq |f(\mathbf{0})| + \sqrt{\frac{2}{3}}\frac{8r}{(R-2r)^3}\Bigg(A_1(r,R)\sup_{|\xi|\leq R}|\mathbf{Sc}\,(f(\xi))\,| \\ &+ A_2(r,R)\sup_{|\xi|\leq R}|\mathbf{Sc}\,(h\mathbf{e}_1(\xi))\,|\Bigg),\end{aligned}$$

where

$$|f|_r = \max_{|\mathbf{x}|=r}|f(\mathbf{x})|$$

and

$$\begin{aligned}A_1(r,R) &= \frac{8r^2(2R-r)}{R-2r} + 6R^2 \\ A_2(r,R) &= 4r^2 - 6rR + 3R^2.\end{aligned}$$

Proof. Considering f written as in (11) and taking into account the maximum modulus principle we have

$$|f|_r \leq |f(\mathbf{0})| + |g|_R + |h|_R.$$

Let us start with the study of the function g. Using the previous lemma it follows that

$$\begin{aligned}|g|_R \leq\ & 2\sqrt{\frac{\pi}{3}}\sqrt{R^3}\sup_{|\xi|\leq R}|\mathbf{Sc}\,(f(\xi))\,|\sum_{n=1}^{\infty}\Bigg[|X_n^{0,\dagger,*_R}|\frac{\|X_n^0\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^0)\|_{L_2(B_R)}} \\ &+ \sum_{m=1}^{n}\left(|X_n^{m,\dagger,*_R}|\frac{\|X_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(X_n^{m,\dagger})\|_{L_2(B_R)}} + |Y_n^{m,\dagger,*_R}|\frac{\|Y_n^{m,\dagger}\|_{L_2(B_R;\mathcal{A};\mathbb{R})}}{\|\mathbf{Sc}(Y_n^{m,\dagger})\|_{L_2(B_R)}}\right)\Bigg].\end{aligned}$$

Applying Proposition 2 and taking into account Remark 5 it follows

$$|g|_R \leq 2\sqrt{\frac{2}{3}}\sup_{|\xi|\leq R}|\mathbf{Sc}\,(f(\xi))\,|\sum_{n=1}^{\infty}\left(\frac{2r}{R}\right)^n(n+1)^2(n+2).$$

In the same way, we can study the function h.

$$|h|_R = |\tilde{h}|_R \leq 2\sqrt{\frac{2}{3}}\sup_{|\xi|\leq R}|\mathbf{Sc}\,(h\mathbf{e_1}(\xi))\,|\sum_{n=1}^{\infty}\left(\frac{2r}{R}\right)^n(n+1)(n+2).$$

Finally, we obtain

$$\begin{aligned}|f|_r \leq |f(\mathbf{0})| &+ 2\sqrt{\frac{2}{3}}\sup_{|\xi|\leq R}|\mathbf{Sc}\,(f(\xi))\,|\sum_{n=1}^{\infty}\left(\frac{2r}{R}\right)^n(n+1)^2(n+2) \\ &+ 2\sqrt{\frac{2}{3}}\sup_{|\xi|\leq R}|\mathbf{Sc}\,(h\mathbf{e_1}(\xi))\,|\sum_{n=1}^{\infty}\left(\frac{2r}{R}\right)^n(n+1)(n+2).\end{aligned}$$

Now, note that the last series are convergent for $0 \leq r < \frac{R}{2}$. □

The previous theorem states that a monogenic L_2-function $f : \Omega \subset \mathbb{R}^3 \longrightarrow \mathcal{A}$ is bounded by a combination of its real part and one of its other components. This result is a more general estimate but, in fact, it is not a complete analogy to the complex case. Therein, an analytic function is only bounded by its real part. However, restricting ourselves to the class of functions which are orthogonal to the subspace of the non-trivial hyperholomorphic constants in $L_2(B_R;\mathcal{A};\mathbb{R})$, we get a stronger result:

Corollary 1. *Let $\tilde{f}$ be a square integrable $\mathcal{A}$-valued monogenic function in B_R orthogonal to the non-trivial hyperholomorphic constants with respect to the inner product* (3)*. Then, for $0 \leq r < \frac{R}{2}$ we have the following inequality:*

$$|\tilde{f}|_r \leq |\tilde{f}(\mathbf{0})| + \frac{8rA(r,R)}{(R-2r)^4} \sup_{|\xi|\leq R} |\mathbf{Sc}(\tilde{f}(\xi))|$$

where

$$A(r,R) \quad = \quad 8r^2(2R-r) + 6R^2(R-2r).$$

Remark 7. *Replacing $\tilde{f}(\mathbf{x})$ by $f(\mathbf{x}) - f(0)$ in the resulting relation from the previous corollary, we arrive at*

$$|f(\mathbf{x}) - f(\mathbf{0})|_r \leq \frac{8rA(r,R)}{(R-2r)^4} \sup_{|\xi|\leq R} |\mathbf{Sc}\left(f(\xi) - f(\mathbf{0})\right)|$$

with $A(r,R)$ as in the previous theorem, which is a refinement of Hadamard's real part theorem.

We observe that in the constants of our estimates, the factor $1/(R-2r)$ occurs and not $R-r$ as it could be expected from the complex case. As we will see in the next section, to explain this is because monogenic functions do not map balls to balls in the small but balls to ellipsoids.

6 First applications

As in the case of holomorphic functions in the complex plane we have to ask if also the growth of the derivative (here of the hypercomplex derivative) can be bounded by the growth of the function. If this is possible then, consequently, the behaviour of the derivative can be estimated by the scalar part of the monogenic function. The following theorem gives a first result.

Theorem 9. *Let f be a square integrable $\mathcal{A}$-valued monogenic function in B_R. Then, for $0 \leq r < \frac{R}{2}$ we have the following inequality:*

$$\left|(\frac{1}{2}\overline{D})f(\mathbf{x})\right|_r \quad \leq \quad \frac{8}{\sqrt{3}} \frac{R^2(2R^2+7rR+2r^2)}{(R-2r)^5} \sup_{|\xi|\leq R} |\mathbf{Sc}\left(f(\xi)\right)|.$$

Proof. We consider f written as in (11) (Lemma 1). Since the referred series is convergent in L_2, it converges uniformly to f in each compact subset of B_R. Also the series of all partial derivatives converges uniformly to the corresponding partial derivatives of f in compact subsets of B_R. Applying the hypercomplex derivative $\frac{1}{2}\overline{D}$ term by term to the series, it follows formally

$$(\frac{1}{2}\overline{D})f = \sum_{n=1}^{\infty}\left[(\frac{1}{2}\overline{D})X_n^{0,\dagger,*_R}\alpha_n^0 + \sum_{m=1}^{n}\left((\frac{1}{2}\overline{D})X_n^{m,\dagger,*_R}\alpha_n^m + (\frac{1}{2}\overline{D})Y_n^{m,\dagger,*_R}\beta_n^m\right)\right.$$
$$\left. + \quad (\frac{1}{2}\overline{D})X_n^{n+1,\dagger,*_R}\alpha_n^{n+1} + (\frac{1}{2}\overline{D})Y_n^{n+1,\dagger,*_R}\beta_n^{n+1}\right]$$

The proof follows from the idea applied in Theorem 8 with the estimates of the Fourier coefficients from Lemma 4 and taking into account the following equalities for the homogeneous monogenic polynomials (7) and their derivatives:

$$\begin{aligned}(\frac{1}{2}\overline{D})X_n^{l,\dagger} &= (n+l+1)X_{n-1}^{l,\dagger}\\ (\frac{1}{2}\overline{D})Y_n^{m,\dagger} &= (n+m+1)Y_{n-1}^{m,\dagger},\end{aligned}$$

for $l = 0, ..., n$ and $m = 1, ..., n$. □

7 $\mathcal{M}$-conformal mappings

The concept of monogenic-conformal mappings described by paravector-valued real differentiable functions in $\Omega \subset \mathbb{R}^{n+1}$ and with values in the Clifford algebra $Cl_{0,n}$ (in the Cauchy-Riemann sense) was introduced by Malonek in [28]. Let $z^* \in S$ be a fixed point and $\{S_m\}$ a regular sequence of subdomains which is shrinking to z^* if m tends to infinity and whereby z^* belongs to all S_m. In [28] it is shown that a function F realizes locally in the neighborhood of a fixed point $z = z^*$ a left $\mathcal{M}$-conformal mapping if and only if F is left monogenic and its left derivative is different from zero.

This result is described by the limit of a "quotient" of a 2-form (surface area) and a 3-form (volume). However, the geometric properties of such a result are not directly visible. Here we show that the description of monogenic functions can be now formulated easily by accessible geometric mapping properties.

For simplicity in what follows we focus our attention on the case of the Dirac operator. Let $\mathbb{R}^{0,3}$ be the real vector space $\mathbb{R}^3$ endowed with a quadratic form of signature $(0,3)$ and let $(\varepsilon_0, \varepsilon_1, \varepsilon_2)$ be an associated orthonormal basis for $\mathbb{R}^{0,3}$. Then $\mathbb{R}^{0,3}$ generates the Clifford algebra $\mathbb{R}_{0,3}$ which is a real linear associative algebra of dimension 2^3 and with identity 1. The multiplication in $\mathbb{R}_{0,3}$ is given according to the multiplication rules

$$\begin{aligned}&\varepsilon_i^2 = -1, \quad i = 0, 1, 2\\ &\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = 0, \quad i \neq j, \quad 0 \leq i, j \leq 2.\end{aligned}$$

We introduce the Dirac operator

$$\partial = \varepsilon_0 \partial_{x_0} + \varepsilon_1 \partial_{x_1} + \varepsilon_2 \partial_{x_2}. \tag{15}$$

We are mainly interested in the case of vector-valued functions $F : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}_{0,3}$ defined as

$$F(x) = F_0(x)\varepsilon_0 + F_1(x)\varepsilon_1 + F_2(x)\varepsilon_2, \tag{16}$$

where its coordinates F_i $(i = 0, 1, 2)$ are real-valued functions defined in Ω. Continuously real-differentiable functions $F : \Omega \longrightarrow \mathbb{R}_{0,3}$ which satisfy

$$\partial F = 0 \Longleftrightarrow \begin{cases} \partial_{x_0} F_0 + \partial_{x_1} F_1 + \partial_{x_2} F_2 = 0 \\ \partial_{x_0} F_1 - \partial_{x_1} F_0 = 0 \\ \partial_{x_0} F_2 - \partial_{x_2} F_0 = 0 \\ \partial_{x_1} F_2 - \partial_{x_2} F_1 = 0 \end{cases}, \tag{17}$$

are said to be (left) monogenic in Ω. Moreover, as $\partial^2 = -\Delta$, where Δ is the Laplace operator in $\mathbb{R}^3$, (left) monogenic functions in Ω are also harmonic in Ω.

Let $F : \Omega \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^{0,3}$ be an arbitrary real-differentiable function. Clearly then,

$$F(\mathbf{x}) := F(\mathbf{0}) + f(\mathbf{x}) + R(\mathbf{x})$$

being f its linear part and R stands for the rest (degree $n \geq 2$). Under the above hypotheses, we claim that f is a general linear function given as in (16), where the coordinates f_i $(i = 0, 1, 2)$ are given by

$$\begin{aligned} f_0(x) &= a_0 x_0 + a_1 x_1 + a_2 x_2 \\ f_1(x) &= b_0 x_0 + b_1 x_1 + b_2 x_2 \\ f_2(x) &= c_0 x_0 + c_1 x_1 + c_2 x_2. \end{aligned} \tag{18}$$

Let us denote by $\mathcal{E}$ the ellipsoid generated by the quadratic form

$$\mathcal{E} := \{(x_0, x_1, x_2) : \frac{x_0^2}{\alpha^2} + \frac{x_1^2}{\beta^2} + \frac{x_2^2}{\gamma^2} = 1\}, \tag{19}$$

where α, β and γ are the lengths of the semi-axes. The next theorem characterizes the local mapping properties of the linear part f of an arbitrary function F.

Theorem 10. *Let f be a linear function. Then, the function f is monogenic if and only if it maps a ball to an ellipsoid centered at the origin with the property that the reciprocal of the length of one semi-axis is equal to the sum of the reciprocals of the lengths of the other two semi-axes.*

Proof. For simplicity we just prove the sufficient condition. The necessary condition can be found in [23]. Suppose that there exists a linear analytic function f which maps the unit ball B to an arbitrarily oriented ellipsoid ε with the referred property. We rotate this ellipsoid transforming it to an ellipsoid ε^* such that the directions of its semi-axes $y^{(0)}$, $y^{(1)}$, $y^{(2)}$ coincide with the directions of the standard coordinate system (y_0, y_1, y_2). Such an ellipsoid is given by

$$\varepsilon^* : \{(y_0, y_1, y_2) : \frac{y_0^2}{\alpha^2} + \frac{y_1^2}{\beta^2} + \frac{y_2^2}{\gamma^2} \leq 1\}.$$

Let $\tilde{f}$ be the function whose image represents ε^*. We denote by $\tilde{D}$ the associated matrix to $\tilde{f}$

$$\tilde{D} = \begin{pmatrix} \tilde{a}_0 & \tilde{a}_1 & \tilde{a}_2 \\ \tilde{b}_0 & \tilde{b}_1 & \tilde{b}_2 \\ \tilde{c}_0 & \tilde{c}_1 & \tilde{c}_2 \end{pmatrix}.$$

We remind that ε^* preserves the orientation, and therefore, it holds the property $\tilde{D}y^{(i)} = \lambda y^{(i)}$ $(i = 0, 1, 2)$. It is easily seen that $\tilde{D}$ is a diagonal matrix and moreover, its elements satisfy the equation $\tilde{a}_0 + \tilde{b}_1 + \tilde{c}_2 = 0$. In this case the associated function is given by

$$\tilde{f}(x) = -(\tilde{b}_1 + \tilde{c}_2)x_0\mathbf{e}_0 + \tilde{b}_1 x_1 \mathbf{e}_1 + \tilde{c}_2 x_2 \mathbf{e}_2.$$

It is easy to check that this function is monogenic (with respect to the Dirac operator). Now, as it was described before, we apply a rotation R to ε^* in order to obtain ε. Roughly speaking, for the rotation

$$R = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix}$$

such that $R^T R = I = RR^T$, we have then that $R\tilde{D}R^T := A$. Note that the original function f is now represented by the symmetric matrix A, in fact $AX = f(x)$ for $X = (x_0 \; x_1 \; x_2)^T$, being its coordinates given by

$$\begin{aligned} f_0(x) &= (\tilde{a}_0 r_{1,1}^2 + \tilde{b}_1 r_{1,2}^2 + \tilde{c}_2 r_{1,3}^2)x_0 + (\tilde{a}_0 r_{1,1}r_{2,1} + \tilde{b}_1 r_{1,2}r_{2,2} + \tilde{c}_2 r_{1,3}r_{2,3})x_1 \\ &+ (\tilde{a}_0 r_{1,1}r_{3,1} + \tilde{b}_1 r_{1,2}r_{3,2} + \tilde{c}_2 r_{1,3}r_{3,3})x_2 \\ f_1(x) &= (\tilde{a}_0 r_{1,1}r_{2,1} + \tilde{b}_1 r_{1,2}r_{2,2} + \tilde{c}_2 r_{1,3}r_{2,3})x_0 + (\tilde{a}_0 r_{2,1}^2 + \tilde{b}_1 r_{2,2}^2 + \tilde{c}_2 r_{2,3}^2)x_1 \\ &+ (\tilde{a}_0 r_{2,1}r_{3,1} + \tilde{b}_1 r_{2,2}r_{3,2} + \tilde{c}_2 r_{2,3}r_{3,3})x_2 \\ f_2(x) &= (\tilde{a}_0 r_{1,1}r_{3,1} + \tilde{b}_1 r_{1,2}r_{3,2} + \tilde{c}_2 r_{1,3}r_{3,3})x_0 \\ &+ (\tilde{a}_0 r_{2,1}r_{3,1} + \tilde{b}_1 r_{2,2}r_{3,2} + \tilde{c}_2 r_{2,3}r_{3,3})x_1 + (\tilde{a}_0 r_{3,1}^2 + \tilde{b}_1 r_{3,2}^2 + \tilde{c}_2 r_{3,3}^2)x_2. \end{aligned}$$

It is easy to check, as desired, that the function f is monogenic. □

Remark 8. *By a simple linear transformation of variables the previous theorem can be extended to a (small) ball with radius R.*

Remark 9. *The composition of a linear function with a translation allows to extend the result to an ellipsoid centered at an arbitrary point $\tilde{x}$. This composition preserves the monogenicity.*

Next one has to show that Theorem 10 can be generalized to arbitrary real-analytic functions which have the described local mapping properties. A monogenic function with non-vanishing linear part will map in the small balls to the special class of ellipsoids. Non-vanishing linear part means that all directional first derivatives of the function are different from zero. Equivalently this can be characterized by the Jacobian determinant. For details and further relations to the hypercomplex derivative see the paper [11].

Theorem 11. *Let F be a real-analytic function. Then, the function F is monogenic if and only if it maps locally a ball to an ellipsoid with the property that the reciprocal of the length of one semi-axis is equal to the sum of the reciprocals of the lengths of the other two semi-axes.*

Proof. If the function is monogenic then there is almost nothing to prove. The local mapping properties at a point x are determined by the linear part of the Taylor expansion at x. Theorem 10 leads to the stated result.

If a real-analytic function has the supposed local mapping properties then we have to expand the function at x in a real Taylor series. Applying ideas from [15], Chapter II, paragraph 5.2.2 we can show after a longer calculation which we will skip here that the linear part of the Taylor expansion satisfies at each point of the domain the Dirac equation and so the function must be monogenic. □

Monogenic functions as null solutions of the Dirac operator can be mapped to monogenic (or anti-monogenic) functions in the sense of satisfied Cauchy-Riemann equations (for details see, e.g., [13]). This transformation is an isometry and so it becomes clear that the here discussed mapping properties of monogenic functions remain true under this transformation.

The result of Theorem 11 allows to describe the monogenic functions as a special class of quasi-conformal mappings. If we visualize quasi-conformal mappings in $\mathbb{R}^3$ by points, given by the lengths of the semi-axes of the associated ellipsoids, then the monogenic functions (with non-vanishing Jacobian determinant) can be seen as a two-dimensional manifold in $\mathbb{R}^3$.

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Wrap groups of fiber bundles and their structure

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ABSTRACT

This article is devoted to the investigation of wrap groups of connected fiber bundles. These groups are constructed with mild conditions on fibers. Their examples are given. It is shown, that these groups exist and for differentiable fibers have the infinite dimensional Lie groups structure, that is, they are continuous or differentiable manifolds and the composition $(f, g) \mapsto f^{-1}g$ is continuous or differentiable depending on a class of smoothness of groups. Moreover, it is demonstrated that in the cases of real, complex, quaternion and octonion manifolds these groups have structures of real, complex, quaternion or octonion manifolds respectively. Nevertheless, it is proved that these groups does not necessarily satisfy the Campbell-Hausdorff formula even locally.

RESUMEN

Este artículo es dedicado a la investigación de grupos Wrap de fibrados conexos. Estos grupos son construidos con condiciones blandas sobre las fibras, ejemplos son dados. Es demostrado que estos grupos existen y para fibras diferenciables tienen una estructura de grupo de Lie infinito dimensional, es decir, son variedades continuas o diferenciables y la composición $(f, g) \mapsto f^{-1}g$ es continua o diferenciable dependiendo de la clase de suavidad de los grupos. Además es demostrado que en el caso de variedades real, compleja, cuaternion y octonion esos grupos tienen una estructura de variedad real, compleja, cuaternion o octonion respectivamente. También es probado que estos grupos no necesariamente satisfacen la fórmula de Campbell-Hausdorff incluso localmente.

1 Introduction.

Wrap groups of fiber bundles considered in this paper are constructed with the help of families of mappings from a fiber bundle with a marked point into another fiber bundle with a marked point over the fields **R**, **C**, **H** and the octonion algebra **O**. Conditions on fibers supplied with parallel transport structures are rather mild here. Therefore, they generalize geometric loop groups of circles, spheres and fibers with parallel transport structures over them. A loop interpretation is lost in their generalizations, so they are called here wrap groups. This paper continues previous works of the author on this theme, where generalized loop groups of manifolds over **R**, **C** and **H** were investigated, but neither for fibers nor over octonions [15, 23, 21, 22].

Loop groups of circles were first introduced by Lefshetz in 1930-th and then their construction was reconsidered by Milnor in 1950-th. Lefshetz has used the C^0-uniformity on families of continuous mappings, which led to the necessity of combining his construction with the structure of a free group with the help of words. Later on Milnor has used the Sobolev's H^1-uniformity, that permitted to introduce group structure more naturally [27]. Iterations of these constructions produce iterated loop groups of spheres. Then their constructions were generalized for fibers over circles and spheres with parallel transport structures over **R** or **C** [4].

Wrap groups of quaternion and octonion fibers as well as for wider classes of fibers over **R** or **C** are defined and investigated here for the first time.

Holomorphic functions of quaternion and octonion variables were investigated in [19, 20, 17]. There specific definition of super-differentiability was considered, because the quaternion skew field has the graded algebra structure. This definition of super-differentiability does not impose the condition of right or left super-linearity of a super-differential, since it leads to narrow class of functions. There are some articles on quaternion manifolds, but practically they undermine a complex manifold with additional quaternion structure of its tangent space (see, for example, [28, 39] and references therein). Therefore, quaternion manifolds as they are defined below were not considered earlier by others authors (see also [17]). Applications of quaternions in mathematics and physics can be found in [6, 9, 10, 14].

In this article wrap groups of different classes of smoothness are considered. Henceforth, we consider not only orientable manifolds M and N, but also nonorientable manifolds.

In particular, geometric loop groups have important applications in modern physical theories (see [11, 24] and references therein). Groups of loops are also intensively used in gauge theory. Wrap groups defined below with the help of families of mappings from a manifold M into another manifold N with a dimension $dim(M) > 1$ can be used in the membrane theory which is the generalization of the string (superstring) theory.

Section 2 is devoted to the definitions of topological and manifold structures of wrap groups. The existence of these groups is proved and that they are infinite dimensional Lie groups not satisfying even locally the Campbell-Hausdorff formula (see Theorems 3, 6, 12, Corollaries 5, 8, 9 and Examples 10). In the cases of complex, quaternion and octonion manifolds it is proved that they have structures of complex, quaternion and octonion manifolds respectively.

All main results of this paper are obtained for the first time.

2 Wrap groups of fibers.

To avoid misunderstandings we first give our definitions and notations.

1.1. Note. Denote by $\mathcal{A}_r$ the Cayley-Dickson algebra such that $\mathcal{A}_0 = \mathbf{R}$, $\mathcal{A}_1 = \mathbf{C}$, $\mathcal{A}_2 = \mathbf{H}$ is the quaternion skew field, $\mathcal{A}_3 = \mathbf{O}$ is the octonion algebra. Henceforth we consider only $0 \le r \le 3$.

1.2. Definition. A canonical closed subset Q of the Euclidean space $X = \mathbf{R^n}$ or of the standard separable Hilbert space $X = l_2(\mathbf{R})$ over $\mathbf{R}$ is called a quadrant if it can be given by the condition $Q := \{x \in X : q_j(x) \ge 0\}$, where $(q_j : j \in \Lambda_Q)$ are linearly independent elements of the topologically adjoint space X^*. Here $\Lambda_Q \subset \mathbf{N}$ (with $card(\Lambda_Q) = k \le n$ when $X = \mathbf{R^n}$) and k is called the index of Q. If $x \in Q$ and exactly j of the q_i's satisfy $q_i(x) = 0$ then x is called a corner of index j.

If X is an additive group and also left and right module over $\mathbf{H}$ or $\mathbf{O}$ with the corresponding associativity or alternativity respectively and distributivity laws then it is called the vector space over $\mathbf{H}$ or $\mathbf{O}$ correspondingly.

In particular $l_2(\mathcal{A}_r)$ consisting of all sequences $x = \{x_n \in \mathcal{A}_r : n \in \mathbf{N}\}$ with the finite norm $\|x\| < \infty$ and scalar product $(x, y) := \sum_{n=1}^{\infty} x_n y_n^*$ with $\|x\| := (x, x)^{1/2}$ is called the Hilbert space (of separable type) over $\mathcal{A}_r$, where z^* denotes the conjugated Cayley-Dickson number, $zz^* =: |z|^2$, $z \in \mathcal{A}_r$. Since the unitary space $X = \mathcal{A}_r^n$ or the separable Hilbert space $l_2(\mathcal{A}_r)$ over $\mathcal{A}_r$ while considered over the field $\mathbf{R}$ (real shadow) is isomorphic with $X_{\mathbf{R}} := \mathbf{R^{2^r n}}$ or $l_2(\mathbf{R})$, then the above definition also describes quadrants in $\mathcal{A}_r^n$ and $l_2(\mathcal{A}_r)$. In the latter case we also consider generalized quadrants as canonical closed subsets which can be given by $Q := \{x \in X_{\mathbf{R}} : q_j(x + a_j) \ge 0, a_j \in X_{\mathbf{R}}, j \in \Lambda_Q\}$, where $\Lambda_Q \subset \mathbf{N}$ ($card(\Lambda_Q) = k \in \mathbf{N}$ when $dim_{\mathbf{R}} X_{\mathbf{R}} < \infty$).

1.2.2. Definition. A differentiable mapping $f : U \to U'$ is called a diffeomorphism if

(i) f is bijective and there exist continuous mappings f' and $(f^{-1})'$, where U and U' are interiors of quadrants Q and Q' in X.

In the $\mathcal{A}_r$ case with $1 \le r \le 3$ we consider bounded generalized quadrants Q and Q' in $\mathcal{A}_r^n$ or $l_2(\mathcal{A}_r)$ such that they are domains with piecewise C^∞-boundaries. We impose additional conditions on the diffeomorphism f in the $1 \le r \le 3$ case:

(ii) $\bar{\partial} f = 0$ on U,

(iii) f and all its strong (Frechét) differentials (as multi-linear operators) are bounded on U, where ∂f and $\bar{\partial} f$ are differential $(1, 0)$ and $(0, 1)$ forms respectively, $d = \partial + \bar{\partial}$ is an exterior derivative, for $2 \le r \le 3$ ∂ corresponds to super-differentiation by z and $\tilde{\partial} = \bar{\partial}$ corresponds to super-differentiation by $\tilde{z} := z^*$, $z \in U$ (see [19, 20]).

The Cauchy-Riemann Condition (ii) means that f on U is the $\mathcal{A}_r$-holomorphic mapping.

1.2.3. Definition and notation. An $\mathcal{A}_r$-manifold M with corners is defined in the usual way: it is a metric separable space modelled on $X = \mathcal{A}_r^n$ or $X = l_2(\mathcal{A}_r)$ respectively and is supposed to be of class C^∞, $0 \le r \le 3$. Charts on M are denoted (U_l, u_l, Q_l), that is, $u_l : U_l \to u_l(U_l) \subset Q_l$ is a C^∞-diffeomorphism for each l, U_l is open in M, $u_l \circ {u_j}^{-1}$ is biholomorphic for $1 \le r \le 3$ from the domain $u_j(U_l \cap U_j) \ne \emptyset$ onto $u_l(U_l \cap U_j)$ (that is, $u_j \circ u_l^{-1}$ and $u_l \circ u_j^{-1}$ are holomorphic and bijective) and $u_l \circ u_j^{-1}$ satisfy conditions $(i - iii)$ from §1.2.2, $\bigcup_j U_j = M$.

A point $x \in M$ is called a corner of index j if there exists a chart (U, u, Q) of M with $x \in U$ and $u(x)$ is of index $ind_M(x) = j$ in $u(U) \subset Q$. A set of all corners of index $j \ge 1$ is called a border ∂M of M, x is called an inner point of M if $ind_M(x) = 0$, so $\partial M = \bigcup_{j \ge 1} \partial^j M$, where $\partial^j M := \{x \in M : ind_M(x) = j\}$.

For a real manifold with corners on the connecting mappings $u_l \circ u_j^{-1} \in C^\infty$ of real charts only Condition 1.2.2(i) is imposed.

1.2.4. Terminology. In an $\mathcal{A}_r$-manifold N there exists an Hermitian metric, which in each analytic system of coordinates is the following $\sum_{j,k=1}^n h_{j,k} dz_j d\bar{z}_k$, where $(h_{j,k})$ is a positive definite Hermitian matrix with coefficients of the class C^∞, $h_{j,k} = h_{j,k}(z) \in \mathcal{A}_r$, z are local coordinates in N.

As real manifolds we shall consider Riemann manifolds.

In accordance with the definition above for internal points of N it is supposed that they can belong only to interiors of charts, but for boundary points ∂N it may happen that $x \in \partial N$ belongs to boundaries of several charts. It is convenient to choose an atlas such that $ind(x)$ is the same for all charts containing this x.

1.3.1. Remark. If M is a metrizable space and $K = K_M$ is a closed subset in M of codimension $codim_{\mathbf{R}}\ N \ge 2$ such that $M \setminus K = M_1$ is a manifold with corners over $\mathcal{A}_r$, then we call M a pseudo-manifold over $\mathcal{A}_r$, where K_M is a critical subset.

Two pseudo-manifolds B and C are called diffeomorphic, if $B \setminus K_B$ is diffeomorphic with $C \setminus K_C$ as for manifolds with corners (see also [4, 26]).

Take on M a Borel σ-additive measure ν such that ν on $M \setminus K$ coincides with the Riemann volume element and $\nu(K) = 0$, since the real shadow of M_1 has it.

The uniform space $H_p^t(M_1, N)$ of all continuous piecewise H^t Sobolev mappings from M_1 into N is introduced in the standard way [21, 22], which induces $H_p^t(M, N)$ the uniform space of continuous piecewise H^t Sobolev mappings on M, since $\nu(K) = 0$, where $\mathbf{R} \ni t \ge [m/2] + 1$, m denotes the dimension of M over $\mathbf{R}$, $[k]$ denotes the integer part of $k \in \mathbf{R}$, $[k] \le k$. Then put $H_p^\infty(M, N) = \bigcap_{t > m} H_p^t(M, N)$ with the corresponding uniformity.

For manifolds over $\mathcal{A}_r$ with $1 \le r \le 3$ take as $H_p^t(M, N)$ the completion of the family of all continuous piecewise $\mathcal{A}_r$-holomorphic mappings from M into N relative to the H_p^t uniformity, where $[m/2]+1 \le t \le \infty$. Henceforth we consider pseudo-manifolds with connecting mappings of charts continuous in M and $H_p^{t'}$ in $M \setminus K_M$ for $0 \le r \le 3$, where $t' \ge t$.

1.3.2. Note. Since the octonion algebra $\mathbf{O}$ is non-associative, we consider a non-associative subgroup G of the family $Mat_q(\mathbf{O})$ of all square $q \times q$ matrices with entries in $\mathbf{O}$. More generally G is a group which has a H_p^t manifold structure over $\mathcal{A}_r$ and group's operations are H_p^t mappings. The G may be non-associative for $r=3$, but G is supposed to be alternative, that is, $(aa)b = a(ab)$ and $a(a^{-1}b) = b$ for each $a, b \in G$.

As a generalization of pseudo-manifolds there is used the following (over $\mathbf{R}$ and $\mathbf{C}$ see [4, 34]). Suppose that M is a Hausdorff topological space of covering dimension $dim\ M = m$ supplied with a family $\{h : U \to M\}$ of the so called plots h which are continuous maps satisfying conditions $(D1-D4)$:

$(D1)$ each plot has as a domain a convex subset U in $\mathcal{A}_r^n$, $n \in \mathbf{N}$;

$(D2)$ if $h : U \to M$ is a plot, V is a convex subset in $\mathcal{A}_r^l$ and $g : V \to U$ is an H_p^t mapping, then $h \circ g$ is also a plot, where $t \ge [m/2]+1$;

$(D3)$ every constant map from a convex set U in $\mathcal{A}_r^n$ into M is a plot;

$(D4)$ if U is a convex set in $\mathcal{A}_r^n$ and $\{U_j : j \in J\}$ is a covering of U by convex sets in $\mathcal{A}_r^n$, each U_j is open in U, $h : U \to M$ is such that each its restriction $h|_{U_j}$ is a plot, then h is a plot. Then M is called an H_p^t-differentiable space.

A mapping $f : M \to N$ between two H_p^t-differentiable spaces is called differentiable if it continuous and for each plot $h : U \to M$ the composition $f \circ h : U \to N$ is a plot of N. A topological group G is called an H_p^t-differentiable group if its group operations are H_p^t-differentiable mappings.

Let E, N, F be $H_p^{t'}$-pseudo-manifolds or $H_p^{t'}$-differentiable spaces over $\mathcal{A}_r$, let also G be an $H_p^{t'}$ group over $\mathcal{A}_r$, $t \le t' \le \infty$. A fiber bundle $E(N, F, G, \pi, \Psi)$ with a fiber space E, a base space N, a typical fiber F and a structural group G over $\mathcal{A}_r$, a projection $\pi : E \to N$ and an atlas Ψ is defined in the standard way [4, 26, 35] with the condition, that transition functions are of $H_p^{t'}$ class such that for $r=3$ a structure group may be non-associative, but alternative.

Local trivializations $\phi_j \circ \pi \circ \Psi_k^{-1} : V_k(E) \to V_j(N)$ induce the $H_p^{t'}$-uniformity in the family W of all principal $H_p^{t'}$-fiber bundles $E(N, G, \pi, \Psi)$, where $V_k(E) = \Psi_k(U_k(E)) \subset X^2(G)$, $V_j(N) = \phi_j(U_j(N)) \subset X(N)$, where $X(G)$ and $X(N)$ are $\mathcal{A}_r$-vector spaces on which G and N are modelled, $(U_k(E), \Psi_k)$ and $(U_j(N), \phi_j)$ are charts of atlases of E and N, $\Psi_k = \Psi_k^E$, $\phi_j = \phi_j^N$.

If $G = F$ and G acts on itself by left shifts, then a fiber bundle is called the principal fiber bundle and is denoted by $E(N, G, \pi, \Psi)$. As a particular case there may be $G = \mathcal{A}_r^*$, where $\mathcal{A}_r^*$ denotes the multiplicative group $\mathcal{A}_r \setminus \{0\}$. If $G = F = \{e\}$, then E reduces to N.

2. Definitions. Let M be a connected H_p^t-pseudo-manifold over $\mathcal{A}_r$, $0 \le r \le 3$ satisfying the following conditions:

(i) it is compact;

(ii) M is a union of two closed subsets over $\mathcal{A}_r$ A_1 and A_2, which are pseudo-manifolds and which are canonical closed subsets in M with $A_1 \cap A_2 = \partial A_1 \cap \partial A_2 =: A_3$ and a codimension over $\mathbf{R}$ of A_3 in M is $codim_{\mathbf{R}} A_3 = 1$, also A_3 is a pseudo-manifold;

(iii) a finite set of marked points $s_{0,1}, ..., s_{0,k}$ is in $\partial A_1 \cap \partial A_2$, moreover, ∂A_j are arcwise connected $j = 1, 2$;

(iv) $A_1 \setminus \partial A_1$ and $A_2 \setminus \partial A_2$ are H_p^t-diffeomorphic with $M \setminus [\{s_{0,1}, ..., s_{0,k}\} \cup (A_3 \setminus Int(\partial A_1 \cap \partial A_2))]$ by mappings $F_j(z)$, where $j = 1$ or $j = 2$, $\infty \geq t \geq [m/2] + 1$, $m = dim_{\mathbf{R}} M$ such that $H^t \subset C^0$ due to the Sobolev embedding theorem [25], where the interior $Int(\partial A_1 \cap \partial A_2)$ is taken in $\partial A_1 \cup \partial A_2$.

Instead of (iv) we consider also the case

(iv') M, A_1 and A_2 are such that $(A_j \setminus \partial A_j) \cup \{s_{0,1}, ..., s_{0,k}\}$ are $C^0([0,1], H_p^t(A_j, A_j))$-retractable on $X_{0,q} \cap A_j$, where $X_{0,q}$ is a closed arcwise connected subset in M, $j = 1$ or $j = 2$, $s_{0,q} \in X_{0,q}$, $X_{0,q} \subset K_M$, $q = 1, ..., k$, $codim_{\mathbf{R}}\ K_M \geq 2$.

Let $\hat{M}$ be a compact connected H_p^t-pseudo-manifold which is a canonical closed subset in $\mathcal{A}_r^l$ with a boundary $\partial \hat{M}$ and marked points $\{\hat{s}_{0,q} \in \partial \hat{M} : q = 1, ..., 2k\}$ and an H_p^t-mapping $\Xi : \hat{M} \to M$ such that

(v) Ξ is surjective and bijective from $\hat{M} \setminus \partial \hat{M}$ onto $M \setminus \Xi(\partial \hat{M})$ open in M, $\Xi(\hat{s}_{0,q}) = \Xi(\hat{s}_{0,k+q}) s_{0,q}$ for each $q = 1, ..., k$, also $\partial M \subset \Xi(\partial \hat{M})$.

A parallel transport structure on a $H_p^{t'}$-differentiable principal G-bundle $E(N, G, \pi, \Psi)$ with arcwise connected E and G for H_p^t-pseudo-manifolds M and $\hat{M}$ as above over the same $\mathcal{A}_r$ with $t' \geq t + 1$ assigns to each H_p^t mapping γ from M into N and points $u_1, ..., u_k \in E_{y_0}$, where y_0 is a marked point in N, $y_0 = \gamma(s_{0,q})$, $q = 1, ..., k$, a unique H_p^t mapping $\mathbf{P}_{\hat{\gamma},u} : \hat{M} \to E$ satisfying conditions $(P1 - P5)$:

$(P1)$ take $\hat{\gamma} : \hat{M} \to N$ such that $\hat{\gamma} = \gamma \circ \Xi$, then $\mathbf{P}_{\hat{\gamma},u}(\hat{s}_{0,q}) = u_q$ for each $q = 1, ..., k$ and $\pi \circ \mathbf{P}_{\hat{\gamma},u} = \hat{\gamma}$

$(P2)$ $\mathbf{P}_{\hat{\gamma},u}$ is the H_p^t-mapping by γ and u;

$(P3)$ for each $x \in \hat{M}$ and every $\phi \in DifH_p^t(\hat{M}, \{\hat{s}_{0,1}, ..., \hat{s}_{0,2k}\})$ there is the equality $\mathbf{P}_{\hat{\gamma},u}(\phi(x)) = \mathbf{P}_{\hat{\gamma}\circ\phi,u}(x)$, where $DifH_p^t(\hat{M}, \{\hat{s}_{0,1}, ..., \hat{s}_{0,2k}\})$ denotes the group of all H_p^t homeomorphisms of $\hat{M}$ preserving marked points $\phi(\hat{s}_{0,q}) = \hat{s}_{0,q}$ for each $q = 1, ..., 2k$;

$(P4)$ $\mathbf{P}_{\hat{\gamma},u}$ is G-equivariant, which means that $\mathbf{P}_{\hat{\gamma},uz}(x) = \mathbf{P}_{\hat{\gamma},u}(x) z$ for every $x \in \hat{M}$ and each $z \in G$;

$(P5)$ if U is an open neighborhood of $\hat{s}_{0,q}$ in $\hat{M}$ and $\hat{\gamma}_0, \hat{\gamma}_1 : U \to N$ are $H_p^{t'}$-mappings such that $\hat{\gamma}_0(\hat{s}_{0,q}) = \hat{\gamma}_1(\hat{s}_{0,q}) = v_q$ and tangent spaces, which are vector manifolds over $\mathcal{A}_r$, for γ_0 and γ_1 at v_q are the same, then the tangent spaces of $\mathbf{P}_{\hat{\gamma}_0,u}$ and $\mathbf{P}_{\hat{\gamma}_1,u}$ at u_q are the same, where $q = 1, ..., k$, $u = (u_1, ..., u_k)$.

Two $H_p^{t'}$-differentiable principal G-bundles E_1 and E_2 with parallel transport structures $(E_1, \mathbf{P}_1)$ and $(E_2, \mathbf{P}_2)$ are called isomorphic, if there exists an isomorphism $h : E_1 \to E_2$ such that $\mathbf{P}_{2,\hat{\gamma},u}(x) = h(\mathbf{P}_{1,\hat{\gamma},h^{-1}(u)}(x))$ for each H_p^t-mapping $\gamma : M \to N$ and $u_q \in (E_2)_{y_0}$, where $q = 1, ..., k$, $h^{-1}(u) = (h^{-1}(u_1), ..., h^{-1}(u_k))$.

Let $(S^M E)_{t,H} : (S^{M,\{s_{0,q}:q=1,...,k\}} E; N, G, \mathbf{P})_{t,H}$ be a set of H_p^t-closures of isomorphism classes of H_p^t principal G fiber bundles with parallel transport structure.

3. Theorems. 1. *The uniform space $(S^M E)_{t,H}$ from §2 has the structure of a topological alternative monoid with a unit and with a cancelation property and the multiplication operation of H_p^l class with $l = t' - t$ ($l = \infty$ for $t' = \infty$). If N and G are separable, then $(S^M E)_{t,H}$ is separable. If N and G are complete, then $(S^M E)_{t,H}$ is complete.*

2. *If G is associative, then $(S^M E)_{t,H}$ is associative. If G is commutative, then $(S^M E)_{t,H}$ is commutative. If G is a Lie group, then $(S^M E)_{t,H}$ is a Lie monoid.*

3. *The $(S^M E)_{t,H}$ is non-discrete, locally connected and infinite dimensional for $dim_{\mathbf{R}}(N \times G) > 1$.*

Proof. If there is a homomorphism $\theta : G \to F$ of $H_p^{t'}$-differentiable groups, then there exists an induced principal F fiber bundle $(E \times^\theta F)(N, F, \pi^\theta, \Psi^\theta)$ with the total space $(E \times^\theta F)(E \times F)/\mathcal{Y}$, where $\mathcal{Y}$ is the equivalence relation such that $(vg, f)\mathcal{Y}(v, \theta(g)f)$ for each $v \in E$, $g \in G$, $f \in F$. Then the projection $\pi^\theta : (E \times^\theta F) \to N$ is defined by $\pi^\theta([v, f]) = \pi(v)$, where $[v, f] := \{(w, b) : (w, b)\mathcal{Y}(v, f), w \in E, b \in F\}$ denotes the equivalence class of (v, f).

Therefore, each parallel transport structure $\mathbf{P}$ on the principal G fiber bundle $E(N, G, \pi, \Psi)$ induces a parallel transport structure $\mathbf{P}^\theta$ on the induced bundle by the formula $\mathbf{P}^\theta_{\hat{\gamma},[u,f]}(x) = [\mathbf{P}_{\hat{\gamma},u}(x), f]$.

Define multiplication with the help of certain embeddings and isomorphisms of spaces of functions. Mention that for each two compact canonical closed subsets A and B in $\mathcal{A}_r^l$ Hilbert spaces $H^t(A, \mathbf{R}^m)$ and $H^t(B, \mathbf{R}^m)$ are linearly topologically isomorphic, where $l, m \in \mathbf{N}$, hence $H_p^t(A, N)$ and $H_p^t(B, N)$ are isomorphic as uniform spaces. Let $H_p^t(M, \{s_{0,1}, ..., s_{0,k}\}; W, y_0) := \{(E, f) : E = E(N, G, \pi, \Psi) \in W, f = \mathbf{P}_{\hat{\gamma},y_0} \in H_p^t : \pi \circ f(s_{0,q}) = y_0 \forall q = 1, ..., k; \pi \circ f = \hat{\gamma}, \gamma \in H_p^t(M, N)\}$ be the space of all $H_p^{t'}$ principal G fiber bundles E with their parallel transport H_p^t-mappings $f = \mathbf{P}_{\hat{\gamma},y_0}$, where W is as in §1.3.2. Put $\omega_0 = (E_0, \mathbf{P}_0)$ be its element such that $\gamma_0(M) = \{y_0\}$, where $e \in G$ denotes the unit element, $E_0 = N \times G$, $\pi_0(y, g) = y$ for each $y \in N$, $g \in G$, $\mathbf{P}_{\hat{\gamma}_0,u} = \mathbf{P}_0$.

The mapping $\Xi : \hat{M} \to M$ from §2 induces the embedding

$\Xi^* : H_p^t(M, \{s_{0,1}, ..., s_{0,k}\}; W, y_0) \hookrightarrow H_p^t(\hat{M}, \{\hat{s}_{0,1}, ..., \hat{s}_{0,2k}\}; W, y_0)$,
where $\hat{M}$ and $\hat{A}_1$ and $\hat{A}_2$ are retractable into points.

Let as usually $A \vee B := \rho(\mathcal{Z})$ be the wedge sum of pointed spaces $(A, \{a_{0,q} : q = 1, ..., k\})$ and $(B, \{b_{0,q} : q = 1, ..., k\})$, where $\mathcal{Z} := [A \times \{b_{0,q} : q = 1, ..., k\} \cup \{a_{0,q} : q = 1, ..., k\} \times B] \subset A \times B$, ρ is a continuous quotient mapping such that $\rho(x) = x$ for each $x \in \mathcal{Z} \setminus \{a_{0,q} \times b_{0,j}; q, j = 1, ..., k\}$ and $\rho(a_{0,q}) = \rho(b_{0,q})$ for each $q = 1, ..., k$, where A and B are topological spaces with marked points $a_{0,q} \in A$ and $b_{0,q} \in B$, $q = 1, ..., k$. Then the wedge product $g \vee f$ of two elements $f, g \in H^t_p(M, \{s_{0,1}, ..., s_{0,k}\}; N, y_0)$ is defined on the domain $M \vee M$ such that $(f \vee g)(x \times b_{0,q}) = f(x)$ and $(f \vee g)(a_{0,q} \times x) = g(x)$ for each $x \in M$, where to f, g there correspond $f_1, g_1 \in H^t_p(\hat{M}, \{\hat{s}_{0,1}, ..., \hat{s}_{0,2k}\}; N, y_0)$ such that $f_1 f \circ \Xi$ and $g_1 = g \circ \Xi$.

Let $(E_j, \mathbf{P}_{\hat{\gamma}_j, u^j}) \in H^t_p(M, \{s_{0,1}, ..., s_{0,k}\}; W, y_0)$, $j = 1, 2$, then take their wedge product $\mathbf{P}_{\hat{\gamma}, u^1} := \mathbf{P}_{\hat{\gamma}_1, u^1} \vee \mathbf{P}_{\hat{\gamma}_2, v}$ on $M \vee M$ with $v_q = u_q g^{-1}_{2,q} g_{1,q+k} = y_0 \times g_{1,q+k}$ for each $q = 1, ..., k$ due to the alternativity of G, $\gamma = \gamma_1 \vee \gamma_2$, where $\mathbf{P}_{\hat{\gamma}_j, u^j}(\hat{s}_{j,0,q}) y_0 \times g_{j,q} \in E_{y_0}$ for every j and q. For each $\gamma_j : M \to N$ there exists $\tilde{\gamma}_j : M \to E_j$ such that $\pi \circ \tilde{\gamma}_j = \gamma_j$. Denote by $\mathbf{m} : G \times G \to G$ the multiplication operation. The wedge product $(E_1, \mathbf{P}_{\hat{\gamma}_1, u^1}) \vee (E_2, \mathbf{P}_{\hat{\gamma}_2, u^2})$ is the principal G fiber bundle $(E_1 \times E_2) \times^{\mathbf{m}} G$ with the parallel transport structure $\mathbf{P}_{\hat{\gamma}_1, u^1} \vee \mathbf{P}_{\hat{\gamma}_2, v}$.

The uniform space $H^t_p(J, A_3; W, y_0) := \{(E, f) \in H^t_p(J, W) : \pi \circ f(A_3) = \{y_0\}\}$ has the H^t_p-manifold structure and has an embedding into
$H^t_p(M, \{s_{0,1}, ..., s_{0,k}\}; W, y_0)$ due to Conditions $2(i - iii)$, where either $J = A_1$ or $J = A_2$. This induces the following embedding $\chi^* : H^t_p(M \vee M, \{s_{0,q} \times s_{0,q} : q = 1, ..., k\}; W, y_0) \hookrightarrow H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0)$.

Analogously considering $H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0) = \{f \in H^t(M, W) : f(X_{0,q}) = \{y_0\}, q = 1, ..., k\}$ and $H^t_p(J, A_3 \cup \{X_{0,q} : q = 1, ..., k\}; W, y_0)$ in the case (iv') instead of (iv) we get the embedding $\chi^* : H^t_p(M \vee M, \{X_{0,q} \times X_{0,q} : q = 1, ..., k\}; W, y_0) \hookrightarrow H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0)$. Therefore, $g \circ f := \chi^*(f \vee g)$ is the composition in $H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0)$.

There exists the following equivalence relation $R_{t,H}$ in $H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0)$: $f R_{t,H} h$ if and only if there exist nets $\eta_n \in Dif H^t_p(M, \{X_{0,q} : q = 1, ..., k\})$, also f_n and $h_n \in H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0)$ with $\lim_n f_n = f$ and $\lim_n h_n = h$ such that $f_n(x) = h_n(\eta_n(x))$ for each $x \in M$ and $n \in \omega$, where ω is a directed set and convergence is considered in $H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0)$. Henceforward in the case $2(iv)$ we get $s_{0,q}$ instead of $X_{0,q}$ in the case $2(iv')$.

Thus there exists the quotient uniform space
$H^t_p(M, \{X_{0,q} : q = 1, ..., k\}; W, y_0)/R_{t,H} =: (S^M E)_{t,H}$. In view of [30, 31] $Dif H^t_p(M)$ is the group of diffeomorphisms for $t \geq [m/2] + 1$. The Lebesgue measure λ in the real shadow of $\hat{M}$ by the mapping Ξ induces the measure λ^{Ξ} on M which is equivalent to ν, since Ξ is the H^t_p-mapping from the compact space onto the compact space, $\lambda(\partial \hat{M}) = 0$ and $\Xi : \hat{M} \setminus \partial \hat{M} \to M$ is bijective.

Due to Conditions $(P1 - P5)$ each element $f = \mathbf{P}_{\hat{\gamma}, u}$ up to a set Q_M of measure zero, $\nu(Q_M) = 0$, is given as $f \circ \Xi^{-1}$ on $M \setminus Q_M$, where $\pi \circ f = \hat{\gamma}$, $\hat{\gamma} = \gamma \circ \Xi$. Denote $f \circ \Xi^{-1}$ also by f. Thus, for each $(E, f) \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0)$ the image $f(M)$ is compact and connected in E.

Therefore, for each partition Z there exists $\delta > 0$ such that for each partition Z^* with $\sup_i \inf_j dist(M_i, M^*{}_j) < \delta$ and $(E, f) \in H^t(M, W; Z)$, $f(s_{0,q}) = u_q$, there exists $(E, f_1) \in H^t(M, W; Z^*)$ with $f_1(s_{0,q}) = u_q$ for each $q = 1, ..., k$ such that $f R_{t,H} f_1$, where M_i and M^*_j are canonical closed pseudo-submanifolds in M corresponding to partitions Z and Z^*, $H^t(M, W; Z)$ denotes the space of all continuous piecewise H^t-mappings from M into W subordinated to the partition Z such that Z and Z^* respect H^t_p structure of M.

Hence there exists a countable subfamily $\{Z_j : j \in \mathbf{N}\}$ in the family of all partitions Υ such that $Z_j \subset Z_{j+1}$ for each j and $\lim_j \tilde{diam} Z_j = 0$. Then

(*i*) $str-ind\{H^t(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0; Z_j); h^{Z_i}_{Z_j}; \mathbf{N}\}/R_{t,H} = (S^M E)_{t,H}$ is separable if N and G are separable, since each space $H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0; Z_j)$ is separable.

The space $str-ind\{H^t(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0; Z_j); h^{Z_i}_{Z_j}; \mathbf{N}\}$ is complete due to Theorem 12.1.4 [29], when N and G are complete. Each class of $R_{t,H}$-equivalent elements is closed in it. Then to each Cauchy net in $(S^M E)_{t,H}$ there corresponds a Cauchy net in $str-ind\{H^t(M \times [0, 1], \{s_{0,q} \times e \times 0; W, y_0; Z_j \times Y_j); h^{Z_i \times Y_i}_{Z_j \times Y_j}; \mathbf{N}\}$ due to theorems about extensions of functions [25, 33, 38], where Y_j are partitions of $[0, 1]$ with $\lim_j \tilde{diam}(Y_j) = 0$, $Z_j \times Y_j$ are the corresponding partitions of $M \times [0, 1]$. Hence $(S^M E)_{t,H}$ is complete, if N and G are complete.

If $f, g \in H^t(M, X)$ and $f(M) \neq g(M)$, then

(*ii*) $\inf_{\psi \in Dif H^t_p(M,\{s_{0,q}:q=1,...,k\})} \|f \circ \psi - g\|_{H^t(M,X)} > 0$. Thus equivalence classes $< f >_{t,H}$ and $< g >_{t,H}$ are different. The pseudo-manifold $\hat{M}$ is arcwise connected. Take $\eta : [0, 1] \to \hat{M}$ an H^t_p-mapping with $\eta(0) = \hat{s}_{0,q}$ and $\eta(1) = \hat{s}_{0,k+q}$, where $1 \leq q \leq k$. Choose in $\hat{M}$ H^t_p-coordinates one of which is a parameter along η. Therefore, for each $g_q, g_{k+q} \in G$ there exists $\mathbf{P}_{\hat{\gamma},u}$ with $\mathbf{P}_{\hat{\gamma},u}(s_{0,q}) = y_0 \times g_q$ and $\mathbf{P}_{\hat{\gamma},u}(s_{0,k+q}) = y_0 \times g_{k+q}$ for each $q = 1, ..., k$. Since E and G are arcwise connected, then N is arcwise connected and $(S^M E)_{t,H}$ is locally connected for $dim_{\mathbf{R}} N > 1$. Thus, the uniform space $(S^M E)_{t,H}$ is non-discrete.

The tangent bundle $TH^t_p(M, E)$ is isomorphic with $H^t_p(M, TE)$, where TE is the $H^{t'-1}_p$ fiber bundle, $t' \geq t + 1$. There is an infinite family of $f_\alpha \in H^t_p(M, TE)$ with pairwise distinct images in TE for different α such that $f_\alpha(M)$ is not contained in $\bigcup_{\beta<\alpha} f_\beta(M)$, $\alpha \in \Lambda$, where Λ is an infinite ordinal. Therefore, $T(S^M E)_{t,H}$ is an infinite dimensional fiber bundle due to (*ii*) and inevitably $(S^M E)_{t,H}$ is infinite dimensional.

Evidently, if $f \vee g = h \vee g$ or $g \vee f = g \vee h$ for $\{f, g, h\} \subset H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0)$, then $f = h$. Thus $\chi^*(f \vee g) = \chi^*(h \vee g)$ or $\chi^*(g \vee f) = \chi^*(g \vee h)$ is equivalent to $f = h$ due to the definition of $f \vee g$ and the definition of equal functions, since χ^* is the embedding. Using the equivalence relation $R_{t,H}$ gives $< f >_{t,H} \circ < g >_{t,H} =< h >_{t,H} \circ < g >_{t,H}$ or $< g >_{t,H} \circ < f >_{t,H} =< g >_{t,H} \circ < h >_{t,H}$ is equivalent to $< h >_{t,H} =< f >_{t,H}$. Therefore, $(S^M E)_{t,H}$ has the cancelation property.

Since G is alternative, then $a_{2,q}[a^{-1}_{2,q}(a_{2,q+k}(a^{-1}_{2,q}a_{1,q+k}))]a_{2,q+k}(a^{-1}_{2,q}a_{1,q+k})$, hence $\mathbf{P}_1 \vee (\mathbf{P}_2 \vee \mathbf{P}_2) = (\mathbf{P}_1 \vee \mathbf{P}_2) \vee \mathbf{P}_2$; also $a_{2,q}[a^{-1}_{2,q}(a_{1,q+k}(a^{-1}_{1,q}a_{1,q+k}))]a_{1,q+k}(a^{-1}_{1,q}a_{1,q+k})$, consequently, $\mathbf{P}_1 \vee (\mathbf{P}_1 \vee \mathbf{P}_2) = (\mathbf{P}_1 \vee \mathbf{P}_1) \vee \mathbf{P}_2$ and inevitably for equivalence classes $(aa)b = a(ab)$ and $b(aa) = (ba)a$ for each $a, b \in (S^M E)_{t,H}$. Thus $(S^M E)_{t,H}$ is alternative.

If G is associative, then the parallel transport structure gives $(f \vee g) \vee h = f \vee (g \vee h)$ on $M \vee M \vee M$ for each $\{f, g, h\} \subset H^t_p(M, \{s_{0,q} : q = 1, ..., k; W, y_0)$. Applying the embedding χ^* and the equivalence relation $R_{t,H}$ we get, that $(S^M E)_{t,H}$ is associative $< f >_\xi \circ (< g >_\xi \circ < h >_\xi) =$

$(< f >_\xi \circ < g >_\xi) \circ < h >_\xi$.

In view of Conditions $2(i - iv)$ there exists an H^t_p-diffeomoprhism of $(A_1 \setminus A_3) \vee (A_2 \setminus A_3)$ with $(A_2 \setminus A_3) \vee (A_1 \setminus A_3)$ as pseudo-manifolds (see §1.3.1). For the measure ν on M naturally the equality $\nu(A_3) = 0$ is satisfied. If M' - is the submanifold may be with corners or pseudo-manifold, accomplishing the partition $Z = Z_f$ of the manifold M, then the codimension M' in M is equal to one and $\nu(M') = 0$. For the point $s_{0,q}$ in $(M \setminus A_3) \cup \{s_{0,q}\}$ there exists an open neighborhood U having the H^t_p-retraction $F : [0,1] \times U \to \{s_{0,q}\}$. Hence it is possible to take a sequence of diffeomorphisms $\psi_n \in DifH^t_p(M, \{s_{0,q} : q = 1, ..., k\})$ such that $\lim_{n\to\infty} diam(\psi_n(U)) = 0$.

Let w_0 be a mapping $w_0 : M \to W$ such that $w_0(M) = \{y_0 \times e\}$. Consider $w_0 \vee (E, f)$ for some $(E, f) \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; W, y_0)$. If $(E, f) \in H^t_p(M, \{s_{0,q} : q = 1, .., k\}; W, y_0)$ with the natural positive $t \in \mathbf{N}$, then f is bounded relative to the uniformity of the uniform space $H^t_p(M; E)$. If U_n is a sequence of bounded open or canonical closed subsets in M such that $\lim_n diam(U_n) = 0$, then $\lim_{n\to\infty} \nu(V_n) = 0$ for the sequence of ν-measurable subsets V_n such that $V_n \subset U_n$. Therefore, for each bounded sequence $\{g_n : g_n \in H^t_p(M; E); n \in \mathbf{N}\}$ there exists the limit $\lim_{n\to\infty} g_n|_{U_n} = 0$ relative to the H^t_p uniformity, where U_n is subordinated to the partition of M into H^t submanifolds. Then if $\{g_n : g_n \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; E, y_0); n \in \mathbf{N}\}$ is a bounded sequence such that g_n converges to $g \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; N, y_0)$ on $M \setminus W_k$ for each k relative to the H^t_p-uniformity, the given open W_k in M, where $k, n \in \mathbf{N}$ and $\lim_{n\to\infty} \nu(W_n \triangle U_n) = 0$, then g_n converges to g in the uniform space $H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; E, y_0)$.

Mention that for each marked point $s_{0,q}$ in M there exists a neighborhood U of $s_{0,q}$ in M such that for each $\gamma_1 \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; N, y_0)$ there exists $\gamma_2 \in H^t_p$ such that they are $R_{t,H}$ equivalent and $\gamma_2|_U = y_0$. Therefore, if C is an arcwise connected compact subset in M of codimension $codim_{\mathbf{R}} C \geq 1$ such that $s_{0,q} \in C$, then the standard proceeding shows that for each $\gamma_1 \in H^t_p$ there exists $\gamma_2 \in H^t_p$ such that $\gamma_1 R_{t,H} \gamma_2$ and $\gamma_2|_C = y_0$. Since C is compact, then each its open covering has a finite subcovering and hence

(Y_0) there exists an open neighborhood U of C in M such that for each γ_1 there exists γ_2 such that $\gamma_1 R_{t,H} \gamma_2$ and $\gamma_2|_U = y_0$.

There exists a sequence $\eta_n \in DifH^t_p(M, \{s_{0,q} : q = 1, ..., k\})$ such that $\lim_{n\to\infty} diam(\eta_n(A_2 \setminus \partial A_2)) = 0$ and $w_n, f_n \in H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; E, y_0)$ with

(iii) $\lim_{n\to\infty} f_n = f$, $\lim_{n\to\infty} w_n = w_0$ and $\lim_{n\to\infty} \chi^*(f_n \vee w_n)(\eta_n^{-1}) = f$ due to $\pi \circ f(s_{0,q}) = s_{0,q}$ in the formula of differentiation of compositions of functions (over $\mathbf{H}$ and $\mathbf{O}$ see it in [19, 20, 17]).

In more details, the sequence η_n as a limit of $\eta_n(A_2)$ produces a pseudo-submanifold B in M of codimension not less than one such that B can be presented with the help of the wedge product of spheres and compact quadrants up to H^t_p-diffeomorphism with marked points $\{s_{0,q} : q = 1, ..., k\}$, but as well B may be a finite discrete set also. Then by induction the procedure can be continued lowering the dimension of B. Particularly there may be circles and curves in the case of the unit dimension. Two quadrants up to an H^t_p quotient mapping gluing boundaries produce a sphere. Thus the consideration reduces to the case of the wedge product of spheres. The case of spheres reduces to the iterated construction with circles, since the reduced product $S^1 \wedge S^n$ is H^t_p homeomorphic with S^{n+1} (see Lemma 2.27 [37] and [4]). For the particular case of the n-dimensional sphere $M_n = S^n$ take $\hat{M}_n = D^n$, where D^n is the unit ball (disk) in $\mathbf{R^n}$ or in a n dimensional over $\mathbf{R}$ subspace in $\mathcal{A}^l_r$, $D_1 = [0,1]$ for $n = 1$. But $S^n \setminus s_0$ has the retraction into the point in S^n, where $s_0 \in S^n$, $n \in \mathbf{N}$.

Therefore, $w_0 \vee (E,f)$ and (E,f) belong to the equivalence class $< (E,f) >_{t,H}$: $\{g \in H^t_p(M,\{s_{0,q} : q = 1,...,k\}; W, y_0) : (E,f)R_{t,H}g\}$ due to (iii) and (Y_0). Thus, $< w_0 >_{t,H} \circ < g >_{t,H} = < g >_{t,H}$.

The pseudo-manifold $M \vee M \setminus \{s_{0,q} \times s_{0,j} : q,j = 1,...,k\}$ has the H^t_p-diffeomorphism ψ (see definition in §1.3.1) such that $\psi(x,y) = (y,x)$ for each $(x,y) \in (M \times M \setminus \{s_{0,q} \times s_{0,j} : q,j = 1,...,k\})$. Suppose now, that G is commutative. Then $(f \vee g) \circ \psi|_{(M\times M \setminus \{s_{0,q}\times s_{0,j}:q,j=1,...,k\})} = g \vee f|_{(M\times M\setminus\{s_{0,q}\times s_{0,j}:q,j=1,...,k\})}$. On the other hand, $< f \vee w_0 >_{t,H} = < f >_{t,H} = < f >_{t,H} \circ < w_0 >_{t,H} = < w_0 >_{t,H} \circ < f >_{t,H}$, hence, $< f \vee g >_{t,H} = < f >_{t,H} \circ < g >_{t,H} = < f \vee w_0 >_{t,H} \circ < w_0 \vee g >_{t,H} = < (f \vee w_0) \vee (w_0 \vee g) >_{t,H} = < (w_0 \vee g) \vee (f \vee w_0) >_{t,H}$ due to the existence of the unit element $< w_0 >_{t,H}$ and due to the properties of ψ. Indeed, take a sequence ψ_n as above. Therefore, the parallel transport structure gives $(g \vee f)(\psi(x,y)) = (g \circ f)(y,x)$ for each $x,y \in M$, consequently, $(f \circ g)R_{t,H}(g \circ f)$ for each $f,g \in H^t_p(M,\{s_{0,q} : q = 1,...,k\}; W, y_0)$. The using of the embedding χ^* gives that $(S^ME)_{t,H}$ is commutative, when G is commutative.

The mapping $(f,g) \mapsto f \vee g$ from $H^t_p(M,\{s_{0,q} : q = 1,...,k\}; W, y_0)^2$ into $H^t_p(M \vee M \setminus \{s_{0,q} \times s_{0,j} : q,j = 1,...,k\}; W, y_0)$ is of class H^t_p. Since the mapping χ^* is of class H^t_p, then $(f,g) \mapsto \chi^*(f \vee g)$ is the H^t_p-mapping. The quotient mapping from $H^t_p(M,\{s_{0,q} : q = 1,...,k\}; W, y_0)$ into $(S^ME)_{t,H}$ is continuous and induces the quotient uniformity, $T^b(S^ME)_{t,H}$ has embedding into $(S^MT^bE)_{t,H}$ for each $1 \le b \le t' - t$, when $t' > t$ is finite, for every $1 \le b < \infty$ if $t' = \infty$, since E is the $H^{t'}_p$ fiber bundle, T^bE is the fiber bundle with the base space N. Hence the multiplication $(< f >_{t,H}, < g >_{t,H}) \mapsto < f >_{t,H} \circ < g >_{t,H} = < f \vee g >_{t,H}$ is continuous in $(S^ME)_{t,H}$ and is of class H^l_p with $l = t' - t$ for finite t' and $l = \infty$ for $t' = \infty$.

4. Definition. The $(S^ME)_{t,H}$ from Theorem 3.1 we call the wrap monoid.

5. Corollary. *Let $\phi : M_1 \to M_2$ be a surjective H^t_p-mapping of H^t_p-pseudo-manifolds over the same $\mathcal{A}_r$ such that $\phi(s_{1,0,q}) = s_{2,0,a(q)}$ for each $q = 1,...,k_1$, where $\{s_{j,0,q} : q = 1,...,k_j\}$ are marked points in M_j, $j = 1,2$, $1 \le a \le k_2$, $l_1 \le k_2$, $l_1 : card\ \phi(\{s_{1,0,q} : q = 1,...,k_1\})$. Then there exists an induced homomorphism of monoids $\phi^* : (S^{M_2}E)_{t,H} \to (S^{M_1}E)_{t,H}$. If $l_1 = k_2$, then ϕ^* is the embedding.*

Proof. Take $\Xi_1 : \hat{M}_1 \to M_1$ with marked points $\{\hat{s}_{1,0,q} : q = 1,...,2k_1\}$ as in §2, then take $\hat{M}_2$ the same $\hat{M}_1$ with additional $2(k_2 - l_1)$ marked points $\{\hat{s}_{2,0,q} : q = 1,...,2k_3\}$ such that $\hat{s}_{1,0,q} = \hat{s}_{2,0,q}$ for each $q = 1,..,k_1$, $k_3 = k_1 + k_2 - l_1$, then $\phi \circ \Xi_1 := \Xi_2 : \hat{M}_2 \to M_2$ is the desired mapping inducing the parallel transport structure from that of M_1. Therefore, each $\hat{\gamma}_2 : \hat{M}_2 \to N$ induces $\hat{\gamma}_1 : \hat{M}_1 \to N$ and to $\mathbf{P}_{\hat{\gamma}_2,u^2}$ there corresponds $\mathbf{P}_{\hat{\gamma}_1,u^1}$ with additional conditions in extra marked points, where $u^1 \subset u^2$. The equivalence class $< (E_2, \mathbf{P}_{\hat{\gamma}_2,u^2}) >_{t,H} \in (S^{M_2}E)_{t,H}$

gives the corresponding elements $< (E_1, \mathbf{P}_{\hat{\gamma}_1,u^1}) >_{t,H} \in (S^{M_1}E)_{t,H}$, since $DifH^t_p(\hat{M}_1, \{\hat{s}_{0,q} : q = 1,...,2k_2\}) \subset DifH^t_p(\hat{M}_1, \{\hat{s}_{0,q} : q = 1,...,2k_3\})$. Then $\phi^*(< (E_2, \mathbf{P}_{\hat{\gamma}_2,u^2}) \vee (E_1, \mathbf{P}_{\hat{\eta}_2,v^2}) >_{t,H}) = \phi^*(< (E_2, \mathbf{P}_{\hat{\gamma}_2,u^2}) >_{t,H})\phi^*(< (E_1, \mathbf{P}_{\hat{\eta}_2,v^2}) >_{t,H})$, since $f_2 \circ \phi(x)$ for each $x \in \Xi_1(\hat{M}_1 \setminus \partial\hat{M}_1)$ coincides with $f_1(x)$, where f_j corresponds to $\mathbf{P}_{\gamma_j,y_0\times e}$ (see also the beginning of §3).

If $l_1 = k_2$, then $\hat{M}_1 = \hat{M}_2$ and the group of diffeomorphisms $DifH^t_p(\hat{M}_1, \{\hat{s}_{0,q} : q = 1,...,2k_1\})$ is the same for two cases, hence ϕ^* is bijective and inevitably ϕ^* is the embedding.

6. Theorems. 1. *There exists an alternative topological group $(W^ME)_{t,H}$ containing the monoid $(S^ME)_{t,H}$ and the group operation of H^l_p class with $l = t' - t$ ($l = \infty$ for $t' = \infty$). If N and G are separable, then $(W^ME)_{t,H}$ is separable. If N and G are complete, then $(W^ME)_{t,H}$ is complete.*

2. *If G is associative, then $(W^ME)_{t,H}$ is associative. If G is commutative, then $(W^ME)_{t,H}$ is commutative. If G is a Lie group, then $(W^ME)_{t,H}$ is a Lie group.*

3. *The $(W^ME)_{t,H}$ is non-discrete, locally connected and infinite dimensional for $dim_{\mathbf{R}}(N \times G) > 1$. Moreover, if there exist two different sets of marked points $s_{0,q,j}$ in A_3, $q = 1,...,k$, $j = 1,2$, then two groups $(W^ME)_{t,H,j}$, defined for $\{s_{0,q,j} : q = 1,...,k\}$ as marked points, are isomorphic.*

4. *The $(W^ME)_{t,H}$ has a structure of an H^t_p-differentiable manifold over $\mathcal{A}_r$.*

Proof. If $\gamma \in H^t_p(M, \{s_{0,q} : q = 1,...,k\}; N, y_0)$, then for $u \in E_{y_0}$ there exists a unique $h_q \in G$ such that $\mathbf{P}_{\hat{\gamma},u}(\hat{s}_{0,q+k}) = u_q h_q$, where $h_q = g_q^{-1}g_{q+k}$, $y_0 \times g_q = \mathbf{P}_{\hat{\gamma},u}(\hat{s}_{0,q})$, $g_q \in G$. Due to the equivariance of the parallel transport structure h depends on γ only and we denote it by $h^{(E,\mathbf{P})}(\gamma) = h(\gamma) = h$, $h = (h_1,...,h_k)$. The element $h(\gamma)$ is called the holonomy of $\mathbf{P}$ along γ and $h^{(E,\mathbf{P})}(\gamma)$ depends only on the isomorphism class of $(E, \mathbf{P})$ due to the use of $DifH^t_p(\hat{M}; \{\hat{s}_{0,q} : q = 1,...,2k\})$ and boundary conditions on $\hat{\gamma}$ at $\hat{s}_{0,q}$ for $q = 1,...,2k$.

Therefore, $h^{(E_1,\mathbf{P}_1)(E_2,\mathbf{P}_2)}(\gamma) = h^{(E_1,\mathbf{P}_1)}(\gamma)h^{(E_2,\mathbf{P}_2)}(\gamma) \in G^k$, where G^k denotes the direct product of k copies of the group G. Hence for each such γ there exists the homomorphism $h(\gamma) : (S^ME)_{t,H} \to G^k$, which induces the homomorphism $h : (S^ME)_{t,H} \to C^0(H^t_p(M, \{s_{0,q} : q = 1,...,k\}; N, y_0), G^k)$, where $C^0(A, G^k)$ is the space of continuous maps from a topological space A into G^k and the group structure $(hb)(\gamma) = h(\gamma)b(\gamma)$ (see also [4] for S^n).

Thus, it is sufficient to construct $(W^MN)_{t,H}$ from $(S^MN)_{t,H}$. For the commutative monoid $(S^MN)_{t,H}$ with the unit and the cancelation property there exists a commutative group $(W^MN)_{t,H}$. Algebraically it is the quotient group F/B, where F is the free commutative group generated by $(S^MN)_{t,H}$, while B is the minimal closed subgroup in F generated by all elements of the form $[f + g] - [f] - [g]$, f and $g \in (S^MN)_{t,H}$, $[f]$ denotes the element in F corresponding to f (see also about such abstract Grothendieck construction in [13, 36]).

By the construction each point in $(S^MN)_{t,H}$ is the closed subset, hence $(S^MN)_{t,H}$ is the topological T_1-space. In view of Theorem 2.3.11 [7] the product of T_1-spaces is the T_1-space. On the other hand, for the topological group G from the separation axiom T_1 it follows, that G is the

Tychonoff space [7, 32]. The natural mapping $\eta : (S^M N)_{t,H} \to (W^M N)_{t,H}$ is injective. We supply F with the topology inherited from the topology of the Tychonoff product $(S^M N)^{\mathbf{Z}}_{t,H}$, where each element z in F has the form $z = \sum_f n_{f,z}[f]$, $n_{f,z} \in \mathbf{Z}$ for each $f \in (S^M N)_{t,H}$, $\sum_f |n_{f,z}| < \infty$. By the construction F and F/B are T_1-spaces, consequently, F/B is the Tychonoff space. In particular, $[nf] - n[f] \in \mathsf{B}$, hence $(W^M N)_{t,H}$ is the complete topological group, if N and G are complete, while η is the topological embedding, since $\eta(f+g) = \eta(f) + \eta(g)$ for each $f, g \in (S^M N)_{t,H}$, $\eta(e) = e$, since $(z+B) \in \eta(S^M N)_{t,H}$, when $n_{f,z} \geq 0$ for each f, and inevitably in the general case $z = z^+ - z^-$, where $(z^+ + B)$ and $(z^- + B) \in \eta(S^M N)_{t,H}$.

Using plots and $H^{t'}_p$ transition mappings of charts of N and $E(N, G, \pi, \Psi)$ and equivalence classes relative to $Dif H^t_p(M, \{s_{0,q} : q = 1, ..., k\})$ we get, that $(W^M E)_{t,H}$ has the structure of the H^t_p-differentiable manifold, since $t' \geq t$.

The rest of the proof and the statements of Theorems 6(1-4) follows from this and Theorems 3(1-3) and [21, 22].

7. Definition. The $(W^M E)_{t,H} = (W^{M,\{s_{0,q}:q=1,...,k\}} E; N, G, \mathbf{P})_{t,H}$ from Theorem 6.1 we call the wrap group.

8. Corollary. *There exists the group homomorphism* $h : (W^M E)_{t,H} \to C^0(H^t_p(M, \{s_{0,q} : q = 1, ..., k\}; N, y_0), G^k)$.

Proof follows from §6 and putting $h^{f^{-1}}(\gamma)(h^f(\gamma))^{-1}$.

9. Corollary. *If M_1 and M_2 and ϕ satisfy conditions of Corollary 5, then there exists a homomorphism $\phi^* : (W^{M_2} E)_{t,H} \to (W^{M_1} E)_{t,H}$. If $l_1 = k_2$, then ϕ^* is the embedding.*

10. Remarks and examples. Consider examples of M which satisfy sufficient conditions for the existence of wrap groups $(W^M E)_{t,H}$. Take M, for example, D^n_R, $S^n_R \setminus V$ with $s_0 \in \partial V$, $D^n_R \setminus Int(D^n_b)$ with $s_0 \in \partial D^n_b$ and $0 < b < R < \infty$, where S^n_R denotes the sphere of the dimension $n > 1$ over $\mathbf{R}$ and radius R, V is H^t_p-diffeomorphic with the interior $Int(D^n_R)$ of the n-dimensional ball $D^n_R := \{x \in \mathbf{R^n} : \sum_{k=1}^n x_k^2 \leq R\}$ or in n dimensional over $\mathbf{R}$ subspace in $\mathcal{A}^l_r$ and is the proper subset in $S^n_R := \{x \in \mathbf{R^{n+1}} : \sum_{k=1}^{n+1} x_k^2 = R\}$. Instead of sphere it is possible to take an H^t_p pseudo-manifold Q^n homeomorphic with a sphere or a disk, particularly, Milnor's sphere. Indeed, divide M by the equator $\{x_1 = 0\}$ into two parts A_1 and A_2 and take $A_3 = \{x \in M : x_1 = 0\} \cup P$, where $s_0 \in \partial A_1 \cap \partial A_2$, while $P = \emptyset$, $P = \partial V$, $P = \partial D^n_b$ correspondingly. Then take also V and D^n_b such that their equators would be generated by the equator $\{x_1 = 0\}$ in S^n_R or D^n_R respectively or more generally Q^n.

Take then $M = Q^n \setminus \bigcup_{k=1}^l V_k$, where V_k are H^t_p-diffeomorphic to interiors of bounded quadrants in $\mathbf{R^n}$ or in n dimensional subspace in $\mathcal{A}^a_r$, where $l > 1$, $l \in \mathbf{N}$, $\partial V_k \cap \partial V_j = \{s_0\}$ and $V_k \cap V_j = \emptyset$ for each $k \neq j$, $diam(V_k) \leq b < R/3$. In more details it is possible make a specification such that if l is even, then $[l/2] - 1$ among V_k are displayed above the equator and the same amount below it, two of V_k have equators, generated by equators $\{x_1 = 0\}$ in Q^n. If l odd, then $[(l-1)/2]$ among V_k are displayed above and the same amount below it, one of V_k has equator generated by that of $\{x_1 = 0\}$ in Q^n, $s_0 \in \bigcap_k \partial V_k \cap \{x \in M : x_1 = 0\}$.

Divide M by the equator $\{x_1 = 0\}$ into two parts A_1 and A_2 and let $A_3 = \{x \in M : x_1 = 0\} \cup P$, where $P = \bigcup_{k=1}^{l} \partial V_k$. Then either $A_1 \setminus A_3$ and $A_2 \setminus A_3$ are H_p^t diffeomorphic as pseudo-manifolds or manifolds with corners and H_p^t diffeomorphic with $M \setminus [\{s_0\} \cup (A_3 \setminus Int(\partial A_1 \cap \partial A_2))] =: D$ or $2(iv')$ is satisfied, since the latter topological space D is obtained from Q^n by cutting a non-void connected closed subset, $n > 1$, consequently, D is retractable into a point.

In a case of a usual manifold M the point $s_0 \in \partial M$ (for $\partial M \neq \emptyset$) may be a critical point, but in the case of a manifold with corners this s_0 is the corner point from ∂M, since for $x \in \partial M$ there is not less than one chart (U, u, Q) such that $u(x) \in \partial Q$, $M \setminus \partial M = \bigcup_k u_k^{-1}(Int(Q_k))$, $\partial M \subset \bigcup_k u_k^{-1}(\partial Q_k)$. Further, if M satisfies Conditions $2(i - v)$ or $(i - iii, iv', v)$, then $M \times D_R^m = P$ also satisfies them for $m \geq 1$, since D_R^m is retractable into the point, taking as two parts $A_j(K) = A_j(M) \times D_R^m$ of P, where $j = 1, 2$, $A_j(M)$ are pseudo-submanifolds of M. Then $A_1(P) \cap A_2(P) = (A_1(M) \cap A_2(M)) \times D_R^m$ and it is possible to take $A_3(P) = A_3(M) \times D_R^m$, $s_0(P) \in s_0(M) \times \{x \in D_R^m : x_1 = 0\}$. In particular, for $M = S^1$ and $m = 1$ this gives the filled torus.

This construction can be naturally generalized for non-orientable manifolds, for example, the Möbius band L, also for $M := L \setminus (\bigcup_{j=1}^{\beta} V_j)$ with the diameter b_j of V_j less than the width of L, where each V_j is H_p^t diffeomorphic with an interior of a bounded quadrant in $\mathbf{R^2}$, $s_{0,q} \in \partial L \cap (\bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_j)$, $a_0 := 0$, $a_1 + ... + a_k = \beta$, $q = 1, ..., k$, since ∂L is diffeomorphic with S^1, also $S^1 \setminus \{s_{0,q}\}$ is retractable into a point, consequently, A_1 and A_2 are retractable into a point. For L take $\hat{M} = I^2$, then take a connected curve $\hat{\eta}$ consisting of the left side $\{0\} \times [0, 1]$ joined by a straight line segment joining points $\{0, 1\}$ and $\{1, 0\}$ and then joined by the right side $\{1\} \times [0, 1]$. This gives the proper cutting of $\hat{M}$ which induces the proper cutting of L and of M with $A_3 \supset \eta \cup \partial L$ up to an H_p^t diffeomorphism, where $\eta := \Xi(\hat{\eta})$, hence the Möbius band L and M satisfy Conditions $2(i - iii, iv', v)$.

Take a quotient mapping $\phi : I^2 \to S^1$ such that $\phi(\{s_{0,1}, s_{0,2}\}) = s_0 \in S^1$, $s_{0,1} = (0, 0)$, $s_{0,2} = (0, 1) \in I^2$, where $I = [0, 1]$, hence there exists the embedding $\phi^* : (W^{S^1, s_0}E)_{t,H} \hookrightarrow (W^{I^2, \{s_{0,1}, s_{0,2}\}}E)_{t,H}$.

The Klein bottle K has $\hat{M} = I^2$ with twisting equivalence relation on ∂I^2 so it satisfies sufficient conditions. Moreover, K is the quotient $\phi : Z \to K$ of the cylinder Z with twisted equivalence relation of its ends S^1 using reflection relative to a horizontal diameter. Thus $A_3 \supset \phi(S^1)$. Therefore, there exists the embedding $\phi^* : (W^{K, \{s_0\}}E)_{t,H} \to (W^{Z, \{s_{0,1}, s_{0,2}\}}E)_{t,H}$, where $s_{0,1}, s_{0,2} \in \partial Z$, $\phi(\{s_{0,1}, s_{0,2}\}) = s_0$.

Take a pseudo-manifold Q^n H_p^t-diffeomorphic with S^n for $n \geq 2$, cut from it β non-intersecting open domains $V_1, ..., V_\beta$ H_p^t-diffeomorphic with interiors of bounded quadrants in $\mathbf{R^n}$, $s_{0,q} \in \bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_j$, $a_0 := 0$, $a_1 + ... + a_k = \beta$, $q = 1, ..., k$. Then glue for $V_1, ..., V_l$, $1 \leq l \leq \beta$, by boundaries of slits H_p^t-diffeomorphic with S^{m-1} the reduced product $L \vee S^{n-2}$, since $\partial L = S^1$, $S^1 \wedge S^{n-2}$ is H_p^t-diffeomorphic with S^{n-1} [37]. We get the non-orientable H_p^t-pseudo-manifold M, satisfying sufficient conditions.

Since the projective space $\mathbf{R}P^n$ is obtained from the sphere by identifying diametrically opposite points. Then take M H^t_p-diffeomorphic with $\mathbf{R}P^n$ for $n > 1$ also M with cut $V_1, ..., V_\beta$ H^t_p-diffeomorphic with open subsets in $\mathbf{R}P^n$, $s_{0,q} \in (\bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_j) \cap \{x \in M : x_1 = 0\}$, $V_j \cap V_l = \emptyset$ for each $j \ne l$, $a_0 := 0$, $a_1 + ... + a_k = \beta$, $q = 1, ..., k$. Then Conditions $2(i - v)$ or $(i - iii, iv', v)$ are also satisfied for $\mathbf{R}P^n$ and M.

In view of Proposition 2.14 [37] about H-groups $[X, x_0; K, k_0]$ there is not any expectation or need on rigorous conditions on a class of acceptable M for constructions of wrap groups $(W^M E)_{t,H}$.

If M_1 is an analytic real manifold, then taking its graded product with generators $\{i_0, ..., i_{2^r-1}\}$ of the Cayley-Dickson algebra gives the $\mathcal{A}_r$ manifold (see [19, 17, 18]). Particularly this gives $l2^r$ dimensional torus in $\mathcal{A}_r^l$ for the l dimensional real torus $\mathbf{T}_2 = (S^1)^l$ as M_1.

Consider $\mathbf{T}_2$. It can be slit along a closed curve (loop) C H^∞_p-diffeomorphic with S^1 and marked points $s_{0,q} \in C \subset \mathbf{T}_2$ such that C rotates on the surface of $\mathbf{T}_2 = S^1_R \times S^1_b$ on angle π around S^1_b while C rotates on 2π around S^1_R, such that C rotates on 4π around S^1_R that return to the initial point on C, where $0 < b < R < \infty$, $q = 1, ..., k$, $k \in \mathbf{N}$. Therefore, the slit along C of $\mathbf{T}_2$ is the non-orientable band which inevitably is the Möbius band with twice larger number of marked points $\{s^L_{0,j} : j = 1, ..., 2k\} \subset \partial L$.

Therefore, for $M = \mathbf{T}_2$ as $\hat{M}$ take a quadrant in $\mathbf{R^2}$ with $2k$ pairwise opposite marked points $\hat{s}_{0,q}$ and $\hat{s}_{0,q+k}$ on the boundary of $\hat{M}$, $q = 1, ..., k$, $k \in \mathbf{N}$. Suitable gluing of boundary points in $\partial\hat{M}$ gives the mapping $\Xi : \hat{M} \to \mathbf{T}_2$, $\Xi(\hat{s}_{0,q}) = \Xi(\hat{s}_{0,q+k}) = s_{0,q}$, $q = 1, ..., k$. Proper cutting of $\hat{M}$ into $\hat{A}_j$, $j = 1, 2$, or of L induces that of $\mathbf{T}_2$. Thus we get a pseudo-submanifold $A_3(\mathbf{T}_2) =: A_3 \supset C$, while A_1 and A_2 are retractable into a marked point $s_{0,q} \in C$ for each q, hence $\mathbf{T}_2$ satisfies Conditions $2(i - iii, iv', v)$. In view of Corollary 9 there exists the embedding $\phi^* : (W^{\mathbf{T}_2,\{s_{0,q}:q=1,...,k\}}E)_{t,H} \to (W^{L,\{s^L_{0,q}:q=1,...,2k\}}E)_{t,H}$, where $\phi : L \to \mathbf{T}_2$ is the quotient mapping with $\phi(\{s^L_{0,q}, s^L_{0,q+k}\})\{s_{0,q}\}$, $q = 1, ..., k$.

For the n-dimensional torus $\mathbf{T}_n$ in $\mathcal{A}^a_r$ with $n > 2$ take a $n-1$-dimensional surface B such that each its projection into $\mathbf{T}_2$ is H^t_p-diffeomorphic with C for a loop C as above. Therefore, the slit along B up to a H^t_p-diffeomorphism gives $M_0 := L \times I^{n-2}$ for even n or $M_0 := S^1 \times I^{n-1}$ for odd n, where $I = [0, 1]$. Since I^m is retractable into a point, where $m \ge 1$. Thus we lightly get for $\mathbf{T}_n$ a pseudo-submanifold $A_3 \supset B$ and two A_1 and A_2 retractable into points and satisfying sufficient Conditions $2(i - iii, iv', v)$, where $\hat{M} = I^n$ up to a H^t_p-diffeomorphism, $s_{0,q} \in B \subset A_3 := A_3(\mathbf{T}_n)$, $\{s^{M_0}_{0,q}, s^{M_0}_{0,q+k}\} \subset \partial M_0$, $q = 1, ..., k$, $k \in \mathbf{N}$. Proper cutting of $\hat{M}$ into $\hat{A}_j$, $j = 1, 2$, induces that of $\mathbf{T}_n$. Thus there exists an H^t_p quotient mapping $\phi : M_0 \to \mathbf{T}_n$ with $\phi(\{s^{M_0}_{0,q}, s^{M_0}_{0,q+k}\}) = \{s_{0,q}\}$ and the embedding $\phi^* : (W^{\mathbf{T}_n,\{s_{0,q}:q=1,...,k\}}E)_{t,H} \hookrightarrow (W^{M_0,\{s^{M_0}_{0,q}:q=1,...,2k\}}E)_{t,H}$ due to Corollary 9.

More generally cut from $\mathbf{T}_n$ open subsets V_j which are H^t_p diffeomorphic with interiors of bounded quadrants in $\mathbf{R^n}$ embedded into $\mathcal{A}^l_r$, $j = 1, ..., \beta$, such that $s_{0,q} \in B \cap (\bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_j)$, $V_j \cap V_i = \emptyset$ for each $j \ne i$, $V_j \cap B = \emptyset$ for each j, where B is defined up to an H^t_p diffeomorhism, $a_0 := 0$, $a_1 + ... + a_k = \beta$, $q = 1, ..., k$, that gives the manifold M_2. Then from M_0 cut analogously corresponding $V_{j,b}$, such that $s_{0,q} \in B \cap (\bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_{j,1})$, $s_{0,q+k} \in B \cap (\bigcap_{j=a_1+...+a_{q-1}+1}^{a_1+...+a_q} \partial V_{j,2})$,

$V_{j,b_1} \cap V_{i,b_2} = \emptyset$ for each $j \ne i$ or $b_1 \ne b_2$, $a_0 := 0$, $a_1 + ... + a_k = \beta$, $q = 1, ..., k$, $j = 1, ..., \beta$, $b = 1, 2$, that produces the manifold M_1. We choose $V_{j,b}$ such that for the restriction $\phi : M_1 \to M_2$ of the mapping ϕ there is the equality $\phi(V_{j,1} \cup V_{j,2}) = V_j$ for each j, $\phi(\{s_{0,q}^{M_1}, s_{0,q+k}^{M_1}\}) = \{s_{0,q}\}$. This gives the embedding $\phi^* : (W^{M_2,\{s_{0,q}:q=1,...,k\}}E)_{t,H} \hookrightarrow (W^{M_1,\{s_{0,q}^{M_1}:q=1,...,2k\}}E)_{t,H}$.

Another example is M_3 obtained from the previous M_2 with $2k$ marked points and 2β cut out domains V_j, when $s_{0,q}$ is identified with $s_{0,q+k}$ and each ∂V_j is glued with $\partial V_{j+\beta}$ for each $j \in \lambda_q \subset \{d : a_1 + ... + a_{q-1} + 1 \le d \le a_1 + ... + a_q\}$, $q = 1, ..., k$, $k \in \mathbf{N}$, by an equvalence relation υ. Such M_3 is obtained from the torus $\mathbf{T}_{n,m}$ with m holes instead of one hole in the standard torus $\mathbf{T}_{n,1} = \mathbf{T}_n$ cutting from it V_j with $j \in \{1, ..., 2\beta\} \setminus (\bigcup_{q=1,...,k} \lambda_q)$, where $m = m_1 + ... + m_k$, $m_q := card(\lambda_q)$. For $\mathbf{T}_n$ and M_2 the surface B is H_p^t diffeomorphic with $(\partial L) \times I^{n-2}$ for even n or $S^1 \times I^{n-1}$ for odd n. Take $A_3 \supset B \cup (\bigcup_{j\in\lambda_q} \upsilon(\partial V_j))$, it is arcwise connected and contains all marked points. Therefore, M_3 satisfies conditions of §2 and there exists the embedding $\upsilon^* : (W^{M_3,\{s_{0,q}^{M_3}:q=1,...,k\}}E)_{t,H} \hookrightarrow (W^{M_2,\{s_{0,q}^{M_2}:q=1,...,2k\}}E)_{t,H}$. This also induces the embedding $(W^{\mathbf{T}_{n,m},\{s_{0,q}^{\mathbf{T}_{n,m}}:q=1,...,k\}}E)_{t,H} \hookrightarrow (W^{\mathbf{T}_n,\{s_{0,q}^{\mathbf{T}_n}:q=1,...,2k-1\}}E)_{t,H}$ such that each element $g \in (W^{\mathbf{T}_{n,m},\{s_{0,q}^{\mathbf{T}_{n,m}}:q=1,...,k\}}E)_{t,H}$ can be presented as a product $g = (..(g_1g_2)...g_m)$ of m elements $g_j \in (W^{\mathbf{T}_n,\{s_{0,q}^{\mathbf{T}_n}:q=1,...,2k-1\}}E)_{t,H}$, $g_j =< f_j >_{t,H}$, $supp(\pi \circ f_j) \subset B_j$, $B_1 \cup ... \cup B_m = \mathbf{T}_n$, $B_i \cap B_j = \partial B_i \cap \partial B_j$ for each $i \ne j$, each B_j is a canonical closed subset in $\mathbf{T}_n$, $s_{0,1} \in B_1$, $s_{0,2q}, s_{2q+1} \in B_d$ for $m_1 + ... + m_0 + 1 \le d \le m_1 + ... + m_q$, $q = 1, ..., k-1$, where $m_0 := 0$.

Evidently, in the general case for different manifolds M and N wrap groups may be non isomorphic. For example, as M_1 take a sphere S^n of the dimension $n > 1$, as M_2 take $M_1 \setminus K$, where K is up to an H_p^t-diffeomorphism the union of non intersecting interiors B_j of quadrants of diameters $d_1, ..., d_s$ much less, than 1, $K = B_1 \cup ... \cup B_l$, $l \in \mathbf{N}$. Let N be a δ-enlargement for M_2 in $\mathbf{R^{n+1}}$ relative to the metric of the latter Euclidean space, where $0 < \delta < \min(d_1, ..., d_l)/2$. Then the groups $(W^{M_1}N)_{t,H}$ and $(W^{M_2}N)_{t,H}$ are not isomorphic. This lightly follows from the consideration of the element $b :=< f >_{t,H} \in (W^{M_2}N)_{t,H}$, where $f : M_2 \to N$ is the identity embedding induced by the structure of the δ-enlargement.

Recall, that for orientable closed manifolds A and B of the same dimension m the degree of the continuous mapping $f : A \to B$ is defined as an integer number $deg(f) \in \mathbf{Z}$ such that $f_*[A] = deg(f)[B]$, where $[A] \in H_m(A)$ or $[B] \in H_m(B)$ denotes a generator, defined by the orientation of A or B respectively [5]. Consider mappings $f_j : S^n \to N$ such that $V_j \supset \partial B_j \cap N$, where V_j is a domain in $\mathbf{R^{n+1}}$ bounded by the hyper-surface $f_j(B_j)$, f_j is w_0 on each B_i with $i \ne j$, while the degree of the mapping f_j from S^n onto $f_j(S^n)$ is equal to one. If there would be an isomorphism $\theta : (W^{M_2}N)_{t,H} \to (W^{M_1}N)_{t,H}$, then $\theta(b)$ would have a non trivial decomposition into the sum of non canceling non zero additives, which is induced by mappings $f_j : S^n \to N$. Nevertheless, an element b in $(W^{M_2}N)_{t,H}$ has not such decomposition.

If two groups G_1 and G_2 are not isomorphic, then certainly $(W^ME; N, G_1, \mathbf{P})_{t,H}$ and $(W^ME; N, G_2, \mathbf{P})_{t,H}$ are not isomorphic.

The construction of wrap groups can be spread on locally compact non compact M satisfying conditions $2(ii-iv)$ or (ii,iii,iv') changing (v) such that $\hat{M}$ is locally compact non-compact H_p^t-domain in $\mathcal{A}_r^l$, its boundary $\partial\hat{M}$ may happen to be void. For this it is sufficient to restrict the family of functions to that of with compact supports $f: M \to W$ relative to $w_0: M \to W$, that is $supp_{w_0}(f) := cl_M\{x \in M : f(x) \neq y_0 \times e\}$ is compact, $cl_M A$ denotes the closure of a subset A in M. Then classes of equivalent elements are given with the help of closures of orbits of the group of all H_p^t diffeomorphisms g with compact supports preserving marked points $DifH_{p,c}^t(M, \{s_{0,q} : q = 1, ..., k\})$ that is $supp_{id}(g) := cl_M\{x \in M : g(x) \neq x\}$ are compact, where $id(x) = x$ for each $x \in M$. Then wrap groups $(W^M E)_{t,H}$ for manifolds M such as hyperboloid of one sheet, one sheet of two-sheeted hyperboloid, elliptic hyperboloid, hyperbolic paraboloid and so on in larger dimensional manifolds over $\mathcal{A}_r$. For non compact locally compact manifolds it is possible also consider an infinite countable discrete set of marked points or of isolated singularities. These examples can be naturally generalized for certain knotted manifolds arising from the given above.

Milnor and Lefshetz have used for $M = S^1$ and $G = \{e\}$ the diffeomorphism group preserving an orientation and a marked point of S^1. So their loop group $L(S^1, N)$ may be non-commutative. The iterated loop group $L(S^1, L(S^{n-1}, N))$ is isomorphic with $L(S^n, N)$, where the latter group is supplied with the uniformity from the iterated loop group, so n times iterated loop group of S^1 gives loop group of S^n [4]. For $dim_{\mathbf{R}} M > 1$ orientation preservation loss its significance. Here above it was used the diffeomorphism group without any demands on orientation preservation of M such that two copies of M in the wedge product already are not distinguished in equivalence classes and for commutative G it gives a commutative wrap group.

Mention for comparison homotopy groups. The group $\pi_q(X)$ for a topological space X with a marked point x_0 in view of Proposition 17.1 (b) [2] is commutative for $q > 1$. For $q = 1$ the fundamental group $\pi_1(X)$ may be non-commutative, but it is always commutative in the particular case, when $X = G$ is an arcwise connected topological group (see §49(G) in [32]).

11. Proposition. *Let $L(S^1, N)$ be an H_p^1 loop group in the classical sense. Then the iterated loop group $L(S^1, L(S^1, N))$ is commutative.*

Proof. Consider two elements $a, b \in L(S^1, L(S^1, N))$ and two mappings $f \in a$, $g \in b$, $(f(x))(y) = f(x,y) \in N$, where $x, y \in I = [0,1] \subset \mathbf{R}$, $e^{2\pi x} \in S^1$. An inverse element d^{-1} of $d \in L(S^1, N)$ is defined as the equivalence class $d^{-1} =< h^- >$, where $h \in d$, $h^-(x) : h(1-x)$. Then

(1) $f(x, 1-y) = (f(x))(1-y) \in a^{-1}$ and $g(x, 1-y) = (g(x))(1-y) \in b^{-1}$ for $L(S^1, L(S^1, N))$ and symmetrically

(2) $(f(y))(1-x) = f(1-x, y) \in a^{-1}$ and $(g(y))(1-x) = g(1-x, y) \in b^{-1}$. On the other hand, $f \vee g$ corresponds to ab, and $g \vee f$ corresponds to ba, where the reduced product $S^1 \wedge S^1$ is H_p^t-diffeomorphic with S^2 in the sense of pseudo-manifolds up to critical subsets of codimension not less than two.

Consider $(S^1 \vee S^1) \wedge (S^1 \vee S^1)$ and $(f \vee w_0) \vee (w_0 \vee g)$ and $(g \vee w_0) \vee (w_0 \vee f)$ and the iterated equivalence relation $R_{1,H}$. This situation corresponds to $\hat{M} = I^2$ divided into four quadrats by segments $\{1/2\} \times [0,1]$ and $[0,1] \times \{1/2\}$ with the corresponding domains for f, g and w_0 in the considered wedge products, where $< f \vee w_0 >=< w_0 \vee f >=< f >$ is the same class of equvalent elements.

Since $G = \{e\}$, $(ab)^{-1} = b^{-1}a^{-1}$, then $g(1-x,y) \vee f(1-x,y)$ is in the same class of equivalent elements as $g(x,1-y) \vee f(x,1-y)$. But due to inclusions $(1,2)$ $< g(1-x,y) \vee f(1-x,y) >=< f(x,y) \vee g(x,y) >^{-1}$ and $< f(x,y) \vee g(x,y) >=< g(x,1-y) \vee f(x,1-y) >^{-1}$ and $< h(x,y) >^{-1} =< h(x,1-y) >=< h(1-x,y) >$ for $h \in ab$, consequently, $< h(x,y) >=< h(1-x,1-y) >$ and $< (f \vee g)(x,1-y) >< f(x,1-y) \vee g(x,1-y) >\in (ab)^{-1}$, since $(x,y) \mapsto (1-x,1-y)$ interchange two spheres in the wedge product $S^2 \vee S^2$. Hence $a^{-1}b^{-1} = b^{-1}a^{-1}$ and inevitably $ab = ba$.

12. Theorem. *Let M and N be connected both either C^∞ Riemann or $\mathcal{A}_r$ holomorphic manifolds with corners, where M is compact and $dimM \geq 1$ and $dimN > 1$. Then $(W^M N)_{t,H}$ has no any nontrivial continuous local one parameter subgroup g^b for $b \in (-\epsilon, \epsilon)$ with $\epsilon > 0$.*

Proof. Suppose the contrary, that $\{g^b : b \in (-\epsilon, \epsilon)\}$ with $\epsilon > 0$ is a local nontrivial one parameter subgroup, that is, $g^b \neq e$ for $b \neq 0$. Then to g^δ for a marked $0 < \delta < \epsilon$ there corresponds $f = f_\delta \in H^\infty_p$ such that $< f >_{t,H}= g^\delta$, where $f \in H^t_p$. If $f(U) = \{y_0 \times e\}$ for a sufficiently small connected open neighborhood U of $s_{0,q}$ in M, then there exists a sequence $f \circ \psi_n$ in the equivalence class $< f >_{t,H}$ with a family of diffeomorphisms $\psi_n \in DifH^t_p(M; \{s_{0,q} : q = 1, ..., k\})$ such that $\lim_{n\to\infty} diam\psi_n(U) = 0$ and $\bigcap_{n=1}^\infty \psi_n(U) = \{s_{0,q}\}$. If $h(x) \neq y_0$, then in view of the continuity of h there exists an open neighborhood P of x in M such that $y_0 \notin h(P)$. Consider the covariant differentiation ∇ on the manifold M (see [12]). The set S_h of points, where $\nabla^k h$ is discontinuous is a submanifold of codimension not less than one, hence of measure zero relative to the Riemann volume element in M. For others points x in M, $x \in M \setminus S_h$, all $\nabla^k h$ are continuous.

Take then open $V = V(f)$ in M such that $V \supset U$ and $\nabla^k_\nu f|_{\partial V} \neq 0$ for some $k \in \mathbf{N}$, where $\nabla_\nu f(x) := \lim_{z\to x, z\in M\setminus V} \nabla_\nu f(z)$, ν is a normal (perpendicular) to ∂V in M at a point x in the boundary ∂V of V in M. Practically take a minimal $k = k(x)$ with such property. Since M is compact and $\partial V := cl(V) \cap cl(M \setminus V)$ is closed in M, then ∂V is compact. The function $x \mapsto k(x) \in \mathbf{N}$ is continuous, since f and $\nabla^l f$ for each l are continuous. But $\mathbf{N}$ is discrete, hence each $\partial_q V := \{x \in \partial V : k(x) = q\}$ is open in V. Therefore, ∂V is a finite union of $\partial_q V$, $1 \leq q \leq q_m$, where $q_m : \max_{x\in\partial V} k(x) < \infty$ for $f = f_\delta$, since ∂V is compact. Thus, there exists a subset $\lambda \subset \{1, ..., q_m\}$ such that $\partial V = \bigcup_{q\in\lambda} \partial_q V$ and $\partial_q V \neq \emptyset$ for each $q \in \lambda$. If $\nabla^l f(x) = 0$ for $l = 1, ..., k(x) - 1$ and $\nabla^{k(x)} f(x) \neq 0$, then $\nabla^{k(x)} f(\psi(y)) = \nabla^{k(x)}(\psi(y)).(\nabla\psi(y))^{\otimes k(x)} \neq 0$ for $y \in M$ such that $\psi(y) = x$, since $\nabla\psi(y) \neq 0$, where $\psi \in DifH^\infty_p(M; \{s_{0,q} : q = 1, ..., k\})$.

We can take $\epsilon > 0$ such that $\{g^b : b \in (-\epsilon, \epsilon)\} \subset U$, where $U = -U$ is a connected symmetric open neighborhood of e in $(W^M N)_{t,H}$. Since $g^{b_1} + g^{b_2} = g^{b_1+b_2}$ for each $b_1, b_2, b_1 + b_2 \in (-\epsilon, \epsilon)$, then $\lim_{t\to 0} g^b = e$ for the local one parameter subgroup and in particular $\lim_{m\to\infty} g^{1/m} = e$, where $m \in \mathbf{N}$. Take $\delta = \delta_m = 1/m$ and $f = f_m \in H^\infty_p$ such that $< f_m >_{t,H}= g^{1/m}$. On the other hand, $jg^{1/m} = g^{j/m}$ for each $j < m\epsilon$, $j \in \mathbf{N}$, hence $f_{j/m}(M) = f_{1/m}(M)$ for each $j < m\epsilon$, since $f \vee h(M \vee M) = f(M) \vee h(M)$ and using embedding η of $(S^M N)_{t,H}$ into $(W^M N)_{t,H}$.

The function $|\nabla_\nu^{k(x)} f_\delta(x)|$ for $x \in \partial V$ is continuous by δ due to the Sobolev embedding theorem [25], $0 < \delta < \epsilon$, consequently, $\inf_{x\in\partial V} |\nabla_\nu^{k(x)} f_\delta(x)| > 0$, since ∂V is compact. We can choose a family f_δ such that $z^{(l)}(\delta, x) := \nabla^l f_\delta(x)$ is continuous for each $0 \le l \le k_0$ by $(\delta, x) \in (-\epsilon, \epsilon) \times M$, since $\{g^b : b \in (-\epsilon, \epsilon)\}$ is the continuous by b one parameter subgroup, where $k_0 := q_m(\delta_0)$. Therefore, for this family there exists a neighborhood $[-\epsilon + c, \epsilon - c]$ such that $\delta_0 \in [-\epsilon+c, \epsilon-c] \subset (-\epsilon, \epsilon)$ with $0 < c < \epsilon/3$ such that $q_m(\delta) \le k_0$ for each $\delta \in [-\epsilon+c, \epsilon-c]$ with a suitable choice of $V(f_\delta)$, since $\mathbf{N}$ is discrete. On the other hand, $\sup_{x\in\partial V(f_\delta), 0<\delta\le\epsilon-c} |\nabla_\nu^{k(x)} f_\delta(x)| \le \sup_{x\in M, 0<\delta\le\epsilon-c} |\nabla_\nu^{k(x)} f_\delta(x)| =: B < \infty$, since M and $[-\epsilon + c, \epsilon - c]$ are compact.

Therefore, for this family there exists a neighborhood $[-\epsilon+c, \epsilon-c]$ such that $\delta_0 \in [-\epsilon+c, \epsilon-c] \subset (-\epsilon, \epsilon)$ with $0 < c < \epsilon/3$ such that $q_m(\delta) \le k_0$ for each $\delta \in [-\epsilon + c, \epsilon - c]$ with a suitable choice of $V(f_\delta)$, since $\mathbf{N}$ is discrete.

Then $\underline{\lim}_{\delta\to 0, \delta>0} |\nabla_\nu^{k(x)} f_\delta(x)| =: b > 0$ for $x \in \partial V$ with a suitable choice of $V = V(f_\delta)$, since M is connected, $dimM \ge 1$ and $\inf_{m\in\mathbf{N}} diam f_{j/m}(M) > 0$ for a marked $\delta_0 = j/m_0 < \epsilon$ with $j, m > m_0 \in \mathbf{N}$ mutually prime, $(j, m) = 1$, $(j, m_0) = 1$. To $< f_{l/m} >_{t,H}$ there corresponds $< f_{1/m} >_{t,H} \vee ... \vee < f_{1/m} >_{t,H} =: < f_{1/m} >_{t,H}^{\vee l}$ which is the l-fold wedge product. Thus there exists $C = const > 0$ for M such that $|\nabla_\nu^{k(x)} f_{l/m}(x)| \ge Cl \inf_{y\in\partial V(f_{1/m})} |\nabla_\nu^{k(y)} f_{1/m}(y)| \ge Clb$, where $C > 0$ is fixed for a chosen atlas $At(M)$ with given transition mappings $\phi_i \circ \phi_j^{-1}$ of charts.

Consider $\delta_0 \le l/m < \epsilon - c$ and m and l tending to the infinity. Then this gives $B \ge Clb$ for each $l \in \mathbf{N}$, that is the contradictory inequality, hence $(W^M N)_{t,H}$ does not contain any non trivial local one parameter subgroup.

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Regular quaternionic functions and conformal mappings

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ABSTRACT

In this paper we study the action of conformal mappings of the quaternionic space on a class of regular functions of one quaternionic variable. We consider functions in the kernel of the Cauchy-Riemann operator

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3},$$

a variant of the Cauchy–Fueter operator. This choice is motivated by the strict relation existing between this type of regularity and holomorphicity w.r.t. the whole class of complex structures on $\mathbb{H}$. For every imaginary unit $p \in \mathbb{S}^2$, let J_p be the corresponding complex structure on $\mathbb{H}$. Let $Hol_p(\Omega, \mathbb{H})$ be the space of holomorphic maps from (Ω, J_p) to $(\mathbb{H}, L_p)$, where L_p is defined by left multiplication by p. Every element of $Hol_p(\Omega, \mathbb{H})$ is regular, but there exist regular functions that are not holomorphic for any p. These properties can be recognized by computing the *energy quadric* of a function. We show that the energy quadric is invariant w.r.t. three–dimensional rotations of $\mathbb{H}$. As an application, we obtain that every rotation of the space $\mathbb{H}$ can be generated by biregular rotations, invertible regular functions with regular inverse. Moreover, we prove that the energy quadric of a regular function can always be diagonalized by means of a three–dimensional rotation.

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RESUMEN

En este artículo estudiamos la acción de aplicaciones conforme del espacio de cuaterniones sobre la clase de funciones regulares de una variable cuaternionica. Nosotros consideramos funciones en el kernel del operador de Cauchy–Riemann

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3},$$

una variante del operador de Cauchy–Fueter. Esta elección es motivada por la relación estricta existente entre este tipo de regularidad y holomorficidad w.r.t. de la clase entera de estructuras complejas sobre $\mathbb{H}$. Para todo imaginario unitario $p \in \mathbb{S}^2$, sea J_p la correspodiente estructura compleja sobre $\mathbb{H}$. Sea $Hol_p(\Omega, \mathbb{H})$ el espacio de aplicaciones holomórficas de (Ω, J_p) a $(\mathbb{H}, L_p)$, donde L_p es definido por multiplicación a la izquierda por p. Todo elemento de $Hol_p(\Omega, \mathbb{H})$ es regular, pero existen funciones regulares que no son holomórficas para cualquer p. Estas propiedades pueden ser reconocidas mediante el cálculo de la *energía cuadrica* de una función. Nosotros mostramos que la energía cuadrica es invariante w.r.t. por rotaciones tres–dimensionales de H. Como aplicación, obtenemos que toda rotación del espacio $\mathbb{H}$ puede ser generada por rotaciones bi regulares, funciones regulares invertibles con inversa regular. Además mostramos que la energía cuadrica de una función regular siempre puede ser diagonalizada por una rotación tres–dimensional.

Key words and phrases: *quaternionic regular functions, hyperholomorphic functions, conformal mappings, Möbius transformations.*

Math. Subj. Class.: *Primary 30G35; Secondary 30A30*

1 Introduction.

The aim of this paper is to analyze the action of the conformal group of the one–point compactification $\mathbb{H}^*$ of $\mathbb{H}$ on a class of regular functions of one quaternionic variable.

Let Ω be a smooth bounded domain in $\mathbb{C}^2$. Let $\mathbb{H}$ be the space of real quaternions $q = x_0 + ix_1 + jx_2 + kx_3$, where i, j, k denote the basic quaternions. We identify $\mathbb{H}$ with $\mathbb{C}^2$ by means of the mapping that associates the quaternion $q = z_1 + z_2 j$ with the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$. We consider the class $\mathcal{R}(\Omega)$ of *left–regular* (also called *hyperholomorphic*) functions $f : \Omega \to \mathbb{H}$ in the kernel of the Cauchy–Riemann operator

$$\mathcal{D} = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}.$$

This differential operator is a variant of the original Cauchy–Riemann–Fueter operator (cf. for example [19] and [5, 5])

$$\frac{\partial}{\partial x_0}+i\frac{\partial}{\partial x_1}+j\frac{\partial}{\partial x_2}+k\frac{\partial}{\partial x_3}.$$

Hyperholomorphic functions have been studied by many authors (see for instance [1, 7, 11, 12, 14, 17, 18]). Many of their properties can be easily deduced from known properties satisfied by Fueter–regular functions, since a function f is regular on Ω if and only if $f\circ\gamma$ is Fueter–regular on $\gamma(\Omega)=\gamma^{-1}(\Omega)$, where γ is the reflection of $\mathbb{C}^2$ defined by $\gamma(z_1,z_2)=(z_1,\bar{z}_2)$. However, regular functions in the space $\mathcal{R}(\Omega)$ have some characteristics that are more intimately related to the theory of holomorphic functions of two complex variables. In particular, the space $\mathcal{R}(\Omega)$ contains the spaces of holomorphic maps with respect to any constant complex structure. This is no longer true if we adopt the original definition of Fueter regularity (see Section 2 for more details).

Let J_1,J_2 be the complex structures on the tangent bundle $T\mathbb{H}\simeq\mathbb{H}$ defined by left multiplication by i and j. Let J_1^*,J_2^* be the dual structures on the cotangent bundle $T^*\mathbb{H}\simeq\mathbb{H}$ and set $J_3^*=J_1^*J_2^*$. For every complex structure $J_p=p_1J_1+p_2J_2+p_3J_3$ (p a imaginary unit in the unit sphere $\mathbb{S}^2$), let

$$\overline{\partial}_p=\frac{1}{2}\left(d+pJ_p^*\circ d\right)$$

be the Cauchy–Riemann operator with respect to the structure J_p. Let us define $Hol_p(\Omega,\mathbf{H})=\operatorname{Ker}\overline{\partial}_p$, the space of holomorphic maps from (Ω,J_p) to $(\mathbf{H},L_p)$, where L_p is the complex structure defined by left multiplication by p. Then every element of $Hol_p(\Omega,\mathbf{H})$ is regular. These subspaces do not fill the whole space of regular functions (cf. [13]). This result is a consequence of a criterion of J_p–holomorphicity, based on the concept of *energy quadric* of a regular function (cf. Section 3.2 for exact definitions).

In Section 4 we come to conformal transformations. >From a theorem of Liouville, the general conformal mapping of $\mathbb{H}^*$ is the composition of a sequence of translations, dilations, rotations and inversions. It can be written as a quaternionic *Möbius transformation*, i.e. a fractional linear map of the form

$$L_A(q)=(aq+b)(cq+d)^{-1},$$

with $A\in GL(2,\mathbb{H})$. For properties of these maps, see for example [2], [5]§6.2, [11] and [19] and the references cited in those papers.

Given a function $f\in C^1(\Omega)$ and a conformal transformation L_A, let f^A be the function

$$f^A(q)=\frac{(c\gamma(q)+d)^{-1}}{|c\gamma(q)+d|^2}f(L'_{\gamma(A)}(q)),$$

where $L'_{\gamma(A)}(q)=\gamma\circ L_A\circ\gamma(q)$. In Theorem 3, we prove that f is regular on Ω if and only if f^A is regular on $\Omega'=(L'_{\gamma(A)})^{-1}(\Omega)$. Moreover, $(f^A)^B=f^{AB}$ for every $A,B\in GL(2,\mathbb{H})$. The first property can be deduced from Theorem 6 of Sudbery [19] using the reflection γ.

We are interested also in the action of conformal mappings on the energy quadric and on the holomorphicity properties of the maps. For a general conformal transformation L_A, the energy and, *a fortiori*, the energy quadric of a regular function is not conserved. However, we show that three–dimensional rotations of $\mathbb{H}$ (those which fix the real numbers) conserve the energy quadric (for translations this it is a trivial fact).

Let $a \in \mathbb{H}$, $a \neq 0$. Let $rot_a(q) = aqa^{-1}$ be the three–dimensional rotation of $\mathbb{H}$ defined by a. In Theorem 4, we prove that the function

$$f^a = rot_{\gamma(a)} \circ f \circ rot_a$$

is regular on $\Omega^a = rot_a^{-1}(\Omega)$ if and only if f is regular on Ω. Moreover, the energy density of f^a is $\mathcal{E}(f^a) = \mathcal{E}(f) \circ rot_a$ and the matrix function $M(f)$ (for f regular $M(f)$ is the energy quadric, cf. Section 3) transforms in the following way

$$M(f^a) = Q_a(M(f) \circ rot_a)Q_a^T,$$

where $Q_a \in SO(3)$ is the orthogonal matrix associated to the rotation $rot_{\gamma(a)}$ of the space $\mathbb{R}^3 = \langle i, j, k \rangle$.

This formula has many consequences. It allows to obtain (Corollary 3) that f^a is J_p–holomorphic if and only if f is $J_{p'}$–holomorphic, with $p' = rot_{\gamma(a)}^{-1}(p)$. Moreover, we get (Corollary 4) that the energy quadric of a regular function can always be diagonalized by means of a three–dimensional rotation. Finally, we obtain a biregularity result about rotations (Proposition 2 and Corollary 5). We prove that every three-dimensional rotation is the composition of (at most) two three-dimensional biregular rotations, and that every four-dimensional rotation is the composition of two biregular rotations.

2 Notations and definitions

2.1 Fueter regular functions

We identify the space $\mathbb{C}^2$ with the set $\mathbb{H}$ of quaternions by means of the mapping that associates the pair $(z_1, z_2) = (x_0 + ix_1, x_2 + ix_3)$ with the quaternion $q = z_1 + z_2 j = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}$. A quaternionic function $f = f_1 + f_2 j \in C^1(\Omega)$ is *(left) regular* (or *hyperholomorphic*) on Ω if

$$\mathcal{D}f = 2\left(\frac{\partial}{\partial \bar{z}_1} + j\frac{\partial}{\partial \bar{z}_2}\right) = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \quad \text{on } \Omega.$$

We will denote by $\mathcal{R}(\Omega)$ the space of regular functions on Ω.

With respect to this definition of regularity, the space $\mathcal{R}(\Omega)$ contains the identity mapping and every holomorphic mapping (f_1, f_2) on Ω (with respect to the standard complex structure) defines a regular function $f = f_1 + f_2 j$. We recall some properties of regular functions, for which we refer to the papers of Sudbery[19], Shapiro and Vasilevski[17] and Nōno[12]:

1. The complex components are both holomorphic or both non–holomorphic.

2. Every regular function is harmonic.
3. If Ω is pseudoconvex, every complex harmonic function is the complex component of a regular function on Ω.
4. The space $\mathcal{R}(\Omega)$ of regular functions on Ω is a *right* $\mathbb{H}$–module with integral representation formulas.
5. f is regular $\Leftrightarrow$ $\dfrac{\partial f_1}{\partial \bar z_1} = \dfrac{\partial \overline{f_2}}{\partial z_2}, \quad \dfrac{\partial f_1}{\partial \bar z_2} = -\dfrac{\partial \overline{f_2}}{\partial z_1}.$

We note that a function $f = f_1 + f_2 j$ is regular on Ω if and only if its Jacobian matrix has the form

$$J(f) = \left(\frac{\partial(f_1, f_2, \bar f_1, \bar f_2)}{\partial(z_1, z_2, \bar z_1, \bar z_2)}\right) = \left(\begin{array}{cc|cc} a_1 & -\bar b_2 & -\bar c_2 & -c_1 \\ a_2 & \bar b_1 & \bar c_1 & -c_2 \\ \hline -c_2 & -\bar c_1 & \bar a_1 & -b_2 \\ c_1 & -\bar c_2 & \bar a_2 & b_1 \end{array}\right)$$

at every $z \in \Omega$, where $a = \left(\dfrac{\partial f_1}{\partial z_1}, \dfrac{\partial f_2}{\partial z_1}\right)$, $b = \left(\dfrac{\partial \bar f_2}{\partial \bar z_2}, -\dfrac{\partial \bar f_1}{\partial \bar z_2}\right)$, $c = \left(\dfrac{\partial \bar f_2}{\partial z_1}, -\dfrac{\partial \bar f_1}{\partial z_1}\right) = -\left(\dfrac{\partial f_1}{\partial \bar z_2}, \dfrac{\partial f_2}{\partial \bar z_2}\right)$. We shall call a matrix of this form a *regular matrix*. Note that a regular matrix can have rank $0, 2, 3$ or 4 but not rank 1.

A definition equivalent to regularity has been given by Joyce[6] in the setting of hypercomplex manifolds. Joyce introduced the module of *q–holomorphic* functions on a hypercomplex manifold.

A hypercomplex structure on the manifold $\mathbb{H}$ is given by the complex structures J_1, J_2 on $T\mathbb{H} \simeq \mathbb{H}$ defined by left multiplication by i and j. Let J_1^*, J_2^* be the dual structures on $T^*\mathbb{H} \simeq \mathbb{H}$. In complex coordinates

$$\begin{cases} J_1^* dz_1 = i\, dz_1, & J_1^* dz_2 = i\, dz_2 \\ J_2^* dz_1 = -d\bar z_2, & J_2^* dz_2 = d\bar z_1 \\ J_3^* dz_1 = i\, d\bar z_2, & J_3^* dz_2 = -i\, d\bar z_1 \end{cases}$$

where we make the choice $J_3^* = J_1^* J_2^*$, which is equivalent to $J_3 = -J_1 J_2$. In real coordinates, the action of the structures is the following

$$\begin{cases} J_1 dx_0 = -dx_1, & J_1 dx_2 = -dx_3, \\ J_2 dx_0 = -dx_2, & J_2 dx_1 = dx_3, \\ J_3 dx_0 = dx_3, & J_3 dx_1 = dx_2. \end{cases}$$

A function f is regular if and only if f is *q–holomorphic*, i.e.

$$df + iJ_1^*(df) + jJ_2^*(df) + kJ_3^*(df) = 0.$$

In complex components $f = f_1 + f_2 j$, we can rewrite the equations of regularity as

$$\overline{\partial} f_1 = J_2^*(\partial \overline{f}_2).$$

The original definition of regularity given by Fueter (cf. [19] or [5]) differs from that adopted here by a real coordinate reflection. Let γ be the transformation of $\mathbb{C}^2$ defined by $\gamma(z_1, z_2) = (z_1, \bar{z}_2)$. Then a C^1 function f is regular on the domain Ω if and only if $f \circ \gamma$ is Fueter–regular on $\gamma(\Omega) = \gamma^{-1}(\Omega)$, i.e. it satisfies the differential equation

$$\left(\frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}\right)(f \circ \gamma) = 0 \quad \text{on } \gamma^{-1}(\Omega).$$

2.2 Biregular functions

A quaternionic function $f \in C^1(\Omega)$ is called *biregular* if f is invertible and f, f^{-1} are regular. If this property holds locally, f is called *locally biregular*. These functions were studied in [8], [9] and [15].

The class $\mathcal{BR}(\Omega)$ of biregular functions is closed with respect to right multiplication by a non-zero quaternion, but it is not closed with respect to composition or sum: even if $f + g$ is invertible and $f, g \in \mathcal{BR}(\Omega)$, the sum can be not biregular.

2.2.0.1 Examples

1. Every biholomorphic map (f_1, f_2) on Ω defines a biregular function $f = f_1 + f_2 j$.

2. The identity function is biregular on $\mathbb{H}$. More generally, the affine functions $f(q) = qa + b$, $a \in \mathbb{H}^*$, $b \in \mathbb{H}$, are biregular on $\mathbb{H}$.

3. $f = \bar{z}_1 + \bar{z}_2 j \in \mathcal{R}(\mathbb{H})$, $f^{-1} = f \in \mathcal{BR}(\mathbb{H})$.

4. The function $f = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$ is regular, but the inverse map

$$f^{-1} = \frac{1}{3}\left(z_1 + z_2 + \bar{z}_1 - 2\bar{z}_2 + (z_1 + z_2 - 2\bar{z}_1 + \bar{z}_2)j\right)$$

   is not regular. Note that in this case the Jacobian determinant is negative. This cannot happen for a biregular function (cf. [15]).

2.3 Holomorphic functions w.r.t. a complex structure J_p

Let $J_p = p_1 J_1 + p_2 J_2 + p_3 J_3$ be the orthogonal complex structure on $\mathbb{H}$ defined by a unit imaginary quaternion $p = p_1 i + p_2 j + p_3 k$ in the sphere $\mathbb{S}^2 = \{p \in \mathbb{H} \mid p^2 = -1\}$. In particular, J_1 is the standard complex structure of $\mathbb{C}^2 \simeq \mathbb{H}$.

Let $\mathbb{C}_p = \langle 1, p\rangle$ be the complex plane spanned by 1 and p and let L_p be the complex structure defined on $T^*\mathbb{C}_p \simeq \mathbb{C}_p$ by left multiplication by p. If $f = f^0 + i f^1 : \Omega \to \mathbb{C}$ is a J_p*–holomorphic* function, i.e. $df^0 = J_p^*(df^1)$ or, equivalently, $df + iJ_p^*(df) = 0$, then f defines a regular function $\tilde{f} = f^0 + pf^1$ on Ω. We can identify $\tilde{f}$ with a holomorphic function

$$\tilde{f} : (\Omega, J_p) \to (\mathbb{C}_p, L_p).$$

We have $L_p = J_{\gamma(p)}$, where $\gamma(p) = p_1 i + p_2 j - p_3 k$. More generally, we can consider the space of holomorphic maps from (Ω, J_p) to $(\mathbb{H}, L_p)$

$$Hol_p(\Omega, \mathbb{H}) = \{f : \Omega \to \mathbb{H} \text{ of class } C^1 \mid \overline{\partial}_p f = 0 \text{ on } \Omega\} = \operatorname{Ker} \overline{\partial}_p$$

where $\overline{\partial}_p$ is the Cauchy–Riemann operator with respect to the structure J_p

$$\overline{\partial}_p = \frac{1}{2}\left(d + pJ_p^* \circ d\right).$$

These functions will be called *J_p–holomorphic maps* on Ω.

For any positive orthonormal basis $\{1, p, q, pq\}$ of $\mathbb{H}$ ($p, q \in \mathbb{S}^2$), let $f = f_1 + f_2 q$ be the decomposition of f with respect to the orthogonal sum

$$\mathbb{H} = \mathbb{C}_p \oplus (\mathbb{C}_p)q.$$

Let $f_1 = f^0 + pf^1$, $f_2 = f^2 + pf^3$, with f^0, f^1, f^2, f^3 the real components of f w.r.t. the basis $\{1, p, q, pq\}$. Then the equations of regularity can be rewritten in complex form as

$$\overline{\partial}_p f_1 = J_q^*(\partial_p \overline{f}_2),$$

where $\overline{f}_2 = f^2 - pf^3$ and $\partial_p = \frac{1}{2}\left(d - pJ_p^* \circ d\right)$. Therefore every $f \in Hol_p(\Omega, \mathbb{H})$ is a regular function on Ω.

Remark 1. *1. The identity map belongs to the space $Hol_i(\Omega, \mathbb{H}) \cap Hol_j(\Omega, \mathbb{H})$ but not to $Hol_k(\Omega, \mathbb{H})$.*

2. For every $p \in \mathbb{S}^2$, $Hol_{-p}(\Omega, \mathbb{H}) = Hol_p(\Omega, \mathbb{H})$.

3. Every $\mathbb{C}_p$–valued regular function is a J_p–holomorphic function.

4. If $f \in Hol_p(\Omega, \mathbb{H}) \cap Hol_q(\Omega, \mathbb{H})$, with $p \neq \pm q$, then $f \in Hol_r(\Omega, \mathbb{H})$ for every $r = \frac{\alpha p + \beta q}{\|\alpha p + \beta q\|}$ ($\alpha, \beta \in \mathbb{R}$) in the circle of $\mathbb{S}^2$ generated by p and q.

If the almost complex structure J_p is not constant, i.e. not compatible with the hyperkähler structure of $\mathbb{H}$, we get a similar result. Note that in this case the structure is not necessarily integrable. Let $f \in C^1(\Omega)$ and assume that $p = p(z) \in \mathbb{S}^2$ varies continuously with z in Ω. We will say that p is *f-equivariant* if $f(z) = f(z')$ implies $p(z) = p(z')$ ($z, z' \in \Omega$). This property allows to define $p^* : f(\Omega) \to \mathbb{S}^2$ such that $p^* \circ f = p$ on Ω. In [15], the following result was proved.

Proposition 1. *If $f \in C^1(\Omega)$ satisfies the equation*

$$\overline{\partial}_{p(z)} f = \frac{1}{2}\left[df(z) + p(z)J_{p(z)}^* \circ df(z)\right] = 0 \tag{1}$$

at every $z \in \Omega$, then f is a regular function on Ω. If, moreover, the structure p is f-equivariant and p^* admits a continuous extension to an open set $U \supseteq f(\Omega)$, then f is a (pseudo)holomorphic map from (Ω, J_p) to (U, L_{p^*}).

Example 1. *$f(z)=\bar z_1+z_2^2+\bar z_2 j$ is regular on $\mathbb{H}$. On $\Omega=\mathbb{H}\setminus\{z_2=0\}$ f is holomorphic w.r.t. the almost complex structure J_p, where*

$$p(z)=\frac{1}{\sqrt{|z_2|^2+|z_2|^4}}\left(|z_2|^2 i+(\operatorname{Im} z_2)j-(\operatorname{Re} z_2)k\right).$$

Note that $p(z)$ can not be continued to $\mathbb{H}$ as a continuous map. Also the inverse map $f^{-1}(z)=\bar z_1-z_2^2+\bar z_2 j$ is regular on $\mathbb{H}$. Then f is biregular on $\mathbb{H}$. But f is also (pseudo)biholomorphic on Ω: $f(\Omega)=\Omega$ and $f^{-1}:(\Omega,J_{p'})\to(\mathbb{H},L_{p'\circ f})$ is holomorphic, where

$$p'(z)=\frac{1}{\sqrt{|z_2|^2+|z_2|^4}}\left(|z_2|^2 i-(\operatorname{Im} z_2)j+(\operatorname{Re} z_2)k\right).$$

Note that $L_{p^*}=L_{p\circ f^{-1}}=J_{p'}$ at $f(z)$ and $L_{p'\circ f}=J_p$ at $z\in\Omega$.

3 A criterion for holomorphicity

3.1 Energy and regularity

In [13] it was proved that on every domain Ω there exist regular functions that are not J_p-holomorphic for any p. A similar result was obtained by Chen and Li[3] for the larger class of *q-maps* between hyperkähler manifolds.

The criterion for holomorphicity is based on an energy-minimizing property of holomorphic maps.

The *energy density* (w.r.t. the euclidean metric) of a function $f:\Omega\to\mathbb{H}$, of class $C^1(\Omega)$, is given by

$$\mathcal{E}(f)=\tfrac{1}{2}\|df\|^2=\tfrac{1}{2}\operatorname{tr}(J(f)\overline{J(f)}^T).$$

After integration on Ω, we get the *energy* of $f\in C^1(\overline{\Omega})$:

$$\mathcal{E}_\Omega(f)=\tfrac{1}{2}\int_\Omega \mathcal{E}(f)dV.$$

Using ideas from [10] and [3], it was proved in [13] that a regular function $f\in C^1(\overline{\Omega})$ minimizes energy in the homotopy class constituted by maps u with $u_{|\partial\Omega}=f_{|\partial\Omega}$ which are homotopic to f relative to $\partial\Omega$:

Now we introduce the Lichnerowicz invariants. Let $A(f)=(a_{\alpha\beta})$ be the 3×3 matrix with entries the real functions $a_{\alpha\beta}=-\langle J_\alpha, f^*L_{i_\beta}\rangle$, where $(i_1,i_2,i_3)=(i,j,k)$. For $f\in C^1(\overline{\Omega})$, we set

$$A_\Omega(f)=\int_\Omega A(f)\,dV \quad\text{and}\quad M_\Omega(f)=\tfrac{1}{2}\left((\operatorname{tr} A_\Omega(f))I_3-A_\Omega(f)\right),$$

where I_3 denotes the identity matrix.

We recall the criterion for regularity and holomorphicity proved in [13].

Theorem 1. *1. $M_\Omega(f)$ is a relative homotopy invariant of f.*

2. f is regular on Ω if and only if $\mathcal{E}_\Omega(f) = \operatorname{tr} M_\Omega(f)$.

3. If $f \in \mathcal{R}(\Omega)$, then $M_\Omega(f)$ is symmetric and positive semidefinite.

4. If $f \in \mathcal{R}(\Omega)$, then f belongs to some space $Hol_p(\Omega, \mathbb{H})$ (for a constant structure J_p) if and only if $\det M_\Omega(f) = 0$.

5. $f \in Hol_p(\Omega, \mathbb{H})$ if and only if $X_p = (p_1, p_2, p_3)$ is a unit vector in the kernel of $M_\Omega(f)$.

>From the criterion it can be seen that almost all regular functions are not holomorphic with respect to any constant complex structure J_p.

Example 2. *$f = \bar{z}_1 + z_2 + \bar{z}_2 j$ is J_p-holomorphic, with $p = \frac{1}{\sqrt{5}}(i - 2k)$, since on the unit ball B (with normalized unit volume)*

$$\mathcal{E}_B(f) = 3 \quad and \quad M_B(f) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & \frac{1}{2} \end{bmatrix}.$$

Example 3. *$f = z_1 + z_2 + \bar{z}_1 + (z_1 + z_2 + \bar{z}_2)j$ is regular, but not holomorphic:*

$$\mathcal{E}_B(f) = 6 \quad and \quad M_B(f) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Example 4. *$f = \bar{z}_1 + \bar{z}_2 j$ is regular and has matrix*

$$M_B(f) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

of rank one. This means that $f \in Hol_j(\mathbb{H}, \mathbb{H}) \cap Hol_k(\mathbb{H}, \mathbb{H})$.

Example 5. *The identity mapping belongs to the space*

$$Hol_i(\mathbb{H}, \mathbb{H}) \cap Hol_j(\mathbb{H}, \mathbb{H}) = \bigcap_{p \in <i,j>} Hol_p(\mathbb{H}, \mathbb{H}).$$

Example 6 (Nonlinear case)**.** *$f = |z_1|^2 - |z_2|^2 + \bar{z}_1 \bar{z}_2 j$ has energy $\mathcal{E}_B(f) = 2$ on the unit ball. The matrix $M_B(f)$ is*

$$M_B(f) = \begin{bmatrix} \frac{4}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

Therefore f is regular but not holomorphic w.r.t. any constant complex structure J_p.

3.2 The energy quadric

In [15], a pointwise version of the criterion for holomorphicity was established.

Theorem 2. *Let Ω be connected and $f \in C^1(\Omega)$. Consider the matrix of real functions on Ω*

$$M(f) = \tfrac{1}{2}\left((\operatorname{tr} A(f))I_3 - A(f)\right).$$

1. *f is regular on Ω if and only if $\mathcal{E}(f) = \operatorname{tr} M(f)$ at every point $z \in \Omega$.*
2. *If $f \in \mathcal{R}(\Omega)$, then $M(f)$ is symmetric and positive semidefinite.*
3. *If $f \in \mathcal{R}(\Omega)$, then $\det M(f) = 0$ on Ω if and only if there exists an open, dense subset $\Omega' \subseteq \Omega$ on which f satisfies equation (1) for some function $p(z) : \Omega' \to \mathbb{S}^2$. Moreover, if $\det M(f) = 0$ and $p(z)$ is f-equivariant, $p^* \circ f = p$ and p^* extends continuously to an open set $U \supseteq f(\Omega)$, then f is a (pseudo)holomorphic map from (Ω', J_p) to (U, L_{p^*}).*

Let

$$a = \left(\frac{\partial f_1}{\partial z_1}, \frac{\partial f_2}{\partial z_1}\right),\ b = \left(\frac{\partial \bar f_2}{\partial \bar z_2}, -\frac{\partial \bar f_1}{\partial \bar z_2}\right),\ c = \left(\frac{\partial \bar f_2}{\partial z_1}, -\frac{\partial \bar f_1}{\partial z_1}\right),\ d = -\left(\frac{\partial f_1}{\partial \bar z_2}, \frac{\partial f_2}{\partial \bar z_2}\right).$$

Then the energy density is given by $\mathcal{E}(f) = |a|^2+|b|^2+|c|^2+|d|^2$. A lengthy but straightforward computation gives the following expression for the matrix $M(f)$:

$$M(f) = \begin{bmatrix} |c|^2+|d|^2 & \operatorname{Im}(\langle d,a\rangle - \langle c,b\rangle) & \operatorname{Re}(\langle d,a\rangle + \langle c,b\rangle) \\ \operatorname{Im}(\langle c,a\rangle - \langle d,b\rangle) & \frac{1}{2}|a-b|^2 + \frac{1}{2}|c-d|^2 & -\operatorname{Im}(\langle a,b\rangle + \langle c,d\rangle) \\ \operatorname{Re}(\langle c,a\rangle + \langle d,b\rangle) & -\operatorname{Im}(\langle a,b\rangle - \langle c,d\rangle) & \frac{1}{2}|a+b|^2 + \frac{1}{2}|c-d|^2 \end{bmatrix}.$$

Then $\mathcal{E}(f) = \operatorname{tr} M(f)$ if and only if $c = d$, i.e. f is regular. In this case the matrix $M(f)$ becomes

$$M(f) = \begin{bmatrix} 2|c|^2 & \operatorname{Im}\langle c, a-b\rangle & \operatorname{Re}\langle c, a+b\rangle \\ \operatorname{Im}\langle c, a-b\rangle & \frac{1}{2}|a-b|^2 & -\operatorname{Im}\langle a,b\rangle \\ \operatorname{Re}\langle c, a+b\rangle & -\operatorname{Im}\langle a,b\rangle & \frac{1}{2}|a+b|^2 \end{bmatrix}.$$

Definition 1. *For a regular function f on Ω, the family of positive semi-definite quadrics with matrices $\{M(f)(z) \mid z \in \Omega\}$ will be called the energy quadric of f.*

Remark 2. *If f is invertible, then every $p(z)$ is f-equivariant. If p is a constant complex structure, then p is f-equivariant for every f.*

Remark 3. *If f is (real) affine, $M(f)$ is a constant matrix. If f is not affine, $\det M(f) = 0$ on Ω does not imply that $\det M_\Omega(f) = 0$, but Theorems 1 and 2 imply that the converse is true.*

Example 7. *The function* $f(z) = \bar{z}_1 + z_2^2 + \bar{z}_2 j$ *is regular (also biregular, cf. Example 1) on* $\mathbb{H}$. *We have*

$$\mathcal{E}(f) = 2 + 4|z_2|^2, \quad M(f) = 2\begin{bmatrix} 1 & -\operatorname{Im} z_2 & \operatorname{Re} z_2 \\ -\operatorname{Im} z_2 & |z_2|^2 & 0 \\ \operatorname{Re} z_2 & 0 & |z_2|^2 \end{bmatrix}.$$

Then the energy quadric of f *is singular on* $\mathbb{H}$. *On the domain* $\Omega' = \mathbb{H} \setminus \{z_2 = 0\}$, *where* $M(f)$ *has maximum rank 2, the kernel of* $M(f)$ *is spanned by the vector* $X = (|z_2|^2, \operatorname{Im} z_2, -\operatorname{Re} z_2)$. *Then* f *is* J_p*-holomorphic on* Ω', *with*

$$p(z) = \frac{1}{\sqrt{|z_2|^2 + |z_2|^4}} \left(|z_2|^2 i + (\operatorname{Im} z_2) j - (\operatorname{Re} z_2) k \right).$$

On the unit ball B, $\mathcal{E}_B(f) = \frac{10}{3}$ *and the matrix*

$$M_B(f) = \int_B M(f) dV = \begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}$$

is non-singular. Therefore, f *is not* J_q*-holomorphic for any constant complex structure* J_q.

Example 8. *The function* $f = |z_1|^2 - |z_2|^2 + \bar{z}_1 \bar{z}_2 j$ *introduced in Example 6 has energy density* $3|z|^2$ *and energy quadric with matrix*

$$M(f) = \begin{bmatrix} 2|z|^2 & 0 & 0 \\ 0 & \frac{1}{2}|z|^2 & 0 \\ 0 & 0 & \frac{1}{2}|z|^2 \end{bmatrix}.$$

Therefore f *is regular but not holomorphic w.r.t. any almost complex structure* J_p. *Note that* $\det M(f) = \frac{1}{2}|z|^6$ *vanishes only at the origin.*

In [15], it was shown that if $f \in \mathcal{BR}(\Omega)$ is a biregular function, then there exists an open, dense subset $\Omega' \subseteq \Omega$, and an almost complex structure $p(z)$ on Ω', such that

$$f : (\Omega', J_p) \to (f(\Omega'), L_{p^*})$$

is a holomorphic map, with holomorphic inverse $f^{-1} : (f(\Omega'), J_{p'}) \to (\Omega', L_{p' \circ f})$. Here $p = p_1 i + p_2 j + p_3 k : \Omega' \to \mathbb{S}^2$, $p^* = p \circ f^{-1}$ and $p' = p_1 i + p_2 j - p_3 k$. In particular, any such map f preserves orientation.

4 Regular functions and conformal mappings

In this section we are going to analyze the action of the conformal group of $\mathbb{H}$ on regular functions. Some of the results we describe can be deduced from [19] Theorem 6 using the reflection $\gamma(z_1, z_2) =$

$(z_1, \bar{z}_2)$ introduced in §2.1, but here we want to investigate also the action on the energy quadric and the holomorphicity properties of the maps.

We recall some definitions and properties of conformal and orientation preserving mappings of the one–point compactification $\widehat{\mathbb{H}}$ of $\mathbb{H}$, for which we refer to [2], [5]§6.2, [11] and [19] and to the references cited in those papers.

The *Dieudonné determinant* of a quaternionic matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the real non–negative number

$$\det_{\mathbb{H}}(A) = \sqrt{|a|^2|d|^2 + |b|^2|c|^2 - 2Re(c\bar{a}b\bar{d})}.$$

It satisfies Binet property $\det_{\mathbb{H}}(AB) = \det_{\mathbb{H}}(A)\det_{\mathbb{H}}(B)$ and a 2×2 matrix A is (left and right) invertible if and only if $\det_{\mathbb{H}} A \neq 0$. Then we can consider the general linear group

$$GL(2, \mathbb{H}) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ quaternionic matrix of order 2} \mid \det_{\mathbb{H}} A \neq 0 \right\}.$$

A theorem of Liouville tells that the general conformal transformation of $\mathbb{H}^*$ is a quaternionic *Möbius transformation*, i.e. a fractional linear map of the form

$$L_A(q) = (aq + b)(cq + d)^{-1},$$

for $A \in GL(2, \mathbb{H})$. The matrix A is determined by L_A up to a real scalar multiple. For every pair of matrices $A, B \in GL(2, \mathbb{H})$, $L_A \circ L_B = L_{AB}$. We have also the alternative representation of conformal mappings

$$L'_A(q) = (qc + d)^{-1}(qa + b), \quad \det_{\mathbb{H}} \bar{A} \neq 0.$$

Theorem 3. *Given a function $f \in C^1(\Omega)$ and a conformal transformation $L_A(q) = (aq + b)(cq + d)^{-1}$, let f^A be the function*

$$f^A(q) = \frac{(c\gamma(q) + d)^{-1}}{|c\gamma(q) + d|^2} f(L'_{\gamma(A)}(q)),$$

where $\gamma(A) = \begin{bmatrix} \gamma(a) & \gamma(b) \\ \gamma(c) & \gamma(d) \end{bmatrix}$. Then f is regular on Ω if and only if f^A is regular on $\Omega' = (L'_{\gamma(A)})^{-1}(\Omega)$. Moreover, $(f^A)^B = f^{AB}$ for every $A, B \in GL(2, \mathbb{H})$.

Proof. We deduce the first statement from the result of Sudbery (cf. [19] Theorem 6), since $f \in \mathcal{R}(\Omega)$ iff $F = f \circ \gamma$ is Fueter–regular on $\gamma(\Omega)$. This last condition is equivalent to the Fueter–regularity of the transformed function

$$F^A(p) = \frac{(cp + d)^{-1}}{|cp + d|^2} F(L_A(p))$$

on $(L_A)^{-1}(\gamma(\Omega))$. Note that this function differs from the one given by Sudbery by a real constant factor. We then obtain that f is regular iff $F^A\circ\gamma$ is regular. We have

$$F^A\circ\gamma(q)=\frac{(c\gamma(q)+d)^{-1}}{|c\gamma(q)+d|^2}f\circ\gamma\circ L_A\circ\gamma(q)=f^A(q),$$

since $\gamma\circ L_A\circ\gamma(q)=L'_{\gamma(A)}(q)$. Now we come to the last statement of the theorem. Let $B=\begin{bmatrix}a' & b'\\ c' & d'\end{bmatrix}$ and $C=AB=\begin{bmatrix}a'' & b''\\ c'' & d''\end{bmatrix}$. Then

$$\begin{aligned}(f^A)^B(q)&=\frac{(c'\gamma(q)+d')^{-1}}{|c'\gamma(q)+d'|^2}f^A(L'_{\gamma(B)}(q))\\ &\frac{(c'\gamma(q)+d')^{-1}}{|c'\gamma(q)+d'|^2}\frac{(c\gamma(L'_{\gamma(B)}(q))+d)^{-1}}{|c\gamma(L'_{\gamma(B)}(q))+d|^2}f((L'_{\gamma(A)}\circ L'_{\gamma(B)})(q))\end{aligned}$$

Let $q'=\gamma(q)$. The last statement of the theorem follows from the equalities

$$L'_{\gamma(A)}\circ L'_{\gamma(B)}=(\gamma\circ L_A\circ\gamma)\circ(\gamma\circ L_B\circ\gamma)=\gamma\circ L_{AB}\circ\gamma=L'_{\gamma(AB)}$$

and

$$\begin{aligned}\overline{(c'q'+d')}\;\overline{(c\gamma(L'_{\gamma(B)}(q))+d)}&=(\overline{q'c'}+\overline{d'})\,((\overline{q'c'}+\overline{d'})^{-1}(\overline{q'a'}+\overline{b'})\bar{c}+\bar{d})\\ &(\overline{q'a'}+\overline{b'})\bar{c}+(\overline{q'c'}+\overline{d'})\bar{d}\\ &=\overline{q'}(\overline{a'}\bar{c}+\overline{c'}\bar{d})+\overline{b'}\bar{c}+\overline{d'}\bar{d}\\ &\overline{c''q'+d''}\end{aligned}$$

□

Remark 4. *If t is a non–zero real number, $f^{tA}=t^{-3}f^A$. Then f^A depends only for a real scalar multiple on the matrix chosen to represent the conformal transformation L_A. We can also restrict the choice of the matrix to the subgroup $SL(2,\mathbb{H})=\{A\in GL(2,\mathbb{H})\mid \det_{\mathbb{H}}(A)=1\}$. In this case, the same conformal transformation gives rise to two functions, f^A and $f^{-A}=-f^A$.*

Every conformal transformation is the composition of a sequence of translations, dilations, rotations and inversions. In order to illustrate the preceding theorem, we now apply it to these basic cases.

Example 9. *The inversion $q\mapsto q^{-1}$ corresponds to the matrix $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ (up to a real scalar multiple) and transforms a regular $f\in\mathcal{R}(\Omega)$ into*

$$f^{inv}(q)=\frac{\gamma(q)^{-1}}{|q|^2}f(q^{-1}),$$

regular on $\Omega'=\{q\in\mathbb{H}\mid q^{-1}\in\Omega\}$.

Example 10. *In particular, the inverted function of the constant function $f = \frac{1}{2\pi^2}$ is the Cauchy–Fueter kernel for the module of regular functions*

$$G(q) = G(z_1 + z_2 j) = \frac{1}{2\pi^2} \frac{\bar{z}_1 - \bar{z}_2 j}{|z|^4}.$$

Example 11. *A translation $q \mapsto q + b$ corresponds to the matrix $A = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$. The transformed function is*

$$f^A(q) = f(L'_{\gamma(A)}(q)) = f(q + \gamma(b)).$$

Example 12. *A dilation $q \mapsto aq$, $a \neq 0$ real, has matrix $A = \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$. A function f transforms into*

$$f^A(q) = f(qa).$$

Example 13. *Given two unit quaternions $a, d \in \mathbb{H}$, the diagonal matrix $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ induces the four–dimensional rotation $q \mapsto aqd^{-1}$. Given a regular function f on Ω, the function*

$$f^A(q) = d^{-1} f(\gamma(d)^{-1} q \gamma(a))$$

is regular on $\Omega' = \gamma(d)\Omega\gamma(a)^{-1}$.

Example 14. *The quaternionic Cayley transformation $\psi(q) = (q+1)(1-q)^{-1}$ maps diffeomorphically the unit ball B to the right half–space $\mathbb{H}^+ = \{q \in \mathbb{H} \mid Re(q) > 0\}$ (see [2] for geometric properties of ψ). It transforms regular functions f on $\mathbb{H}^+$ into*

$$f^\psi(q) = 2^{3/2} \frac{(1-\gamma(q))^{-1}}{|1-\gamma(q)|^2} f(\psi(q)),$$

regular on B. The inverse mapping $\psi^{-1}(q) = (q-1)(1+q)^{-1}$ transforms $f \in \mathcal{R}(B)$ into

$$f^{\psi^{-1}}(q) = 2^{3/2} \frac{(1+\gamma(q))^{-1}}{|1+\gamma(q)|^2} f(\psi^{-1}(q)) \in \mathcal{R}(\mathbb{H}^+).$$

The factor $2^{3/2}$ in the formulas has been chosen to get $(f^\psi)^{\psi^{-1}} = f$.

If we take the identity map, which is regular on $\mathbb{H}$, as f, from Theorem 3 we get the following:

Corollary 1. *For every conformal transformation $L_A(q) = (aq+b)(cq+d)^{-1}$, the function*

$$\frac{(c\gamma(q)+d)^{-1}}{|c\gamma(q)+d|^2} L'_{\gamma(A)}(q),$$

is regular on $\{q \in \mathbb{H} \mid c\gamma(q) + d \neq 0\}$.

4.1 The quadric energy of rotated regular functions

A unit quaternion d defines the *three–dimensional rotation* $q \mapsto rot_d(q) := dqd^{-1}$, which gives rise to the function (cf. Example 13)

$$f^A(q) = d^{-1} f(\gamma(d)^{-1} q \gamma(d)),$$

where A is the scalar matrix $A = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$. Taking $d = \gamma(a)^{-1}$ and multiplying by $\gamma(a)^{-1}$ on the right, we obtain the function $f^a = rot_{\gamma(a)} \circ f \circ rot_a$. From Theorem 3 we immediately get:

Corollary 2. *Let $f \in C^1(\Omega)$ and let $a \in \mathbb{H}$, $a \neq 0$. Let $rot_a(q) = aqa^{-1}$ be the three–dimensional rotation of $\mathbb{H}$ defined by a. Then the function*

$$f^a = rot_{\gamma(a)} \circ f \circ rot_a$$

is regular on $\Omega^a = rot_a^{-1}(\Omega) = a^{-1}\Omega a$ if and only if f is regular on Ω.

Remark 5. *The rotated function f^a has the following properties:*

1. $(f^a)^b = f^{ab}$ *and* $(f+g)^a = f^a + g^a$.
2. $(f^a)^{a^{-1}} = f$.
3. $f^{-a} = f^a$.
4. *If* $b \in \mathbb{H}$, *then* $(fb)^a = f^a \, rot_{\gamma(a)}(b)$.

Now we analyze the action of rotations on the energy quadric. We obtain in this way a new proof of the preceding result and we get new holomorphicity properties of rotated regular functions.

Theorem 4. *Let $f \in C^1(\Omega)$ and let $a \in \mathbb{H}$, $a \neq 0$. Let $f^a = rot_{\gamma(a)} \circ f \circ rot_a$. Then the energy density of f^a is $\mathcal{E}(f^a) = \mathcal{E}(f) \circ rot_a$ and the matrix function $M(f)$ defined in Section 3 transforms in the following way*

$$M(f^a) = Q_a (M(f) \circ rot_a) Q_a^T,$$

where Q_a is the orthogonal matrix in $SO(3)$ associated to the rotation $rot_{\gamma(a)}$ of the space $\langle i, j, k \rangle$.

Before coming to the theorem, we prove a simple result about holomorphicity of rotations.

Lemma 1. *For every $p \in \mathbb{S}^2$, the three-dimensional rotation $rot_a(q) = aqa^{-1}$ is a holomorphic map from $(\mathbb{H}, J_{\gamma(p)})$ to $(\mathbb{H}, L_{rot_a(p)})$.*

Proof. Let $\mathcal{B} = \{p, p', pp'\}$ be a positive orthonormal base of $\mathbb{R}^3 = \langle i, j, k \rangle$. Let $X_p = (p_1, p_2, p_3)$, $X_{p'} = (p'_1, p'_2, p'_3)$, $X_r = (r_1, r_2, r_3)$, with $r = pp' = r_1 i + r_2 j + r_3 k$. Given the transition matrix A

with columns $X_p, X_{p'}, X_r$, the coordinates x'_α $(\alpha = 1, 2, 3)$ of $q = x_0 + x_1 i + x_2 j + x_3 k$ w.r.t. $\mathcal{B}$ are given by the product $(x'_1\ x'_2\ x'_3)^T = A^T(x_1\ x_2\ x_3)^T$. Then

$$x'_1 = \sum_\alpha p_\alpha x_\alpha, \quad x'_2 = \sum_\alpha p'_\alpha x_\alpha, \quad x'_3 = \sum_\alpha r_\alpha x_\alpha.$$

>From this we get that the functions $g_1 = x_0 + x'_1 rot_a(p)$ and $g_2 = x'_2 + x'_3 rot_a(p)$ are holomorphic from $(\mathbb{H}, J_{\gamma(p)})$ to $(\mathbb{H}, L_{rot_a(p)})$, since

$$J_{\gamma(p)}(dx_0) = (p_1 J_1 + p_2 J_2 - p_3 J_3)(dx_0) = -\sum_\alpha p_\alpha dx_\alpha = -dx'_1$$

and

$$\begin{aligned} J_{\gamma(p)}(dx'_2) &= \sum_\alpha p'_\alpha (p_1 J_1 + p_2 J_2 - p_3 J_3)(dx_\alpha) \\ &= \sum_\alpha p_\alpha p'_\alpha dx_0 - (p_2 p'_3 - p_3 p'_2) dx_1 - (p_3 p'_1 - p_1 p'_3) dx_2 - (p_1 p'_2 - p_2 p'_1) dx_3 \\ &= -r_1 dx_1 - r_2 dx_2 - r_3 dx_3 = -dx'_3. \end{aligned}$$

The lemma now follows from the equality

$$\begin{aligned} rot_a(q) &= a(x_0 + x'_1 p + x'_2 p' + x'_3 r) a^{-1} \\ &= (x_0 + x'_1 rot_a(p)) + (x'_2 + x'_3 rot_a(p)) rot_a(p') = g_1 + g_2\, rot_a(p') \end{aligned}$$

□

If in the preceding lemma p is replaced by $\gamma(p)$, we get that the map $rot_a(q)$ is holomorphic also from $(\mathbb{H}, J_p)$ to $(\mathbb{H}, L_{rot_a(\gamma(p))}) = (\mathbb{H}, J_{p'})$, where $p' = \gamma(rot_a(\gamma(p))) = \gamma(a)^{-1} p \gamma(a) = rot^{-1}_{\gamma(a)}(p)$. Replacing a with $\gamma(a)$ we also get that $rot_{\gamma(a)}$ is holomorphic from $(\mathbb{H}, L_{p'}) = (\mathbb{H}, J_{rot_a(\gamma(p))})$ to $(\mathbb{H}, L_{rot_{\gamma(a)}(p')}) = (\mathbb{H}, L_p)$. Then we can draw a commutative diagram with holomorphic vertical maps

$$\begin{array}{ccc} (\mathbb{H}, J_{p'}) & \xrightarrow{\ f\ } & (\mathbb{H}, L_{p'}) \\ {\scriptstyle rot_a}\uparrow & & \downarrow{\scriptstyle rot_{\gamma(a)}} \\ (\mathbb{H}, J_p) & \xrightarrow[f^a]{} & (\mathbb{H}, L_p) \end{array} \tag{2}$$

Proof of Theorem 4. Let J be the real Jacobian matrix of $f \circ rot_a$. Then the real Jacobian matrix of f^a is the product $Q_a J$. It follows that $\mathcal{E}(f^a) = \frac{1}{2}\operatorname{tr}(Q_a J J^T Q_a^T) = \frac{1}{2}\operatorname{tr}(J J^T) = \mathcal{E}(f \circ rot_a)$. A similar computation gives $\mathcal{E}(f \circ rot_a) = \mathcal{E}(f) \circ rot_a$.

For the second statement of the theorem, it is sufficient to prove the equality

$$A(f^a) = Q_a (A(f) \circ rot_a) Q_a^T, \tag{3}$$

for the matrix functions $A(f)$ and $A(f^a)$ defined in Section 3, since then the matrices $A(f^a)$ and $A(f)\circ rot_a$ have the same trace and therefore

$$\begin{aligned} Q_a(M(f)\circ rot_a)Q_a^T &= \frac{1}{2}\left(\operatorname{tr} A(f)\circ rot_a\right) I_3 - \frac{1}{2}A(f^a) \\ &= \frac{1}{2}\left(\operatorname{tr} A(f^a) I_3 - A(f^a)\right) = M(f^a). \end{aligned}$$

It remains to prove (3). Let $p = p_1 i + p_2 j + p_3 k \in \mathbb{S}^2$ and $p' = rot_{\gamma(a)}^{-1}(p)$. Let us define the *p–holomorphic energy* of f

$$\mathcal{I}_p(f) = \frac{1}{2}\|df + L_p \circ df \circ J_p\|^2 = \frac{1}{2}\|df + p\, df \circ J_p\|^2 = 2\|\overline{\partial}_p f\|^2.$$

Then we obtain, as in [3],

$$\mathcal{E}(f) + \langle J_p, f^* L_p\rangle = \frac{1}{4}\mathcal{I}_p(f).$$

If $X = (p_1, p_2, p_3)$, then

$$\begin{aligned} XA(f^a)X^T &= \sum_{\alpha,\beta} p_\alpha p_\beta a_{\alpha\beta} = -\langle \sum_\alpha p_\alpha J_\alpha, (f^a)^* \sum_\beta p_\beta L_{i_\beta}\rangle \\ &= -\langle J_p, (f^a)^* L_p\rangle = \mathcal{E}(f^a) - \frac{1}{4}\mathcal{I}_p(f^a). \end{aligned}$$

Now let $X' = (p'_1, p'_2, p'_3) = XQ_a$. A similar computation gives

$$XQ_a A(f\circ rot_a)Q_a^T X^T = X' A(f\circ rot_a)X'^T = \mathcal{E}(f)\circ rot_a - \frac{1}{4}\mathcal{I}_{p'}(f)\circ rot_a.$$

>From the first statement of the theorem and the arbitrariness of X, equation (3) is equivalent to the equality, for any $p \in \mathbb{S}^2$, of the holomorphic energies

$$\mathcal{I}_{p'}(f)\circ rot_a = \mathcal{I}_p(f^a). \tag{4}$$

>From Lemma 1 (cf. diagram (2)) and rotational invariance of the norm we get

$$\begin{aligned} 2\mathcal{I}_p(f^a) &= \|df^a + L_p \circ df^a \circ J_p\|^2 \\ &= \|rot_{\gamma(a)} \circ df \circ drot_a + L_p \circ rot_{\gamma(a)} \circ df \circ drot_a \circ J_p\|^2 \\ &= \|rot_{\gamma(a)} \circ df \circ drot_a + rot_{\gamma(a)} \circ L_{p'} \circ df \circ J_{p'} \circ drot_a\|^2 \\ &= \|df + L_{p'} \circ df \circ J_{p'}\|^2 \circ rot_a = 2\mathcal{I}_{p'}(f)\circ rot_a. \end{aligned}$$

Then the equality (4) is true and the theorem is proved. □

Corollary 3. *Let $f \in C^1(\Omega)$ and let $a \in \mathbb{H}$, $a \neq 0$. Let $f^a = rot_{\gamma(a)} \circ f \circ rot_a$. Let $Q_a \in SO(3)$ be associated to the rotation $rot_{\gamma(a)}$ of the space $\langle i, j, k\rangle$. Then*

1. *f is regular on Ω if and only if f^a is regular on $\Omega^a = rot_a^{-1}(\Omega) = a^{-1}\Omega a$.*

2. f^a *is* J_p*–holomorphic if and only if* f *is* $J_{p'}$*–holomorphic, with* $p' = rot^{-1}_{\gamma(a)}(p)$.

3. If $f \in C^1(\overline{\Omega})$*, then (cf. Theorem 1)*

$$M_{\Omega^a}(f^a) = Q_a M_\Omega(f) Q_a^T.$$

Proof. 1) From Theorem 4 we get that $\operatorname{tr} M(f^a) = \operatorname{tr} M(f) \circ rot_a$ and $\mathcal{E}(f^a) = \mathcal{E}(f) \circ rot_a$. The first statement follows from Theorem 2, which tells that f is regular iff $\mathcal{E}(f) = \operatorname{tr} M(f)$.

2) It is an immediate consequence of equality (4), since a function is J_p–holomorphic iff its p–holomorphic energy vanishes.

3) It follows easily by integration of $M(f^a)$ on Ω^a. □

Corollary 4. *For every* $f \in \mathcal{R}(\Omega)$*, there exists* $a \in \mathbb{H}$*,* $a \neq 0$*, such that the matrices* $M(f^a)$ *and* $M_{\Omega^a}(f^a)$ *are diagonal, with non–negative entries.*

Proof. It follows immediately from Theorems 4 and 2, since when f is regular $M(f)$ is symmetric and positive semidefinite. □

Remark 6. *For a general conformal transformation* L_A*, the energy and,* a fortiori*, the energy quadric of a regular function is not conserved. For example, the constant function* 1 *has zero energy, while* $\mathcal{E}(2\pi^2 G) \neq 0$ *and* $1^{inv} = 2\pi^2 G$ *(cf. Example 10).*

The same happens for J_p*–holomorphicity. For example, the identity function is in the spaces* $Hol_i(\mathbb{H})$ *and* $Hol_j(\mathbb{H})$*, while*

$$id^{inv}(q) = \frac{\gamma(q)^{-1} q^{-1}}{|q|^2} \in \mathcal{R}(\mathbb{H} \setminus \{0\})$$

is not holomorphic w.r.t. any structure J_p*. This can be seen by computing the energy quadric* $M(id^{inv})$*. Since* $\det M(id^{inv}) = 32/|q|^{30}$ *is always non–zero, it follows from Theorem 2 that* id^{inv} *is not* J_p*–holomorphic, for any* p *(even non–constant). The rank of* id^{inv} *is three, because its image is contained in the space* $\langle 1, i, j \rangle$*, and the function can not have rank less than three, otherwise its quadric energy would have zero determinant (cf. [15] Theorem 7).*

A simpler example is given again by the function 1^{inv}*, since the energy quadric of the kernel* G *is* $M(G) = 2/|q|^8 I_3$*.*

4.2 Biregular rotations

If in Theorem 4 and its corollaries we take as f the identity map we get the following:

Proposition 2. *For every* $a \in \mathbb{H}$*,* $a \neq 0$*, the three–dimensional rotation* $rot_{\gamma(a)a}$ *is a biregular function on* $\mathbb{H}$*, with energy quadric* $M(rot_{\gamma(a)a})$ *of rank 1. This means that* $rot_{\gamma(a)a}$ *is holomorphic w.r.t. a circle of structures* $p \in \mathbb{S}^2$*. More precisely,* $rot_{\gamma(a)a} \in Hol_p(\mathbb{H})$ *for every* $p \in \langle rot_{\gamma(a)}(i), rot_{\gamma(a)}(j) \rangle \cap \mathbb{S}^2$*.*

Proof. We have $rot_{\gamma(a)a} = id^a$ (cf. Theorem 4). Then

$$M(rot_{\gamma(a)a}) = Q_a M(id) Q_a^T = Q_a \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} Q_a^T$$

has rank 1. Its kernel gives the structures with respect to which the rotation is holomorphic. From Corollary 3(2), these structures are generated by $rot_{\gamma(a)}(i)$ and $rot_{\gamma(a)}(j)$, since $id \in Hol_i(\mathbb{H}) \cap Hol_j(\mathbb{H})$.

Biregularity follows from $(\gamma(a)a)^{-1} = a^{-1}\gamma(a^{-1})$, which implies the equality $(id^a)^{-1} = id^{\gamma(a^{-1})} \in \mathcal{R}(\mathbb{H})$. □

Remark 7. *Not every rotation is a regular function, since the quaternion $\gamma(a)a$ is a reduced quaternion, with fourth component zero. These quaternion numbers correspond to rotations of $\mathbb{R}^3 = \langle i, j, k \rangle$ with axis orthogonal to the k axis. However, every quaternion is the product of two reduced quaternions and the map $a \mapsto \gamma(a)a$ is surjective from $\mathbb{H}$ to the space $\mathbb{H}_r$ of reduced quaternions.*

The surjectivity of $a \mapsto \gamma(a)a$ can be seen explicitly, or can be deduced from a property of the regular function id^{inv} (cf. Remark 6). Its restriction to the unit sphere S^3 is the map $q \mapsto \gamma(\bar{q})\bar{q} \in S^3 \cap \mathbb{H}_r$. It is surjective since id^{inv} has rank three.

Corollary 5.
1. *The left–multiplication map $l_{a'}(q) = a'q$ is biregular for every reduced quaternion $a' = \gamma(a)a \neq 0$.*
2. *Every three-dimensional rotation is the composition of two three-dimensional biregular rotations.*
3. *Every four-dimensional rotation is the composition of two biregular rotations.*

Proof. 1) $l_{a'}(q) = \gamma(a)aq = rot_{\gamma(a)a}(q)(a^{-1}\gamma(a)^{-1})$ has the same regularity and holomorphicity properties of $rot_{\gamma(a)a}$, since $\mathcal{R}(\Omega)$ is a right $\mathbb{H}$–module for every Ω and $\overline{\partial}_p(fb) = (\overline{\partial}_p f)b$ for every f and every $b \in \mathbb{H}$.

2) It follows from what has been said in the above remark: if $c = a'b'$, with $a' = \gamma(a)a$, $b' = \gamma(b)b \in \mathbb{H}_r$, then $rot_c = rot_{a'} \circ rot_{b'} = rot_{\gamma(a)a} \circ rot_{\gamma(b)b}$.

3) A four–dimensional rotation $rot_{c,d}(q) = cqd^{-1}$, with $|cd^{-1}| = 1$, can be decomposed as

$$rot_{c,d}(q) = cqc^{-1}\,(cd^{-1}) = rot_c(q)\,(cd^{-1}) = (rot_{a'} \circ rot_{b'})(q)\,(cd^{-1}),$$

where $c = a'b'$ as before. Let $f(q) = rot_{a'}(q)\,(cd^{-1}) \in \mathcal{BR}(\mathbb{H})$. Then $rot_{c,d} = f \circ rot_{b'}$. □

The pair of biregular functions in the corollary can be chosen in the same space $Hol_p(\mathbb{H})$. This comes from Proposition 2, because the two great circles of complex structures in $\mathbb{S}^2$ coincide or intersect in two antipodal points defining a space $Hol_p(\mathbb{H})$. Note that this space is not closed under composition, unless $J_p = L_p$, which happens only when $p = \gamma(p)$ is a reduced quaternion.

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On a new notion of holomorphy and its applications

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ABSTRACT

This paper devotes a new general notion of holomorphy which works in the continous and discrete cases. With the help of methods of a general operator theory the so called L-holomorphy is introduced. Realizations of this calculus follow. New versions of Taylor- and Taylor–Gontcharov formulae are deduced. The results are applied for the solution of higher order systems of differential equations.

RESUMEN

Este artículo es dedicado a una nueva noción de holomorfía la cual funciona en los casos continuo y discreto. Con la ayuda de métodos de la teoría general de operadores la llamada L-holomorfia es presentada. Realizaciones de este cálculo siguen. Nuevas versiones de fórmulas de Taylor-y Taylor-Gontcharov son deducidas. Los resultados son aplicados para la solución de sistemas de orden superior de ecuaciones diferenciales.

Key words and phrases: *Generalized holomorphic functions, Taylor-Gontcharov formulae, Plemelj projections, higher order boundary value problems.*

1 Introduction.

The aim of this article is to introduce a very general notion of holomorphy by the help of three general operators in Banach spaces which have to satisfy some conditions. This introduction is oriented at the theory of right invertible operators. We refer to the well-known book of V.S. Ryabenskij [11] (1987), W. Schempp and F.J. Delvos [2] (1990) and the article by M. Tasche [16] (1981). The advantage of our approach is the fact that holomorphy can be considered in the continuous and discrete case within one calculus. We continue the line of action we have followed in books [6],[7],[5]. In the second part we present a large number of realisations. Here we use above all results of the common research with K. Gürlebeck confer again in [6], [7] and [4]. Finally, some classes of boundary value problems of higher order will be considered. In that connection new formulae of Taylor- and Taylor-Gontcharov type are obtained. All our considerations take place in the scale of Sobolev and Besov spaces as well as its discrete analogue.

2 A general holomorphy

Let X, Y, Z be Banach spaces. We introduce the bounded linear operators T, Tr and P with the following properties

(i) $T : X \to \operatorname{im} T \subset Y$ is injective.

(ii) $Tr : Y \to Z$ is a generalized trace operator .

(iii) The operator $P : \operatorname{im} Tr \cap Y \to Y$ satisfies the property $PTrPu = Pu$.

Furthermore, we assume

(i) $\operatorname{im} Tr\, T \subset \ker P$,

(ii) $\operatorname{im} T \cap \ker Tr = \{0\}$.

Remark 1. We also have $\operatorname{im} T \cap \operatorname{im} P = \{0\}$. Indeed, let $u \in \operatorname{im} T \cap \operatorname{im} P = \{0\}$ then $u = Pw = Tv$ and

$$u = Pw = PTrPw = PTrTv = 0.$$

Theorem 1. (Mean value formula) *Set* $\operatorname{im} T \oplus \operatorname{im} P =: Y_1 \subset Y$. *There is a unique linear operator* L *with* $\mathcal{D}(L) = Y_1$ *and* $L : \mathcal{D}(L) \to X$, *such that*

$$u = PTru + TLu.$$

Proof. Let $u \in \mathcal{D}(L)$. Then u permits the representation

$$u = Pv + Tw,$$

with $v \in \mathrm{im}Tr \cap Y$, and $w \in X$. Applying PTr from the left it follows

$$PTr\,u = PTrP\,v + PTrT\,w = P\,v.$$

In this way the first item of the desired formula is obtained. In order also to obtain the second item we have to use the injectivity of the operator T. On the linear set $\mathrm{im}\,T$ there exists a linear operator $\tilde{L}$ with

$$\tilde{L}Tw = w.$$

The operator $\tilde{L}$ can be extended to an linear operator L on Y_1 setting

$$Lz := \tilde{L}z_1,$$

where $z = z_1 + z_2$ with $z_1 \in \mathrm{im}\,T$ and $z_2 \in \mathrm{im}\,P$. The additivity follows from

$$L(z + z') = L(z_1 + z_2 + z_1' + z_2') = \tilde{L}(z_1 + z_1') = \tilde{L}z_1 + \tilde{L}z_1' = Lz + Lz'.$$

The monogeneity with a real constant λ is also fulfilled. Indeed, we have

$$L(\lambda z) = \tilde{L}(\lambda z_1) = \lambda\tilde{L}z_1 = \lambda Lz.$$

Now we obtain easily $Lu = LPTr\,u + LT\,w = w$ and our decomposition formula is completely proved. The uniqueness follows from

$$TLu - TL_1u = 0 \quad \text{leads to} \quad Lu = L_1u,$$

where L_1 is another linear operator which has to fulfil the decomposition formula. #

Corollary 1. *The following relations between the operators L, P and T are valid:*

(i) The operator L is the left-inverse to the operator T, i.e. $LT = I$.

(ii) Set $R := TL$ then R is a projection onto Y_1 with $\mathrm{im}R = \mathrm{im}T$.

(iii) It holds $\ker L = \mathrm{im}PTr$.

Proof. The relation (i) follows by the definition of L. Indeed, let $v \in X$, then

$$LTv = \tilde{L}Tv = v.$$

(ii) Obviously, TL fulfils the idempotential property and so we have $R^2 = R$. It is immediately clear that

$$\mathrm{im}\,R \subset \mathrm{im}\,T.$$

Conversely, let $v \in \mathrm{im}\,T$ then $v = Tw$ and

$$Rv = RTw = TLTw = Tw = v,$$

i.e. $\operatorname{im} T \subset \operatorname{im} R$. To prove the relation (iii) we have to argue as follows: Let $u \in \ker L$, then

$$u = PTr\, u + TLu = PTr\, u \in \operatorname{im} PTr.$$

On the other hand it follows from $u \in \operatorname{im} PTr$ that $u = PTr\, v$ with $v \in Y$ and

$$u = PTr\, v = PTrPTr\, v + TLu = PTr\, v + TLu,$$

which leads to $TLu = 0$ and because of the injectivity of the operator $T : X \to \operatorname{im} T$ we conclude $Lu = 0$, i.e. $u \in \ker L$. #

Definition 1. *Elements* $u \in \ker L \cap Y$ *are called* L-holomorphic. *The operator* L *is called* algebraic derivative. *The operator* PTr *is called the* initial projection *and the operator* T *is denoted as* general Teodorescu transform. *From the point of view of a general operator theory* T *is also called* algebraical integral.

Corollary 2. *Set* $P_r := TrP : \operatorname{im}Tr \cap Y \to Z$ *and* $Q_r := I - P_r$. *The following properties are valid:*

(i) The operators P_r, Q_r *are idempotent, i.e. we have* $P_r^2 = P_r$ *and* $Q_r^2 = Q_r$ *and furthermore* $Q_rP_r = P_rQ_r = 0$.

(ii) An element $\xi \in Z$ *is the generalized trace of an element* u *from* $\ker L$ *if and only if* $P_r\xi = \xi$.

(iii) We have $Q_r\xi = TrTL\, u$.

Proof. (i). It is sufficient to show

$$P_r^2\xi = TrPTrP\xi = TrP\xi = P_r\xi,$$

with $\xi \in Z$. In order to prove (ii) let $\xi = Tr\, u \in Z$ and $u \in \ker L$. Then we have

$$u = PTr\, u + TL\, u = PTr\, u = P\xi.$$

It now follows $\xi = Tr\, u = TrP\, \xi = P_r\xi$. Conversely, let us assume $\xi = P_r\xi$, then

$$Tr\, u = \xi = P_r\xi = TrP\xi = TrPTr\, u.$$

On the other hand Theorem 1 yields

$$Tr\, u = TrPTr\, u + TrTLu.$$

Hence $TrTL\, u = 0$. Because of $\operatorname{im}T \cap \ker Tr = \{0\}$ follows $TLu = 0$ and such $Lu = 0$, i.e. $u \in \ker L$. For (iii) we have

$$Tr\, u = TrPTr\, u + TrTL\, u.$$

Therfore, it holds

$$TrTL\, u = Tr\, u - TrPTr\, u = \xi - TrP\, \xi = \xi - P_r\xi = Q_r\xi. \quad \#$$

Denotation The operators P_r, Q_r are called general Plemelj projections.

Remark 2. The condition im$TL \cap$ ker$Tr = \{0\}$ can be seen as a very general formulation of a maximum principle.

3 Types of L–holomorphy

3.1 L-holomorphy in $\mathbb{R}^1$

A trivial example is given by consideration of all functions $u \in C^1[0,1]$ with

$$L := \frac{d}{dt}\ ,\ T := \int_0^t \cdot d\tau\ ,$$

$P := I$ and $Tr : C^1[0,1] \to \mathbb{R}^1$ with $Tr\, u = u(0)$. Then we get the well-known mean-value theorem:

$$u(t) = u(0) + \int_0^t \dot{u}(\tau)d\tau = PTr\, u + TLu.$$

This is just the main-theorem of differential-integral calculus. The class of all L-holomorphic functions consist of all real constants.

Also a slightly modification of the trace operator and the generalized Teodorescu transfrom does not change the triviality of the class of L-holomorphic functions. Indeed, let $u \in C^1[0,1]$, take $L := \frac{d}{dt}$, $P := I$ and $Tr\, u := \frac{1}{2}[u(0) + u(1)]$, then

$$(Tu)(t) := \int_0^t u(\tau)d\tau - \frac{1}{2}\int_0^1 u(\tau)d\tau\ .$$

Because of im$PTr = \ker L$ we have again the space of all constants for the class of L-holomorphic functions.

By using the so-called Riemann-Liouville integral of order α (cf. [14],[9]) we obtain a more interesting example. For this reason let $u \in C[0,1],\ 0 < \alpha < 1$. We consider the absolut continuous function

$$(I^{\alpha}_{a+}u)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t \frac{1}{(t-\tau)^{1-\alpha}}u(\tau)d\tau\ ,$$

which has almost everywhere a derivative in $L_1[0,1]$. Take now

$$(Lu)(t) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}(I^{1-\alpha}_{a+}u)(t)\ ,\ (Tu)(t) := (I^{\alpha}_{a+}u)(t)$$

and with $n = [\alpha] + 1$

$$(PTru)(t) := \sum_{k=0}^{n-1} \frac{(t-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \frac{d^{n-k-1}}{dt^{n-k-1}} I_{a+}^{n-\alpha} u(t).$$

$(I_{a+}^{\alpha} u)(t)$ is called Riemann-Liouville fractional integral and $(D_{a+}^{\alpha} u)(t)$ is denoted by Riemann–Liouville fractional derivative. The main-value theorem holds again.

3.2 Notions of holomorphy in the complex plane

The original notion of the holomorphy forms in natural way a class of L-holomorphic function. We have only to set

$$L := \partial_{\overline{z}} \; .$$

In more detailed we have the following: Let $G \subset \mathbb{C}$ be a bounded domain with sufficient smooth boundary curve then the mean-value formula is written as

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{u(t)}{t-z} \, dt - \frac{1}{2\pi i} \int_{G} \frac{1}{t-z} (\overline{\partial} u)(t) d\xi \, d\eta = \begin{cases} u(z) & , \quad z \in G \\ 0 & , \quad z \in \mathbb{C} \setminus \overline{G} \end{cases} .$$

We have only to identify

$$L := \overline{\partial} = \frac{1}{2} \left(\partial_{\xi} + i \partial_{\eta} \right) \qquad (t\xi + i\eta) \; ,$$
$$T := -\frac{1}{2\pi i} \int_{G} \frac{1}{t-z} \cdot d\xi \, d\eta \; , \qquad P := \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{t-z} \cdot d\Gamma_t \; .$$

The trace operator Tr is defined as non-tangential limit from inner points tending to the boundary Γ.

Remark 3. It is quite curious that the initial projection acts on the boundary. It seems that "initial values" are "smudged" over the surface.

Another example in the complex plane can be given by

$$L := \overline{\partial} \;\; , \; (T\cdot)(z) = -\frac{1}{2\pi i} \int_{G} \left[\frac{1}{t-z} - \frac{1}{t+z} \right] \cdot d\xi \, d\eta$$

and $$(P\cdot)(z) = -\frac{1}{2\pi i} \int_{\Gamma} \left[\frac{1}{t-z} - \frac{1}{t+z} \right] \cdot d\Gamma_t \; .$$

The trace operator is definded as before. This model goes back to J. Ryan (cf. [8]).

3.3 L-holomorphy models generated by matrices

A further model for L-holomorphy is given by: Let $\{E_i\}_{i=1}^n$ be a family of orthogonal matrices of order n with entries $0, 1, -1$ as well as the property

$$E_i^* E_j + E_j^* E_i = 0 \qquad (i \neq j)$$

Furthermore, set $E(a) = \sum_{i=1}^{n} E_i a_i$, $a = (a_1, ..., a_n)^T$ and $E^*(a) = \sum_{i=1}^{n} E_i^* a_i$ and $\nabla = (\partial_1, ..., \partial_n)^T$.

Take $L := D(\nabla)$, $T := \frac{1}{\sigma_n} \int\limits_G \frac{D^*(y-x)}{|y-x|^n} \cdot dy$ and $P := \frac{-1}{\sigma_n} \int\limits_\Gamma \frac{D^*(y-x)}{|y-x|^n} \cdot d\Gamma_y$ then it holds

$$(Pu)(x) + TL(\nabla)u(x) = \begin{cases} u(x) & , \quad x \in G \\ 0 & , \quad x \in \mathbb{R}^k \setminus \overline{G} \end{cases} .$$

Here σ_n denotes the area of the n-dimensional unit sphere. (cf. [13],[15]).

3.4 Dzuraev's model

Also Dzuraev's model from 1982 [3] is worthy of being mentioned:

Let $u := (u_1, u_2)$, $z = x_2 + ix_3$ and $\frac{\partial}{\partial \overline{z}} := \frac{1}{2}(\frac{\partial}{\partial x_2} + i\frac{\partial}{\partial x_3})$, $y = y_2 + iy_3$. Further, let

$$\overline{\partial}_x = \begin{pmatrix} \frac{\partial}{\partial x_1} & 2\,\frac{\partial}{\partial z} \\ -2\,\frac{\partial}{\partial \overline{z}} & \frac{\partial}{\partial x_1} \end{pmatrix} , \qquad E(y-x) = \frac{-1}{|y-x|^3} \begin{pmatrix} y_1 - x_1 & -(\overline{y} - \overline{z}) \\ y - z & y_1 - x_1 \end{pmatrix}$$

and $n(y) = \begin{pmatrix} n_1 & n_2 - in_3 \\ -(n_2 + in_3) & n_1 \end{pmatrix}$. Then take

$$L := \overline{\partial}_x \ , \ T := \frac{1}{\sigma_3} \int\limits_G E(y-x) \cdot dy \quad \text{and} \quad P := \frac{1}{\sigma_3} \int\limits_\Gamma E(y-x) n(y) \cdot d\Gamma_y \ .$$

The trace operator Tr means in both cases the non-tangential limit to the boundary Γ from inside of G.

4 Quaternionic holomorphic functions

Real Quaternions: The algebra of real quaternions $\mathbb{H}$ is defined by the basis elements

$e_0 = 1$, e_1, e_2, e_3,

which obey the arithmetic rules:

$$e_0^2 = 1 \ , \ e_1e_2 = -e_2e_1 = e_3 \ , \ e_2e_3 = -e_3e_2 = e_1 \ , \ e_3e_1 = -e_1e_3 = e_2 \ .$$

Each quaternion $a \in \mathbb{H}$ permits the representation

$$a = \sum_{k=0}^{3} a_k e_k \qquad (a_k \in \mathbb{R} \ ; \ k = 0,1,2,3) \ .$$

Addition and multiplication in $\mathbb{H}$ turn it into a non-commutative number field. The main-involution in $\mathbb{H}$ is called quaternionic conjugation and defined by

$$\overline{e_0} = e_0 \ , \ \overline{e_k} = -e_k \qquad (k = 1,2,3) \ .$$

which can be extended onto $\mathbb{H}$ by $\mathbb{R}$-linearity. Therefore we have

$$\overline{a} = a_0 - \sum_{k=1}^{3} a_k e_k = a_0 - \mathbf{a}.$$

Note that

$$a\overline{a} = \overline{a}a = \sum_{k=1}^{3} a_k^2 =: |a|_{\mathbb{H}}^2 \ .$$

If $a \in \mathbb{H} \setminus \{0\}$ then the quaternion

$$a^{-1} := \frac{\overline{a}}{|a|^2}$$

is the inverse to a. For $a, b \in \mathbb{H}$ we have $\overline{abba}$.

Complex quaternions: The set of complex quaternions, which we also need, is denoted by $\mathbb{H}(\mathbb{C})$ and consist of all elements of the form

$$a = \sum_{k=0}^{3} a_k e_k \qquad (a_k \in \mathbb{C} \ ; \ \ k = 0,1,2,3) \ .$$

By definition we state: $ie_k = e_k i, \ k = 0,1,2,3$. Here i denotes the usual imaginary unit in $\mathbb{C}$. Elements of $\mathbb{H}(\mathbb{C})$ can also be represented in the form

$$a = a^1 + \ ia^2 \qquad (a^k \in \mathbb{H}; \ \ k = 1,2).$$

Notice that the quaternionic conjugation acts only on the quaternionic units and not on the pure complex number i.

Let $X = W_p^k(G), Y = W_p^{k+1}(G), Z = W_p^{k-(1/p)+1}(\Gamma); k = 0, 1, 2, ...; 1 < p < \infty$. Further, let

$$
\begin{aligned}
&L := D = \sum_{i=1}^{3} \partial_i e_i \quad \text{(Dirac operator (mass zero))},\\
&(Tu)(x) := -\frac{1}{\sigma_3} \int_G e(x-y)u(y)dy \quad \text{(Teodorescu transform)},\\
&(Pu)(x) := (F_\Gamma u)(x) = \frac{1}{\sigma_3} \int_\Gamma e(x-y)n(y)u(y)d\Gamma_y \ \text{(Cauchy − Fueter operator)},\\
&(Tru)(\xi) := n.t. - \lim_{\substack{z \to \xi \in \Gamma \\ z \in G}} u(z),
\end{aligned}
$$

with $e(x) = D\frac{1}{|x|}$ and $n = \sum_{i=1}^{3} e_i n_i$ the outward pointing unit vector of the normal.

The class of L-holomorphic functions are just the solutions of the Mosil–Teodorescu system.

We now consider so called Dirac operators with mass. We will use the same spaces as above. Then the general operators L, T and P are given by

$$
\begin{aligned}
&L := D + i\alpha \quad \text{(Dirac operator with mass)},\\
&(Tu)(x) := -\frac{1}{\sigma_3} \int_G e_{i\alpha}(x-y)u(y)dy \quad \text{(Teodorescu type transform)},\\
&(Pu)(x) := \frac{1}{\sigma_3} \int_\Gamma e_{i\alpha}(x-y)n(y)u(y)d\Gamma_y \ \text{(Cauchy − Fueter − typeoperator)},\\
&(Tru)(\xi) := n.t. - \lim_{\substack{z \to \xi \in \Gamma \\ z \in G}} u(z).
\end{aligned}
$$

For the description of the kernel function of this new Teodorescu transform we have to use Bessel-functions of third kind so called MacDonald functions. We have

$$
e_{i\alpha(x)} := -\left(\frac{i\alpha}{2\pi}\right)^{(3/2)} \left[|x|^{-1/2} K_{3/2}(i\alpha|x|)\omega - K_{1/2}(i\alpha|x|)\right],
$$

where $\omega \in S^2$ and $K_\mu(t)$ denotes.

5 Discrete quaternionic holomorphic functions

One advandage of our notion of L-holomorphy is its applicability also on lattices. We will present a calculus which was obtained by K. Guerlebeck in 1988 [4] (cf. also [6]). For this reason we have to represent the domain on the lattice and to define what are inner and outer points relatively to the "discrete boundary" and to say what the discrete boundary means. This boundary has to approximate the original domain. It is necessary to disdinguish between a right and a left parts of the boundary. The approximating discrete domain is here always an axes–parallel polyeder with side faces, edges and corner points. More exactly holds

$$\mathbb{R}^3_h := \{(ih, jh, kh) : i, j, k \; integer, \; h > 0\}, \quad G_h := G \cap \mathbb{R}^3_h,$$
$$\Gamma_h := \{x \in G_h : \operatorname{dist}(x, \operatorname{co}G_h) \leq \sqrt{3}h\}.$$

Let $V^{\pm}_{i,h}x$ the translation of x by $\pm h$ in x_i-direction, then

$$\Gamma_{h,\ell(r)} := \{x \in \Gamma_h : \exists i : V^{\pm}_{i,h}x \notin G_h\} \quad \text{(left(right) side planes)},$$
$$\Gamma_{h,\ell(r);i} := \{x \in \Gamma_h : V^{\pm}_{i,h}x \notin G_h\},$$
$$\Gamma_{h,\ell(r);i,j} := \Gamma_{h,\ell(r);i} \cap \Gamma_{h,\ell(r);j} \quad \text{(left(right) edges)},$$
$$\Gamma_{h,\ell(r);i,j,k} := \Gamma_{h,\ell(r);i,j} \cap \Gamma_{h,\ell(r);k} \quad \text{(left(right) corners)}.$$

Let be $X = W^1_{2,h}(G_h)$, $Y = L_{2,h}(G_h)$, $Z = W^{\frac{1}{2}}_{2,h}(G_h)$. Then

$$(Lu)(x) := (D^{\pm}_h u)(x) = \sum_{i=1}^{3} e_i[u(V^{\pm}_{i,h}x) - u(x)]\frac{1}{h} \text{ (discr. Dirac operator)},$$
$$(Tu)(x) := (T^{\pm}_h u)(x) \quad \text{(discrete Teodorescu transform)}$$
$$= \left(\sum_{\operatorname{int}G_h \cup \Gamma_{h,\ell(r)}} + \sum_{left(right)\;corners} - \sum_{left(right)\;edges} \right) e^{\pm}_h (x-y)u(y)h^3,$$

where $e^{\pm}_h$ are the discrete fundamental solutions of $D^{\pm}_h$. The discrete Cauchy–Fueter operator is introduced as follows

$$(Pu)(x) := (F^{\pm}_h u)(x) = \sum_{i=1}^{3} \left(-\sum_{s_i} + \sum_{s_{ij}} - \sum_{s_{ijk}} \right) e^{\pm}_h (x - V^{\mp}_{i,h}y) n(y) u(y) h^2$$
$$+ \sum_{i=1}^{3} \sum_{\substack{y \in \Gamma_{h,\ell(r);m,j,k} \\ m \neq j \neq k}} h^{\pm}(x-y) e_i u(y) h^2,$$

where $s_i = \Gamma_{h,\ell;i} \cup \Gamma_{h,r;i}$, $s_{ij} := \Gamma_{h,\ell;j} - V^{+}_{i,h}\Gamma_{h,\ell}$, $\quad s_{ijk} := \Gamma_{h,\ell;j,k} - V^{+}_{i,h}\Gamma_{h,\ell;i,k}$.

The corresponding mean value formulae are given as follows

$$u(x) = (F^{\pm}_h u)(x) + T^{\pm}_h D^{\pm}_h u(x)$$

Much more complicated is to find a suitable discrete fundamental solution, which is given by $E_h(x)$ as solution of a suitable difference equation

$$-\Delta_h E_h(x) = -\sum_{i=1}^{3} D^{-}_{i,h} D^{+}_{i,h} E_h(x) = \delta_h(x) = \begin{cases} h^{-3}, x = 0 \\ 0, x \in \mathbb{R}^3_h \setminus \{0\} \end{cases}$$

expressed by using the Fourier-Transform we have

$$E_h(x) = \frac{1}{\sqrt{2\pi}^3} R_h F\left(\frac{1}{d^2}\right).$$

The function d is defined as follows

$$d^2 = \frac{4}{h^2}\left(\sin^2\frac{h\xi_1}{2} + \sin^2\frac{h\xi_2}{2} + \sin^2\frac{h\xi_3}{2}\right)$$

and $R_h u$ is the restriction of the continuous function u onto the lattice $\mathbb{R}^3_h$. We have $|E_h| \leq C|x|^m$ with a certain $m > 0$ depending on the properties of the difference operator

$$e_h^{\pm}(x) := D^{\mp}_{j,h} E_h(x).$$

6 L-holomorphy on the sphere

Meanwhile is also existing the notion of holomorphy on the sphere. A good reference is doctoral thesis of P. Van Lancker [17] The following operators has to be used $\Gamma_S + \alpha \quad \alpha \in \mathbb{C} \setminus \mathbb{N} \cup (-\mathbb{N})$.

$$\begin{aligned} L_\alpha : &= \omega(\Gamma_S + \alpha) && \text{(Günter's gradient)}, \\ T_\alpha : &= -\int_\Omega E_\alpha(\omega, \xi) \cdot dS(\omega) && \text{(Teodorescu transform)}, \\ P_{C,\alpha} : &= -\int_{-C} E_\alpha(\omega, \xi) n(\omega) \cdot dC(\omega) && \text{(Cauchy-Fueter type operator)}. \end{aligned}$$

A corresponding Borel-Pompeiu formula is given by

$$P_{C,\alpha} u + T_\alpha D_\alpha u = \begin{cases} u & \text{in} \quad \Omega \\ 0 & \text{in} \quad S \setminus \overline{\Omega} \end{cases}.$$

We will consider the fundamental solution of Günter's gradient. Let $\alpha \in \mathbb{C} \setminus \mathbb{N} \cup \{-2 - \mathbb{N}\}$. Then

$$E_\alpha(\omega, \xi) = \frac{\pi}{\sigma_3 \sin \pi\alpha} K_\alpha(-\xi, \omega)\omega,$$

where σ_3 is the surface area of the unit sphere. Further, we define

$$K_\alpha(-\xi, \omega)\omega = C^{3/2}_\alpha(\omega \cdot \xi) + \xi\omega C^{3/2}_{\alpha-1}(\omega \cdot \xi),$$

with the so-called Gegenbauer polynomials $C^\mu_\alpha(t)$.

Using Kummer's function ${}_2F_1(a, b; c; z)$ we get the representation

$$C^{3/2}_\alpha(z) = \frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)} \frac{1}{4}\, {}_2F_1(-\alpha, \alpha+3; 2; \frac{1-z}{z}) \quad z \in \mathbb{C} \setminus \{-\infty, 1\}.$$

Kummer's function is for $|z| < 1$ defined by

$$ {}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a_k)(a_k)}{(c)_k} \frac{z^k}{k!}, \qquad (a)_k \quad = \quad \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)}. $$

Solutions of $D_\alpha u = 0$ in Ω are called inner spherical holomorphic functions of order α in Ω. We have

$$ D_\alpha E_\alpha(\omega,\xi) = \delta(\xi - \omega)\ . $$

A good reference for this topic is [1]. Further we introduce a singular integral operator of Bitzadse's type

$$ \begin{aligned} (S_{C,\alpha}u)(\xi) : &= 2 \lim_{\varepsilon \to 0} \int\limits_{C \setminus B_\varepsilon(\xi)} E_\alpha(\omega,\xi) n(\omega) u(\omega) dS(\omega) \\ &= 2v.p. \int\limits_C E_\alpha(\omega,\xi) n(\omega) u(\omega) dS(\omega). \end{aligned} $$

One can prove the algebraical identity $S^2_{C,\alpha} = I$. Let $\Omega^+ := \Omega$, $\Omega^- :=$ coΩ. Applying the general trace operator as non-tangential limit on the sphere towards the boundary C we get Plemelj-Sokkotzkij-type formulae.

$$ n.t. - \lim_{\substack{t\to\xi \\ t\in\Omega^\pm}} (F_{C,\alpha}u)(t) \frac{1}{2}[\pm I + S_{C,\alpha}]u(\xi) =: \begin{cases} P_{C,\alpha}u(\xi), t \in \Omega^+ \\ -Q_{C,\alpha}u(\xi), t \in \Omega^- \end{cases} . $$

The operators

$$ Q_{C,\alpha} := \frac{1}{2}[I - S_{C,\alpha}], \qquad P_{C,\alpha} : \frac{1}{2}[I + S_{C,\alpha}] $$

are called Plemelj projections. The space $L_2(\Gamma)$ is now decomposed into the Hardy spaces

$$ \begin{array}{c} L_2(C) = HS^\alpha(\Omega^+) \oplus HS^\alpha(\Omega^-) \\ \qquad\quad \uparrow \qquad\qquad \uparrow \\ \qquad\quad P_{C,\alpha} \qquad Q_{C,\alpha} \end{array} $$

(cf. [12]).

7 Taylor type formula

Using ideas of the theory of right invertible operators (cf. D. Przeworska-Rolewicz, [10]) one has with $Y_m = \mathcal{D}(L^m) \subset Y$ (m is a natural number) the operators

$$ \begin{aligned} &L^j : Y_m \to X_{m-j}, \quad P : Z_{m-j} \to Y_{m-j}, \quad PTr : Y_{m-j} \to Y_{m-j}, \\ &T^j : X_{m-j} \to Y_m \quad (0 \le j \le m-1). \end{aligned} $$

Here we have $Y_m \subseteq \ldots \subseteq Y_2 \subseteq Y_1$ and $L^0 = T^0 = I$.

Proposition 1. *The following properties are fulfiled*

(i) The operators $T^jPTrL^j \quad (0 \le j \le m-1)$ *are projections on* Y_m.

(ii) The projections $T^jPTrL^j \quad (0 \le j \le m-1)$ *are complementary on* Y_m, *i.e.* $(T^jPTrL^j)(T^kPTrL^k) = (T^kPTrL^k)(T^jPTrL^j) = 0$ *for all* $0 \le j, k \le m-1$ *and* $k \ne j$.

Proof. (i) Indeed, using the assumption $PTrP = P$ and corollary 1 we obtain

$$(T^jPTrL^j)(T^jPTrL^j) = T^jPTrL^jT^jPTrL^j = T^jPTrPTrL^j = T^jPTrL^j,$$

i.e. T^jPTrL^j are projections on Y_m. To prove property (ii) we also use corollary 1. It is immediately clear that $L^jT^j = I$ from $LT = I$. Because of $PTrT = 0$ and $L^jT^j = I$ follows for $j < k$:

$$(T^jPTrL^j)(T^kPTrL^k) = T^jPTrL^jT^kPTrL^k = T^jPTrT^{k-j}PTrL^k = 0,$$

i.e.

$$(T^jPTrL^j)(T^kPTrL^k) = 0 \quad (0 \le j < k \le m).$$

Taking into account relation in the corollary from above, the commutative property is obtained. Indeed, from property $LPTr = 0$ we have

$$(T^kPTrL^k)(T^jPTrL^j) = T^kPTrL^kT^jPTrL^j = T^kPTrL^{k-j}PTrL^j = 0,$$

i.e.

$$(T^kPTrL^k)(T^jPTrL^j) = 0 \quad (0 \le j < k \le m).$$

Hence all $T^jPTrL^j (0 \le j \le m)$ are complementary on Y_m. #

Then the next corollary is clear.

Corollary 3. *The operator*

$$P_m := \sum_{j=0}^{m-1} T^jPTrL^j = T^0PTrL^0 \oplus T^1PTrL^1 \oplus \ldots \oplus T^{m-1}PTrL^{m-1}$$

is a projection on Y_{m-1}.

Corollary 4. *The operators* P_m, T^m *and* L^m *have the following relations*

(i) The operator T^m *is the right-inverse to the operator* L^m, *i.e.* $L^mT^m = I$.

(ii) The operators L^m, P_m *satisfy the property* $L^mP_m = 0$.

(iii) It holds $P_mT^m = 0$.

Proof. The relation (i) is simple to be obtained from corollary 1. To prove (ii), one use assumption $LPTr = 0$ and $L^jT^j = I$ for $0 \le j \le m-1$ as mentioned above then

$$L^mP_m = \sum_{j=0}^{m-1} L^mT^jPTrL^j = \sum_{j=0}^{m-1} L^{m-j}PTrL^j = 0.$$

The same for relation (iii) with assumption $PTrT = 0$:

$$P_mT^m = P_m := \sum_{j=0}^{m-1} T^jPTrL^jT^m = P_m := \sum_{j=0}^{m-1} T^jPTrT^{m-j} = 0.$$

Theorem 2. (The Taylor type formula) *Let L be a right invertible operator that defined from an injection T and an initial operator P. Then for $m = 1, 2, ...$ the following identity holds on Y_m*

$$u = \sum_{j=0}^{m-1} T^jPTrL^ju + T^mL^mu.$$

Proof. We have $\ker T^m = \{0\}$ by assumption T is an injection and $\operatorname{im} T^m \subset Y_m = \mathcal{D}(L^m)$. Corollary 3 shows that P_m is a projection and $P_mT^m = 0$. Furthermore, it is simple to show that $\operatorname{im} T^m \cap \operatorname{im} P_m = \{0\}$. Indeed, let $u \in \operatorname{im} T^m \cap \operatorname{im} P_m$ then

$$u = P_mv = T^mw, \quad (v \in Y_{m-1}, w \in X).$$

Since $P_mT^m = 0$ we get

$$u = P_mv = P_mP_mv = P_mT^mw = 0.$$

Let B be the (unique) right inverse to T^m then (from the mean value formula)

$$u = P_mu + T^mBu \quad \text{with} \quad \mathcal{D}(B) := \operatorname{im}T^m \oplus \operatorname{im}P_m.$$

Now we will show that L^m also satisfies above formula. By applying the mean value formula for L^ju we get

$$L^ju = PTrL^ju + TL^{j+1}u \quad (0 \le j \le m-1)$$

Rewrite in more detail and acting operators $T^j \quad (0 \le j \le m-1)$ to both sides we have

$$\begin{aligned} T^0L^0u &= T^0PTrL^0u + TLu, \\ TLu &= TPTrLu + T^2L^2u, \\ &\dots \\ T^{m-1}L^{m-1}u &= T^{m-1}PTrL^{m-1}u + T^mL^mu. \end{aligned}$$

Sum up all equabilities we obtain

$$\begin{aligned} u = T^0 L^0 u &= T^0 P Tr L^0 u + T P Tr L u + \ldots + T^{m-1} P Tr L^{m-1} u + T^m L^m u \\ &= P_m u + T^m L^m u. \end{aligned}$$

Then the property of uniqueness of right inverse operator leads to

$$B = L^m.$$

This completes the proof of our theorem.

Example 15. (Realisation in R^1) We continue the first example in section 3.1.For all functions $u \in C^1[0,1]$, recall that

$$L := \frac{d}{dt}\ ,\ T := \int_0^t \cdot d\tau\ ,$$

$P := I$ and $Tr : C^1[0,1] \to R^1$ with $Tr\, u = u(0)$. Then we have

$$T^j P Tr (L^j u)(t) = (L^j u)(0) \frac{t^j}{j!}$$

and

$$(T^m u)(t) = \int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!} u(\tau) d\tau\ .$$

Hence the theorem 2 yields the classical Taylor's formula

$$u(t) = \sum_{j=0}^{m-1} (L^j u)(0) \frac{t^j}{j!} + \int_0^t \frac{(t-\tau)^{m-1}}{(m-1)!} (L^m u)(\tau) d\tau\ .$$

Example 16. (Taylor formula for fractional operators) In [9] J.D. Munkhammar gave Taylor's formula based on fractional caculus. Let $u(t) \in C^1([a,b])$ then the Riemann-Liouville fractional integral of order α is

$$(Tu)(t) := I^{\alpha}_{a+} u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{u(s)}{(t-s)^{1-\alpha}} ds\ ,$$

and the Riemann–Liouville fractional derivative of order α as follow

$$(Lu)(t) := D^{\alpha}_{a+} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{u(s)}{(t-s)^{\alpha}} ds$$

where $\alpha \in]0,1[$ and Γ is a well known Gamma function. Hence

$$D^{\alpha}_{a+} I^{\alpha}_{a+} = I.$$

Let $\alpha > 0$, $m \in Z^+$ and $u(t) \in C^{[\alpha]+m+1}([a,b])$, the Taylor formula is

$$u(t) = \sum_{k=-m}^{m-1} \frac{D^{\alpha+k}_{a+} u(t_0)}{\Gamma(\alpha+k+1)} (t-t_0)^{\alpha+k} + I^{\alpha+m}_{a+} D^{\alpha+m}_{a+} u(t)$$

for all $a \leq t_0 < t \leq b$.

8 Taylor-Gontcharov's formula for high order genaralized Dirac operators

Corollary 5. (The Taylor-Gontcharov's formula) *A generalization of the Taylor formula leads to*

$$u = \sum_{j=0}^{m-1} T_0 T_1 ... T_j P_j L_j ... L_1 L_0 u + T_1 ... T_m L_m ... L_1 u$$

with $L_0 = T_0 = I$.

Example 17. (Realisation on a lattice) Let G_h be the lattice of the bounded domain G and $\Delta_h = D^+_h D^-_h$ be the discretized Laplace operator. We consider the following problem

$$\begin{aligned} \Delta_h u &= f \quad \text{on} \quad G_h, \\ tr_\Gamma P_{\Gamma_h} u &= g_0 \quad \text{on} \quad \Gamma_h, \\ tr_{\Gamma_h} D^-_h u &= g_1 \quad \text{on} \quad \Gamma_h. \end{aligned}$$

Γ_h is the *"‘numerical"’ boundary* of G for a meshwidth h. The unique solution is then given by

$$u = F^-_h g_0 + T^-_h F^+_h (tr_{\Gamma_h} T^-_h F^+_h)^{-1} T^-_h D^-_h g_1 + T^-_h \mathbf{Q}_h T^+_h f$$

with Bergman projection

$$\mathbf{P}_h = F^+_h (tr_{\Gamma_h} T^-_h F^+_h)^{-1} tr_{\Gamma_h} T^-_h$$

The operators in Taylor-Gontcharov’ s formula are chosen as follows

$$L_1 := D^-_h,\ L_2 := D^+_h,\ P_1 := F^-_h,\ P_2 := F^+_h,\ T_1 := T^-_h,\ T_2 := T^+_h$$

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