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SUMMARY

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Partial Fractions and q-Binomial Determinant Identities

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ABSTRACT

Partial fraction decomposition method is applied to evaluate a general determinant of shifted factorial fractions, which contains several Gaussian binomial determinant identities.

RESUMEN

El método de descomposición en fracción parciales aplicado para evaluar un determinante general de fracciones factoriales trasladadas, la cual contiene varias identidades determinante binomial Gaussiano.

Key words and phrases: *The Cauchy double alternant, Partial fractions, q-Binomial coefficients.*

Math. Subj. Class.: *15A15, 11C20.*

Binomial determinant evaluation plays an important role in combinatorial enumeration, particularly in plane partitions. This paper will establish a very general determinant identity through partial fraction decomposition method. It will be shown to be useful in q-binomial determinant evaluations with several interesting known and new formulae being exemplified.

1 Partial Fraction Decomposition

For two sequences $\{\alpha_k,\gamma_k\}_{k\ge 0}$, define the generalized shifted factorials by

$$(x|\alpha)_0=1 \quad\text{and}\quad (x|\alpha)_n=\prod_{k=0}^{n-1}(1-x\alpha_k) \quad\text{with}\quad n\in\mathbb{N}, \tag{1a}$$

$$(y|\gamma)_0=1 \quad\text{and}\quad (y|\gamma)_n=\prod_{k=0}^{n-1}(1-y\gamma_k) \quad\text{with}\quad n\in\mathbb{N}. \tag{1b}$$

When $\alpha_k=\gamma_k=q^k$ for $k\in\mathbb{N}_0$, they will reduce to the usual shifted factorials

$$(x;q)_0=1 \quad\text{and}\quad (x;q)_n=(1-x)(1-qx)\cdots(1-q^{n-1}x) \quad\text{with}\quad n\in\mathbb{N}. \tag{2}$$

For the triangular matrix given by $\alpha=[\alpha_{ij}]_{0\le i\le j<\infty}$, denote its j-th column by $\alpha_j=(\alpha_{0j},\alpha_{1j},\alpha_{2j},\cdots,\alpha_{jj})$. Then the main result may be stated as follows.

Theorem 1 (Generalized Cauchy determinant)**.** *Let $\{x_k\}_{k=0}^n$ be distinct complex numbers. Then there holds the following determinant identity:*

$$\det_{0\le i,j\le n}\Big[\frac{(x_i|\alpha_j)_j}{(x_i|\gamma)_{j+1}}\Big]=\frac{\prod_{0\le i<j\le n}(x_i-x_j)(\alpha_{ij}-\gamma_j)}{\prod_{k=0}^n(x_k|\gamma)_{n+1}}.$$

The very special case of this theorem with $\alpha_{ij}=\gamma_i$ for $i,j\in\mathbb{N}_0$ results in the celebrated Cauchy's double alternant (cf. [6, 7]):

$$\det_{0\le i,j\le n}\Big[\frac{1}{1-x_i\gamma_j}\Big]=\frac{\prod_{0\le i<j\le n}(x_i-x_j)(\gamma_i-\gamma_j)}{\prod_{0\le i,j\le n}(1-x_i\gamma_j)}. \tag{3}$$

Proof. Expanding the rational function in partial fractions, we have

$$\frac{(x_i|\alpha_j)_j}{(x_i|\gamma)_{j+1}} = \frac{\prod_{\iota=0}^{j-1}(1-x_i\alpha_{\iota j})}{\prod_{k=0}^{j}(1-x_i\gamma_k)} = \sum_{k=0}^{j}\frac{w_{kj}}{1-x_i\gamma_k}$$

where the connected coefficients are determined by the following limit relation

$$w_{kj} = \lim_{x_i\to\frac{1}{\gamma_k}}(1-x_i\gamma_k)\frac{(x_i|\alpha_j)_j}{(x_i|\gamma)_{j+1}} = \frac{\prod_{\iota=0}^{j-1}(\alpha_{\iota j}-\gamma_k)}{\prod_{\iota=0:\iota\neq k}^{j}(\gamma_\iota-\gamma_k)}.$$

This leads us to the following determinant factorization

$$\det_{0\le i,j\le n}\left[\frac{(x_i|\alpha_j)_j}{(x_i|\gamma)_{j+1}}\right] = \det_{0\le i,k\le n}\left[\frac{1}{1-x_i\gamma_k}\right]\times\det_{0\le k,j\le n}\left[w_{kj}\right].$$

For the matrix $\left[w_{kj}\right]_{0\le k,j\le n}$ is upper triangular, its determinant is equal to the product of its diagonal entries:

$$\det_{0\le k,j\le n}\left[w_{kj}\right] = \prod_{j=0}^{n}w_{jj} = \prod_{0\le i<j\le n}\frac{\alpha_{ij}-\gamma_j}{\gamma_i-\gamma_j}.$$

While the first determinant can be evaluated by Cauchy's double alternant (2). Their combination yields the determinant identity stated in Theorem 1. □

Shifting the γ-parameters by $\gamma_k\to\gamma_{k-1}$, we may state the determinant identity in Theorem 1 in the following more convenient form.

Proposition 2 (Determinant identity)**.** *Let* $\{x_k\}_{k=0}^{n}$ *be distinct complex numbers. Then there holds the following determinant identity:*

$$\det_{0\le i,j\le n}\left[\frac{(x_i|\alpha_j)_j}{(x_i|\gamma)_j}\right] = \frac{\prod_{0\le i<j\le n}(x_i-x_j)(\alpha_{ij}-\gamma_{j-1})}{\prod_{k=0}^{n}(x_k|\gamma)_n}.$$

Letting $\alpha_{ij}=p^i y_j$ and $\gamma_k=q^k$ further in Proposition 2, we have the identity.

Corollary 3 (Bibasic determinant evaluation formula)**.**

$$\det_{0\le i,j\le n}\left[\frac{(x_iy_j;p)_j}{(x_i;q)_j}\right] = q^{2\binom{n+1}{3}}\prod_{0\le i<j\le n}(x_j-x_i)\prod_{k=0}^{n}\frac{(q^{1-k}y_k;p)_k}{(x_k;q)_n}.$$

From this corollary, we can derive numerous q-binomial determinant identities.

2 q-Binomial Determinant Identities

Define the Gaussian binomial coefficients by

$$\begin{bmatrix}x\\n\end{bmatrix} = \frac{(q^{1+x-n};q)_n}{(q;q)_n}\quad\text{where}\quad n\in\mathbb{N}_0\quad\text{and}\quad x\in\mathbb{C}.$$

Applying Corollary 3, we show now ten classes of q-binomial determinant identities.

2.1 Expressing the q-binomial coefficient in terms of shifted factorials

$$\begin{bmatrix} X_i - j \\ A \end{bmatrix} = q^{-Aj} \begin{bmatrix} X_i \\ A \end{bmatrix} \frac{(q^{A-X_i};q)_j}{(q^{-X_i};q)_j}$$

we derive the corresponding determinant formula

$$\det_{0\le i,j\le n} \left[\begin{bmatrix} X_i - j \\ A \end{bmatrix} \right] = \prod_{0\le i<j\le n} (q^{-X_j} - q^{-X_i})(1-q^{1+A+i-j}) \tag{4a}$$

$$\times \frac{q^{2\binom{n+1}{3}-A\binom{n+1}{2}}}{(q;q)_n^{n+1}} \prod_{k=0}^{n} \begin{bmatrix} X_k \\ A \end{bmatrix} \begin{bmatrix} n-1-X_k \\ n \end{bmatrix}^{-1}. \tag{4b}$$

2.2 Rewriting the q-binomial coefficient in terms of shifted factorials

$$\begin{bmatrix} A \\ X_i - j \end{bmatrix} = (-1)^j q^{-\binom{j}{2}+jX_i} \begin{bmatrix} A \\ X_i \end{bmatrix} \frac{(q^{-X_i};q)_j}{(q^{1+A-X_i};q)_j}$$

we get the corresponding determinant identity

$$\det_{0\le i,j\le n} \left[q^{-jX_i} \begin{bmatrix} A \\ X_i - j \end{bmatrix} \right] = \prod_{0\le i<j\le n} (q^{-X_i} - q^{-X_j})(1-q^{-A+i-j}) \tag{5a}$$

$$\times \frac{q^{\binom{n+1}{3}+(1+A)\binom{n+1}{2}}}{(q;q)_n^{n+1}} \prod_{k=0}^{n} \begin{bmatrix} A \\ X_k \end{bmatrix} \begin{bmatrix} A+n-X_k \\ n \end{bmatrix}^{-1}. \tag{5b}$$

2.3 Reformulating the q-binomial coefficient in terms of shifted factorials

$$\begin{bmatrix} A + X_i - j \\ X_i - j \end{bmatrix} = q^{-Aj} \begin{bmatrix} A + X_i \\ A \end{bmatrix} \frac{(q^{-X_i};q)_j}{(q^{-A-X_i};q)_j}$$

we obtain the following determinant evaluation formula

$$\det_{0\le i,j\le n} \left[\begin{bmatrix} A + X_i - j \\ X_i - j \end{bmatrix} \right] = \prod_{0\le i<j\le n} (q^{-X_j} - q^{-X_i})(1-q^{1+A+i-j}) \tag{6a}$$

$$\times \frac{q^{2\binom{n+1}{3}-2A\binom{n+1}{2}}}{(q;q)_n^{n+1}} \prod_{k=0}^{n} \begin{bmatrix} A + X_k \\ A \end{bmatrix} \begin{bmatrix} -1-A+n-X_k \\ n \end{bmatrix}^{-1}. \tag{6b}$$

2.4 Applying the q-binomial relation

$$\begin{bmatrix} X_i + j \\ A \end{bmatrix} = \begin{bmatrix} X_i \\ A \end{bmatrix} \frac{(q^{1+X_i};q)_j}{(q^{1-A+X_i};q)_j}$$

we find the corresponding determinant formula

$$\det_{0\le i,j\le n}\left[\begin{bmatrix} X_i+j \\ A \end{bmatrix}\right] = \prod_{0\le i<j\le n}(q^{X_j}-q^{X_i})(1-q^{1+A+i-j}) \tag{7a}$$

$$\times\frac{q^{2\binom{n+1}{3}+(1-A)\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\begin{bmatrix} X_k \\ A \end{bmatrix}\begin{bmatrix} X_k-A+n \\ n \end{bmatrix}^{-1}. \tag{7b}$$

2.5 Observing the q-binomial relation

$$\begin{bmatrix} A \\ X_i+j \end{bmatrix} = (-1)^j q^{(A-X_i)j-\binom{j}{2}}\begin{bmatrix} A \\ X_i \end{bmatrix}\frac{(q^{-A+X_i};q)_j}{(q^{1+X_i};q)_j}$$

we recover the determinant identity due to Carlitz [4] (cf. Chu [5] also)

$$\det_{0\le i,j\le n}\left[q^{jX_i}\begin{bmatrix} A \\ X_i+j \end{bmatrix}\right] = \prod_{0\le i<j\le n}(q^{X_i}-q^{X_j})(1-q^{-A+i-j}) \tag{8a}$$

$$\times\frac{q^{\binom{n+1}{3}+(1+A)\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\begin{bmatrix} A \\ X_k \end{bmatrix}\begin{bmatrix} X_k+n \\ n \end{bmatrix}^{-1}. \tag{8b}$$

2.6 By invoking the q-binomial relation

$$\begin{bmatrix} A+X_i+j \\ X_i+j \end{bmatrix} = \begin{bmatrix} A+X_i \\ A \end{bmatrix}\frac{(q^{1+A+X_i};q)_j}{(q^{1+X_i};q)_j}$$

we recover another determinant identity due to Carlitz [4] (see Menon [9] also)

$$\det_{0\le i,j\le n}\left[\begin{bmatrix} A+X_i+j \\ X_i+j \end{bmatrix}\right] = \prod_{0\le i<j\le n}(q^{X_j}-q^{X_i})(1-q^{1+A+i-j}) \tag{9a}$$

$$\times\frac{q^{2\binom{n+1}{3}+\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\begin{bmatrix} A+X_k \\ A \end{bmatrix}\begin{bmatrix} X_k+n \\ n \end{bmatrix}^{-1} \tag{9b}$$

which reduces, for $q\to1$, to the binomial determinant of Ostrowski [10].

Furthermore for $\delta=0,1$, we can show the following determinant identity

$$\det_{0\le i,j\le n}\left[C^{(\delta)}_{X_i+j}(q)\right] = (2q)^{(1+n)(1+n+\delta)+2\sum_{\imath=0}^{n}X_\imath}q^{n(n+1)(1+2n+6\delta)/6} \tag{10a}$$

$$\times\prod_{k=0}^{n}\frac{(q;q^2)_{1+k}(q;q^2)_{\delta+X_k}}{(q^2;q^2)_{1+\delta+X_k+n}}\prod_{0\le i<j\le n}(q^{2X_i}-q^{2X_j}) \tag{10b}$$

where the q-Catalan numbers due to Andrews [2] has been slightly extended by

$$C_n^{(\delta)}(q):=\frac{(2q)^{1+\delta+2n}}{1-q^{2+2\delta+2n}}\begin{bmatrix} \delta+2n \\ n \end{bmatrix}\frac{1-q}{(-q;q)_n(-q;q)_{\delta+n}}. \tag{11}$$

When $x_k = k+\ell$, we get the following Hankel determinant identity

$$\det_{0\le i,j\le n}\left[C^{(\delta)}_{i+j+\ell}(q)\right] = (2q)^{(1+n)(1+\delta+2n+2\ell)}q^{n(n+1)(4n+6\ell+6\delta-1)/6} \tag{12a}$$

$$\times \prod_{k=0}^{n}\frac{(q;q)_{1+2k}(q;q^2)_{\delta+k+\ell}}{(q^2;q^2)_{1+\delta+k+n+\ell}}. \tag{12b}$$

Letting $\delta=0$ and $q\to 1$, we recover further the related results [1, 8, 11] on the classical Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$:

$$\det_{0\le i,j\le n}\left[C_{i+j}\right]=1,\quad \det_{0\le i,j\le n}\left[C_{i+j+1}\right]=1,\quad \det_{0\le i,j\le n}\left[C_{i+j+2}\right]=n+2. \tag{13}$$

2.7 By means of the q-binomial relation

$$\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A+X_i\\ j\end{bmatrix}^{-1} = q^{(Y_j-A)j}\frac{(q^{-X_i-Y_j};q)_j}{(q^{-A-X_i};q)_j}$$

we get the following determinant identity

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A+X_i\\ j\end{bmatrix}^{-1}\right] = \frac{q^{2\binom{n+1}{3}-\sum_{k=0}^{n}(2kA+nX_k-kY_k)}}{(q;q)_n^{n+1}\prod_{k=0}^{n}\begin{bmatrix}n-1-A-X_k\\ n\end{bmatrix}} \tag{14a}$$

$$\times\prod_{0\le i<j\le n}(q^{X_i}-q^{X_j})(1-q^{1+A-Y_j+i-j}). \tag{14b}$$

2.8 In view of the q-binomial relation

$$\begin{bmatrix}X_i+Y_j+j\\ Y_j\end{bmatrix}\begin{bmatrix}X_i+Y_j\\ Y_j\end{bmatrix}^{-1} = \frac{(q^{1+X_i+Y_j};q)_j}{(q^{1+X_i};q)_j}$$

we obtain the corresponding determinant formula

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}X_i+Y_j+j\\ Y_j\end{bmatrix}\begin{bmatrix}X_i+Y_j\\ Y_j\end{bmatrix}^{-1}\right] = \frac{q^{2\binom{n+1}{3}+\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\begin{bmatrix}X_k+n\\ n\end{bmatrix}^{-1} \tag{15a}$$

$$\times\prod_{0\le i<j\le n}(q^{X_j}-q^{X_i})(1-q^{1+Y_j+i-j}). \tag{15b}$$

2.9 According to the q-binomial relation

$$\begin{bmatrix}A+X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}X_i+j\\ j\end{bmatrix}^{-1} = \frac{(q^{1+A+X_i+Y_j-j};q)_j}{(q^{1+X_i};q)_j}$$

we derive the corresponding determinant identity

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}A+X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}X_i+j\\ j\end{bmatrix}^{-1}\right]=\frac{q^{2\binom{n+1}{3}+\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\begin{bmatrix}X_k+n\\ n\end{bmatrix}^{-1} \tag{16a}$$

$$\times\prod_{0\le i<j\le n}(q^{X_j}-q^{X_i})(1-q^{1+A+Y_j+i-2j}). \tag{16b}$$

2.10 Similarly, the q-binomial relation

$$\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A+X_i-j\\ n-j\end{bmatrix}=q^{(Y_j-A)j}\begin{bmatrix}n\\ j\end{bmatrix}\begin{bmatrix}A+X_i\\ n\end{bmatrix}\frac{(q^{-X_i-Y_j};q)_j}{(q^{-A-X_i};q)_j}$$

leads us to the following binomial determinant evaluation formulae

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A+X_i-j\\ n-j\end{bmatrix}\right]=\prod_{0\le i<j\le n}(q^{-X_j}-q^{-X_i})(1-q^{1+A+i-j-Y_j}) \tag{17a}$$

$$\times\frac{q^{\sum_{k=0}^{n}(k-1-2A+Y_k)k}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}A+X_k\\ n\end{bmatrix}}{\begin{bmatrix}-1-A+n-X_k\\ n\end{bmatrix}}, \tag{17b}$$

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}X_i+j\\ j\end{bmatrix}\begin{bmatrix}A+X_i+Y_j\\ n-j\end{bmatrix}\right]=\prod_{0\le i<j\le n}(q^{-X_i}-q^{-X_j})(1-q^{1+n-A-Y_{n-j}+i-j}) \tag{18a}$$

$$\times\frac{q^{\sum_{k=0}^{n}(k-1+A-2n+Y_{n-k})k}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}n+X_k\\ n\end{bmatrix}}{\begin{bmatrix}-1-X_k\\ n\end{bmatrix}}; \tag{18b}$$

where the last identity is derived from the first one under substitution $j\to n-j$ on the column index.

3 Duplicate Determinant Identities

Performing the parameter replacements in Proposition 2

$$\begin{aligned}x_k &\to ax_k+c/x_k,\\ \gamma_k &\to d\gamma_k/(1+acd^2\gamma_k^2),\\ \alpha_{ij} &\to b\alpha_{ij}/(1+ab^2c\alpha_{ij}^2);\end{aligned}$$

and then applying factorizations

$$\begin{aligned}
x_i - x_j &\quad\to\quad (x_i - x_j)(a - c/x_i x_j),\\
\alpha_{ij} - \gamma_k &\quad\to\quad \frac{(b\alpha_{ij} - d\gamma_k)(1 - abcd\,\alpha_{ij}\gamma_k)}{(1 + ab^2c\alpha_{ij}^2)(1 + acd^2\gamma_k^2)},\\
1 - x_i\gamma_k &\quad\to\quad \frac{(1 - ad\gamma_k x_i)(1 - cd\gamma_k/x_i)}{1 + acd^2\gamma_k^2},\\
1 - x_k\alpha_{ij} &\quad\to\quad \frac{(1 - abx_k\alpha_{ij})(1 - bc\alpha_{ij}/x_k)}{1 + ab^2c\alpha_{ij}^2};
\end{aligned}$$

we find the following duplicate determinant identity.

Proposition 4. *Let $\{x_k\}_{k=0}^n$ be distinct complex numbers. Then there holds the following determinant identity:*

$$\det_{0\le i,j\le n}\left[\frac{(abx_i|\alpha_j)_j(bc/x_i|\alpha_j)_j}{(adx_i|\gamma)_j(cd/x_i|\gamma)_j}\right] = \prod_{0\le i<j\le n}(b\alpha_{ij} - d\gamma_{j-1})(1 - abcd\,\alpha_{ij}\gamma_{j-1})$$
$$\times\ \frac{\prod_{0\le i<j\le n}(x_i - x_j)(a - c/x_i x_j)}{\prod_{k=0}^n (adx_k|\gamma)_n (cd/x_k|\gamma)_n}.$$

This identity contains the following three determinant evaluations.

Corollary 5 ($a = b = 1$ and $\gamma_k \to 0$ in Proposition 4)**.**

$$\det_{0\le i,j\le n}\Big[(x_i|\alpha_j)_j(c/x_i|\alpha_j)_j\Big] = \prod_{0\le i<j\le n}\Big\{\alpha_{ij}(x_i - x_j)(1 - c/x_i x_j)\Big\}.$$

Corollary 6 ($d = 1$ and $\alpha_{ij} \to 0$ in Proposition 4)**.**

$$\det_{0\le i,j\le n}\left[\frac{1}{(ax_i|\gamma)_j(c/x_i|\gamma)_j}\right] = \frac{\prod_{0\le i<j\le n}(x_j - x_i)(a - c/x_i x_j)}{\prod_{k=0}^n (ax_k|\gamma)_n (c/x_k|\gamma)_n}\prod_{\ell=1}^n \gamma_{\ell-1}^{\ell}.$$

Putting $\alpha_{ij} = p^i y_j$ and $\gamma_k = q^k$ in Proposition 4, we find the following determinant evaluation formula of factorial fractions with two different bases.

Corollary 7 (Bibasic determinant identity)**.**

$$\det_{0\le i,j\le n}\left[\frac{(abx_i y_j;p)_j(bcy_j/x_i;p)_j}{(adx_i;q)_j(cd/x_i;q)_j}\right] = d^{\binom{n+1}{2}}\prod_{0\le i<j\le n}(x_j - x_i)(a - c/x_i x_j)$$
$$\times\ q^{2\binom{n+1}{3}}\prod_{k=0}^n \frac{(q^{1-k}by_k/d;p)_k(q^{k-1}abcdy_k;p)_k}{(adx_k;q)_n(cd/x_k;q)_n}.$$

When $p = q$ and $y_k = 1$, it reduces to the following determinant identity

$$\det_{0\le i,j\le n}\left[\frac{(abx_i;q)_j(bc/x_i;q)_j}{(adx_i;q)_j(cd/x_i;q)_j}\right] = b^{\binom{n+1}{2}}\prod_{0\le i<j\le n}(x_i-x_j)(a-c/x_ix_j) \tag{19a}$$

$$\times\ q^{\binom{n+1}{3}}\prod_{k=0}^{n}\frac{(d/b;q)_k(q^{k-1}abcd;q)_k}{(adx_k;q)_n(cd/x_k;q)_n}. \tag{19b}$$

The determinant evaluation formulae established in this section contain numerous q-binomial determinant identities as special cases, which will be illustrated by the following five examples.

3.1 Expressing the q-binomial coefficients in terms of shifted factorials

$$\frac{\begin{bmatrix}X_i+A\\ j\end{bmatrix}\begin{bmatrix}X_i-B-C\\ n-j\end{bmatrix}}{\begin{bmatrix}X_i+B\\ j\end{bmatrix}\begin{bmatrix}X_i-A-C\\ n-j\end{bmatrix}} = q^{(A-B)j}\frac{\begin{bmatrix}X_i-B-C\\ n\end{bmatrix}}{\begin{bmatrix}X_i-A-C\\ n\end{bmatrix}}\times\frac{(q^{1+X_i-A-C-n};q)_j(q^{-X_i-A};q)_j}{(q^{1+X_i-B-C-n};q)_j(q^{-X_i-B};q)_j}$$

we establish from Corollary 7 the determinant evaluation formula

$$\det_{0\le i,j\le n}\left[\frac{\begin{bmatrix}X_i+A\\ j\end{bmatrix}\begin{bmatrix}X_i-B-C\\ n-j\end{bmatrix}}{\begin{bmatrix}X_i+B\\ j\end{bmatrix}\begin{bmatrix}X_i-A-C\\ n-j\end{bmatrix}}q^{\binom{j}{2}}\right] = \prod_{0\le i<j\le n}(q^{X_i}-q^{X_j})(1-q^{n-1+C-X_i-X_j}) \tag{20a}$$

$$\times\frac{q^{B\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}n+A+B+C-k\\ k\end{bmatrix}\begin{bmatrix}B-A\\ k\end{bmatrix}\begin{bmatrix}X_k-B-C\\ n\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}X_k+B\\ n\end{bmatrix}\begin{bmatrix}n-1+B+C-X_k\\ n\end{bmatrix}\begin{bmatrix}X_k-A-C\\ n\end{bmatrix}} \tag{20b}$$

which contains, as special case, the following q-binomial determinant identity

$$\det_{0\le i,j\le n}\left[\frac{\begin{bmatrix}\lambda i+A\\ j\end{bmatrix}\begin{bmatrix}\lambda i-B\\ n-j\end{bmatrix}}{\begin{bmatrix}\lambda i+B\\ j\end{bmatrix}\begin{bmatrix}\lambda i-A\\ n-j\end{bmatrix}}q^{\binom{j}{2}}\right] = \prod_{0\le i<j\le n}(q^{\lambda i}-q^{\lambda j})(1-q^{n-1-\lambda i-\lambda j}) \tag{21a}$$

$$\times\frac{q^{B\binom{n+1}{2}}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}n+A+B-k\\ k\end{bmatrix}\begin{bmatrix}B-A\\ k\end{bmatrix}\begin{bmatrix}\lambda k-B\\ n\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}\lambda k+B\\ n\end{bmatrix}\begin{bmatrix}n-1+B-\lambda k\\ n\end{bmatrix}\begin{bmatrix}\lambda k-A\\ n\end{bmatrix}}. \tag{21b}$$

3.2 Rewriting the q-binomial coefficients in terms of shifted factorials

$$\frac{\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A-X_i+Y_j\\ j\end{bmatrix}}{\begin{bmatrix}B+X_i\\ j\end{bmatrix}\begin{bmatrix}A+B-X_i\\ j\end{bmatrix}} = q^{2j(Y_j-B)}\frac{(q^{-X_i-Y_j};q)_j(q^{X_i-A-Y_j};q)_j}{(q^{-X_i-B};q)_j(q^{X_i-A-B};q)_j}$$

we recover from Corollary 7 the determinant identity due to Joris Van Jeugt

$$\det_{0\le i,j\le n}\left[\frac{\begin{bmatrix}X_i+Y_j\\ j\end{bmatrix}\begin{bmatrix}A-X_i+Y_j\\ j\end{bmatrix}}{\begin{bmatrix}B+X_i\\ j\end{bmatrix}\begin{bmatrix}A+B-X_i\\ j\end{bmatrix}}q^{\binom{j}{2}}\right] = \prod_{0\le i<j\le n}(q^{X_i}-q^{X_j})(1-q^{A-X_i-X_j}) \tag{22a}$$

$$\times\frac{q^{\sum_{k=0}^{n}kY_k}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}1+A+B+Y_k-k\\ k\end{bmatrix}\begin{bmatrix}B-Y_k\\ k\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}B+X_k\\ n\end{bmatrix}\begin{bmatrix}A+B-X_k\\ n\end{bmatrix}}. \tag{22b}$$

This identity can further be specialized to the q-binomial determinant evaluation

$$\det_{0\le i,j\le n}\left[\frac{\begin{bmatrix}A+\lambda i+j\\ j\end{bmatrix}\begin{bmatrix}A-\lambda i+j\\ j\end{bmatrix}}{\begin{bmatrix}B+\lambda i\\ j\end{bmatrix}\begin{bmatrix}B-\lambda i\\ j\end{bmatrix}}q^{\binom{j}{2}}\right]=\prod_{0\le i<j\le n}(q^{-\lambda i}-q^{-\lambda j})(1-q^{\lambda i+\lambda j}) \tag{23a}$$

$$\times\frac{q^{\sum_{k=0}^{n}k(A+k)}}{(q;q)_n^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}1+A+B\\ k\end{bmatrix}\begin{bmatrix}B-A-k\\ k\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}B+\lambda k\\ n\end{bmatrix}\begin{bmatrix}B-\lambda k\\ n\end{bmatrix}}. \tag{23b}$$

3.3 Reformulating the q-binomial coefficients in terms of shifted factorials

$$\frac{\begin{bmatrix}X_i+Y_j+j\\ X_i-Y_j-A-j\end{bmatrix}}{\begin{bmatrix}X_i+Y_j\\ X_i-Y_j-A\end{bmatrix}\begin{bmatrix}X_i+C+j\\ X_i-A-C-j\end{bmatrix}}=q^{(C-Y_j)j}\frac{\begin{bmatrix}A+2C+2j\\ 2C-2Y_j\end{bmatrix}}{\begin{bmatrix}A+2C\\ A+2Y_j\end{bmatrix}\begin{bmatrix}C+X_i\\ A+2C\end{bmatrix}}\times\frac{(q^{1+X_i+Y_j};q)_j(q^{A-X_i+Y_j};q)_j}{(q^{1+X_i+C};q)_j(q^{A-X_i+C};q)_j}$$

we derive from Corollary 7 the following determinant formula

$$\det_{0\le i,j\le n}\left[\frac{\begin{bmatrix}X_i+Y_j+j\\ X_i-Y_j-A-j\end{bmatrix}}{\begin{bmatrix}X_i+Y_j\\ X_i-Y_j-A\end{bmatrix}\begin{bmatrix}X_i+C+j\\ X_i-A-C-j\end{bmatrix}}\right]=\prod_{0\le i<j\le n}(q^{X_i}-q^{X_j})(1-q^{A-1-X_i-X_j}) \tag{24a}$$

$$\times\frac{q^{2\binom{n+2}{3}+\binom{n+1}{2}(C-A)+n\sum_{k=0}^{n}X_k}}{(q;q)_n^{n+1}\begin{bmatrix}A+2C+2n\\ 2n\end{bmatrix}^{n+1}\begin{bmatrix}2n\\ n\end{bmatrix}^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}A+2C+2k\\ 2C-2Y_k\end{bmatrix}\begin{bmatrix}Y_k-C\\ k\end{bmatrix}\begin{bmatrix}-k-A-C-Y_k\\ k\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}A+2C\\ A+2Y_k\end{bmatrix}\begin{bmatrix}X_k+C+n\\ A+2C+2n\end{bmatrix}} \tag{24b}$$

which reduces, for $X_i=bi$ and $Y_j=0$, to the q-binomial determinant identity:

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}bi+j\\ 2j\end{bmatrix}\begin{bmatrix}bi+c+j\\ 2c+2j\end{bmatrix}^{-1}\right]=\prod_{0\le i<j\le n}(q^{-bi}-q^{-bj})(1-q^{1+bi+bj}) \tag{25a}$$

$$\times\frac{q^{\sum_{k=0}^{n}k(nb+c+k)}}{(q;q)_n^{n+1}\begin{bmatrix}2c+2n\\ 2n\end{bmatrix}^{n+1}\begin{bmatrix}2n\\ n\end{bmatrix}^{n+1}}\prod_{k=0}^{n}\frac{\begin{bmatrix}2c+2k\\ 2c\end{bmatrix}\begin{bmatrix}-c\\ k\end{bmatrix}\begin{bmatrix}-k-c\\ k\end{bmatrix}}{\begin{bmatrix}n\\ k\end{bmatrix}\begin{bmatrix}bk+c+n\\ 2c+2n\end{bmatrix}}. \tag{25b}$$

3.4 According to Corollary 6, the q-binomial relation

$$q^{jX_i}\begin{bmatrix}X_i+A-j\\ X_i+j\end{bmatrix}=\frac{(-1)^j q^{\binom{j}{2}-Aj}(q;q)_{X_i+A}}{(q;q)_{X_i}(q;q)_{A-2j}}\times\frac{1}{(q^{1+X_i};q)_j(q^{-A-X_i};q)_j}$$

yields the determinant evaluation formula

$$\det_{0\le i,j\le n}\left[q^{jX_i}\begin{bmatrix}X_i+A-j\\ X_i+j\end{bmatrix}\right]=\prod_{0\le i<j\le n}(q^{-X_i}-q^{-X_j})(1-q^{1+A+X_i+X_j}) \tag{26a}$$

$$\times\prod_{k=0}^{n}\begin{bmatrix}X_k+A-n\\ X_k+n\end{bmatrix}\frac{q^{nX_k}}{(q^{1+A-2n};q)_{2k}}. \tag{26b}$$

In particular for $X_i = c + bi$, it becomes the q-binomial determinant identity

$$\det_{0\le i,j\le n}\left[q^{bij}\begin{bmatrix}a+bi-j\\c+bi+j\end{bmatrix}\right]\prod_{0\le i<j\le n}(q^{-bi}-q^{-bj})(1-q^{1+a+c+bi+bj}) \tag{27a}$$

$$\times\, q^{nb\binom{n+1}{2}}\prod_{k=0}^{n}\frac{(q;q)_{a+bk-n}}{(q;q)_{a-c-2k}(q;q)_{c+bk+n}} \tag{27b}$$

which is the q-analogue of the determinant evaluated by Amdeberhan and Zeilberger [3, Eq 2].

3.5 In view of Corollary 5, the q-binomial relation

$$q^{-jX_i}\begin{bmatrix}X_i+Y_j+j\\X_i-Y_j+A-j\end{bmatrix}\begin{bmatrix}X_i-Y_j+A\\X_i+Y_j\end{bmatrix}=\frac{(-1)^j q^{(A-Y_j)j-\binom{j}{2}}}{(q;q)_{2Y_j-A+2j}(q;q)_{A-2Y_j}}\times(q^{1+X_i+Y_j};q)_j(q^{Y_j-X_i-A};q)_j$$

leads to the following determinant evaluation

$$\det_{0\le i,j\le n}\left[q^{-jX_i}\begin{bmatrix}X_i+Y_j+j\\X_i-Y_j+A-j\end{bmatrix}\begin{bmatrix}X_i-Y_j+A\\X_i+Y_j\end{bmatrix}\right] \tag{28a}$$

$$=\frac{\prod_{0\le i<j\le n}(q^{-X_j}-q^{-X_i})(1-q^{1+A+X_i+X_j})}{\prod_{k=0}^{n}(q;q)_{A-2Y_k}(q;q)_{2Y_k-A+2k}}. \tag{28b}$$

In particular for $X_i = a + bi$ and $Y_j = 0$, the last identity gives

$$\det_{0\le i,j\le n}\left[q^{-bij}\begin{bmatrix}a+bi+j\\c+bi-j\end{bmatrix}\right]=\frac{\prod_{0\le i<j\le n}(q^{-bj}-q^{-bi})(1-q^{1+a+c+bi+bj})}{\prod_{k=0}^{n}(q^{1+a+bk};q)_{c-a}(q;q)_{a-c+2k}} \tag{29}$$

which results in the q-analogue of the binomial determinant identity due to Amdeberhan and Zeilberger [3, Eq 1].

Similarly, letting $x_i = q^{a+bi}$ and $\alpha_{ij} = q^{dj-i}$, we find from Corollary 5 another determinant identity

$$\det_{0\le i,j\le n}\left[\begin{bmatrix}a+bi+dj\\j\end{bmatrix}\begin{bmatrix}c-bi+dj\\j\end{bmatrix}q^{\binom{j}{2}}\right] \tag{30a}$$

$$=\prod_{k=0}^{n}\frac{q^{k(c+dk)}}{(q;q)_k^2}\prod_{0\le i<j\le n}(q^{-bi}-q^{-bj})(1-q^{a-c+bi+bj}) \tag{30b}$$

which is the q-analogue of the result in Amdeberhan and Zeilberger [3, Eq 14]. The list of examples can be endless. However, we are not going further to prolong it due to the space limitation.

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A Family of Stationary Solutions to the Euler Equations and Generalized Solutions

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ABSTRACT

In this work, we present a interesting family of stationary solutions for the Euler equations, which behaves in the same way that the approximated solutions presented in [6].

RESUMEN

En este trabajo, presentamos una familia interesante de soluciones estacionarias para las ecuaciones de Euler, que se comportan de la misma manera que las soluciones aproximadas presentadas en [6].

Key words and phrases: *Euler equations, incompressible flows, generalized solutions.*

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1 Introduction

An ideal incompressible fluid moving inside $D \subset \mathbb{R}^n$ is classically described by a velocity field $u(t,x)$ and a pressure field $p(t,x)$, subject to the classical Euler equations:

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = 0 \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

with the boundary condition that u is tangent to ∂D.

In classical continuous mechanics [1], the motion of an incompressible inviscid fluid in a compact domain $D \subset \mathbb{R}^n$ can be seen as a geodesic on the group of all diffeomorphisms of D with unit jacobian determinant, $G(D)$. This set is included in $S(D)$ the semigroup of all Borel maps h of D that satisfy

$$\int_D f(h(x))dx = \int_D f(x)dx, \qquad \forall f \in C^0(D).$$

For more details see [1], [2] or [6].

We will denote

$$V := \Big\{ u : [0,T] \times D \longrightarrow \mathbb{R}^n \;\text{ such that }\; u \in C^0(Q),\;\; u(t,\cdot) \in Lip(D)$$

$$\text{uniformly in }\; 0 \le t \le T, \text{div } u = 0,\;\; u(t,\cdot)\cdot \widehat{n}\Big|_{\partial D} = 0 \Big\}.$$

Note that the flow $(t,x) \mapsto g(t,x)$ describing the motion of fluid particles is defined by

$$\begin{cases} \partial_t g(t,x) = u(t, g(t,x)) \\ g(0,x) = x. \end{cases} \tag{1.2}$$

By Cauchy-Lipschitz theorem, for each $u \in V$, there is a unique solution to (1.2) and for each time t the map $g(t,x) = g(t,\cdot) \in G(D)$. Then, by elementary calculations, the Euler equations can be replaced by the following equivalent equations:

$$\begin{cases} \partial^2_{tt} g(t,x) + \nabla p(t, g(t,x)) = 0 \\ \det D_x g(t,x) = 1. \end{cases} \tag{1.3}$$

Lets call (1.3) by the “Lagrangian formulation” of the Euler equations.

From a geometrical point of view, different from the natural PDE point of view which consists in adressing the Euler equations as an evolution equation with prescribed initial velocity field, it is natural to solve the problem to minimize the action

$$A(g) = \frac{1}{2}\int_0^T \int_D |\partial_t g(t,x)|^2 dxdt,$$

among all trajectories on $G(D)$ connecting $g(0,\cdot) = Id$ and $g(T,\cdot) = h$.

The corresponding system of PDE's is the lagrangian formulation of the Euler equations (1.3).

Ebin and Marsden showed local existence and uniqueness for this problem, namely, if h and I are sufficiently close in a sufficiently high order Sobolev norm, then there is a unique geodesic connecting Id to h, see [9]. In the large, uniqueness can fail. However, in [12], A. I. Shnirelman shows that existence of minimal geodesics may fail to a class of data.

To solve the problem to find minimal geodesics in a generalized sense, in particular for data h in Shnirelman class, was introduced suitable "Young measure", (see [15] and [18]) of different ways as in [4], [6], [12] and [13]. In [4], was used a concept that takes into account the dynamics of the particles. To each path $t \in [0,T] \longmapsto z(t) \in D$, one associates the probability that it is followed by some material particle. More precisely, was proposed a notion of the generalized flow, as been a probability measure on set $\Omega = D^{[0,T]}$ of all curves $t \in [0,T] \longrightarrow z(t) \in D$, namely, a Borel probability measure μ, on product space $\Omega = D^{[0,T]}$, such that each projection μ_t for $0 \le t \le T$ is a Lebesgue measure on D. The action in this context is express by

$$\int_\Omega \int_0^T \frac{1}{2}|z^{'}(t)|^2 d\mu_t(z)dt.$$

Brenier showed the existence of generalized solutions and, later in [5], the existence and uniqueness of the pressure gradient linked to them through a suitable Poisson equation, but did not obtain for them a complete set of equations beyond the classical Euler equations. However, in [6] it was possible. The problem to find minimal geodesics was reformulated in terms of a pair of measures associated to the field u, solution of the Euler equations, in the following way: Given a smooth trajectory $t \in [0,T] \mapsto g(t,x) \in G(D)$, we define the measures (respectively nonnegative and vector-valued)

$$c(t,x,a) = \delta(x - g(t,a)), \qquad m(t,x,a) = \partial_t g(t,a)\delta(x - g(t,a)), \tag{1.4}$$

defined on $Q^{'} = [0,T] \times D \times D$. These measures satisfy

$$\int_D c(t,x,a)da = 1, \tag{1.5}$$

$$\partial_t c + \nabla_x \cdot m = 0, \tag{1.6}$$

$$c(0,x,a) = \delta(x-a); \qquad c(T,x,a) = \delta(x - h(a)). \tag{1.7}$$

Moreover, the measure m is absolutely continuous with respect to c, with a density $v \in L^2(Q^{'}, dc)$, so that $m = cv$, and the action is given by

$$A(g) = \frac{1}{2}\int_0^1 \int_D |v(t,x,a)|^2 c(t,x,a)dxda,$$

or equivalently, $A(g) = K[c,m]$, where

$$K[c,m] := \sup_X \{\langle c, F\rangle + \langle m, \Phi\rangle\}, \tag{1.8}$$

where (c,m) is of the form (1.4) and

$$X = \left\{(F,\Phi) \in C^0(Q^{'}) \times \left(C^0(Q^{'})\right)^n; \;\; F(t,x,a) + \frac{1}{2}|\Phi(t,x,a)|^2 \leq 0\right\}.$$

Then, Brenier defined the relaxed problem, as the problem to look for pairs of measures (c,m) that minimize $K[c,m]$ and are admissible in the sense of (1.5), (1.6) and (1.7), but do not necessarily satisfy (1.4).

Also was showed that, for $D = [0,1]^n$ and each data $h \in S(D)$, the relaxed problem always has solutions (c,m) and that there is a unique locally bounded measure $\nabla_x p$ in the interior of $Q = [0,T] \times D$, depending only h, such that

$$\partial_t(cv) + \nabla_x(cv \otimes v) + \underline{c}\nabla_x p = 0,$$

holds in the sense of distributions on the interior of $Q^{'}$, where $\underline{c}$ is a extension of c for which the product $\underline{c}(t,x,a)\nabla_x p(t,x)$ is well-defined. Moreover, was showed that for any $h \in S([0,1]^3)$ of the form

$$h(x_1,x_2,x_3) = (H(x_1,x_2),x_3),$$

that , for any $\varepsilon > 0$, there is a $u_\varepsilon \in V$ such that

$$K(u_\varepsilon) + \frac{1}{2\varepsilon}||g_{u_\varepsilon}(T,\cdot) - h||^2_{L^2(D)} \leq I_\varepsilon(h) + \varepsilon,$$

where $I_\varepsilon(h) = \inf\limits_{u \in v}\left\{K(u) + \frac{1}{2\varepsilon}||g_u(T,\cdot) - h||^2_{L^2(D)}\right\}$ and

$$k(u_\varepsilon) = \frac{1}{2}\int_0^T \int_D |u_\varepsilon(t,x)|^2 dxdt = \frac{1}{2}\int_0^T \int_D |\partial_t g_\varepsilon(t,x)|^2 dxdt = A(g_\varepsilon).$$

In addition, the measures $(c_\varepsilon, m_\varepsilon)$ associated with u_ε, through (1.4), converge, as $\varepsilon \to 0$ to the generalized solutions of the relaxed problem. Moreover, the fields u_ε satisfy

$$\nabla_x \cdot u_\varepsilon = 0, \qquad \partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla)u_\varepsilon \to -\nabla p,$$

weakly, as ε tends to zero. As observed in [6], with each solution (c,m) we may associated a measure-valued solution μ, in the sense of DiPerna and Majda, by setting

$$\int_{Q\times\mathbb{R}^d} f(t,x,\xi)d\mu(t,x,\xi) = \int_{Q'} f(t,x,v(t,x,a))dc(t,x,a),$$

for any continuous function $f \in Q\times\mathbb{R}^d$ with at most quadratic growth as $\xi\to\infty$. For more details, see [6] and [7].

In [4], Brenier shows explicit examples of non trivial generalized solutions. A typical example is when D is the unit disk in $2D$ and $h(x) = -x$. We know that the problem of the minimal action has two trivial solutions $g_+(t,x) = e^{i\pi t}x$ and $g_-(t,x) = e^{-i\pi t}x$ with the same pressure field $p(x) = \dfrac{\pi^2|x|^2}{2}$. We have another (generalized) solution (c,m) to the same problem which is given by

$$\int_{Q'} f(t,x,a)c(t,x,a)dtdxda = \int_{[0,1]\times D}\int_0^1 f(t,G(t,a,\theta),a)d\theta dtda,$$

$$\int_{Q'} f(t,x,a)m(t,x,a)dtdxda = \int_{[0,1]\times D}\int_0^1 \partial_t G(t,a,\theta)f(t,G(t,a,\theta),a)d\theta dtda,$$

for all continuous function f, where

$$G(t,\theta,a) = a\cos(\pi t) + \left(1-|a|^2\right)^{\frac{1}{2}} e^{2i\pi\theta}\sin(\pi t) \in D.$$

Note that each particle initially located at $a \in D$ splits up along a circle of radius $\left(1-|a|^2\right)^{\frac{1}{2}}$ $\sin(\pi t)$, with center $a\cos(\pi t)$, that moves across the unit disk and shrinks down to the point $-a$ when $t = 1$.

In general, is very difficult to obtain explicit examples of non trivial generalized solutions and the explicit examples constructed by Brenier, are based on the model presented in [4], which takes in account a concept purely Lagrangian of Young measures, the so-called generalized flows. Beyond supplying an application of the model developed in [6], the results of this paper give an interesting information for the limit of a sequence of the stationary solutions, showing that they are associated with measures that satisfy the Euler equations in a specified weak sense. For another point of view, the results of the paper give example of as a sequence of highly oscillatory solutions still can have a limit that is solution in some sense. Namely, we exhibited a family u_ε (which behavior as the“approximated solutions" argued above) such that when $\varepsilon\to 0$, the velocity field gets more and more oscillatory, but the measures $(c_\varepsilon, m_\varepsilon)$ associated to the field u_ε converges to a solution (c,m) of the equations

$$\int_D c(t,x,a)da = 1, \qquad \partial_t c + \nabla_x\cdot m = 0,$$

and

$$\partial_t(cv) + \nabla_x(cv \otimes v) + c\nabla_x p = 0.$$

Equivalent vector fields to those ones treaties in this work have been studied as application of high order essentially no oscillatory (ENO) schemes for smooth solutions of Navier-Stokes and Euler equations, (see [14]), in problems involving the Taylor-Green vortex, (see [8], [3] and [10]) and to explore a discrete singular convolution algorithm (DSC) for solving certain mechanics problems, (see [16] and [17]).

2 A Family of (Stationary) Solutions

In this work we consider the following family of stationary solutions to the Euler equations:

$$u_n(x,y) = \left(-\cos(x)\sin(ny), \frac{1}{n}\sin(x)\cos(ny), 0\right),$$

$$p_n(x,y) = -\frac{1}{4}\left(\cos(2x) + \frac{1}{n^2}\cos(2ny)\right).$$

Note that, $|Dp_n(x,y)| \le C$, and when n goes to infinity the pressure field strongly converges to $p(x,y) = -\frac{1}{4}\cos(2x)$. Then, is easy verify that $(u_n \cdot \nabla)u_n + \nabla p \to 0$, when $n \to \infty$.

For a moment, let us observe the behavior of family u_n. For $n = 1$ we have,

$$u_1(x,y) = (-\cos(x)\sin(y), \sin(x)\cos(y), 0)$$

and

$$\begin{cases} \dot{x} = -\cos(x)\sin(y) \\ \dot{y} = \sin(x)\cos(y). \end{cases} \tag{2.1}$$

Then, we have $(\dot{x},\dot{y}) = (0,0) \Leftrightarrow (x,y) = \left((2k+1)\frac{\pi}{2}, (2l+1)\frac{\pi}{2}\right)$ or $(k\pi, l\pi)$, where $k,l \in \mathbb{Z}$, (see Figure 1).

For $n = 2$ we have, $(\dot{x},\dot{y}) = (0,0) \Leftrightarrow (x,y) = \left((2k+1)\frac{\pi}{2}, (2l+1)\frac{\pi}{4}\right)$ or $\left(k\pi, \frac{l\pi}{2}\right)$, where $k,l \in \mathbb{Z}$, (see Figure 2).

Thus, for n we have, $(\dot{x},\dot{y}) = (0,0) \Leftrightarrow (x,y) = \left((2k+1)\frac{\pi}{2}, (2l+1)\frac{\pi}{2n}\right)$ or $\left(k\pi, \frac{l\pi}{n}\right)$, where $k,l \in \mathbb{Z}$.

Note that, when $n \to \infty$ the velocity field gets more and more oscillatory. In the next section we will show that the measures (c_n, m_n) defined by $c_n(t,x,a) = \delta\left(x - g_{u_n}(t,a)\right)$, $m_n(t,x,a) = u_n(t,x)\delta(x - g_{u_n}(t,a))$ converges to the solution (c,m) of the equations:

$$\int_D c(t,x,a)\,da = 1, \tag{2.2}$$

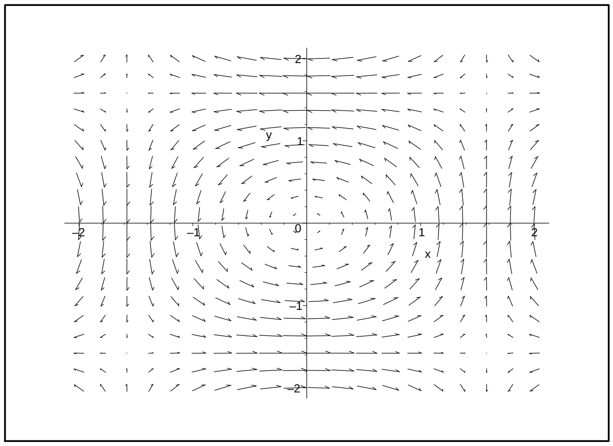

Figure 1: Phase portrait of the velocity field $u_1(x,y) = (-\cos(x)\sin(y), \sin(x)\cos(y), 0)$

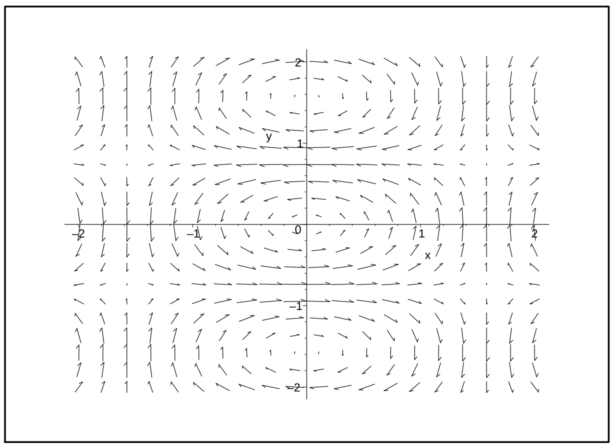

Figure 2: Phase portrait of the velocity field $u_2(x,y) = \left(-\cos(x)\sin(2y),\ \frac{1}{2}\sin(x)\cos(2y), 0\right)$

$$\partial_t c + \nabla_x \cdot m = 0 \tag{2.3}$$

$$\partial_t(cv) + \nabla_x \cdot (cv \otimes v) + \underline{c}\nabla_x p = 0, \tag{2.4}$$

By the consistency theorem in [6], or its generalization for variable density in [11], we know that if u_n is a solution of the Euler equations, then the pair of the measures (c_n, m_n) defined as below satisfy the equations (2.2), (2.3), (2.4).

3 The Limite (c,m)

In this section we build explicitly the limite (c,m). For this, in first we rewrite the field

$$u_n(x,y) = \left(-\cos(x)\sin(ny), \frac{1}{n}\sin(x)\cos(ny), 0\right)$$

as

$$u_n(x,y) = \left(u_1^1(x,ny), \frac{1}{n}u_1^2(x,ny), 0\right),$$

where

$$u_1^1(x,ny) = -\cos(x)\sin(ny) \quad \text{and} \quad u_1^2(x,ny) = \sin(x)\cos(ny).$$

Of here in ahead, we will omit third coordinate of the fields $u_n^{'}$s. We also observe that the field has period 2π. Now, we define

$$\begin{cases} x = x(t,\gamma,\delta) \\ y = y(t,\gamma,\delta) \end{cases}$$

the solution of

$$\begin{cases} \dfrac{dx}{dt} = u_1^1(x,y) = \cos(x)\sin(y) \\ \dfrac{dy}{dt} = u_1^2(x,y) = \sin(x)cos(y) \\ x(0,\gamma,\delta) = \gamma \\ y(0,\gamma,\delta) = \delta. \end{cases} \tag{3.1}$$

Let be $0 \le i \le n-1$, $0 \le \alpha_1 \le 2\pi$, and $\dfrac{2\pi i}{n} \le \alpha_2 \le \dfrac{2\pi}{n}(i+1)$, where i, $n \in \mathbb{N}$. This is:

- for $n = 1$, $i = 0$ and we have $0 \le \alpha_2 \le 2\pi$.
- for $n = 2$, $0 \le i \le 1$ and we have

$$\begin{cases} 0 \le \alpha_2 \le \pi, & \text{if i} = 0 \\ \pi \le \alpha_2 \le 2\pi, & \text{if i} = 1. \end{cases}$$

- for $n = k$, $0 \le i \le k-1$ and we have

$$\begin{cases} 0 \le \alpha_2 \le \dfrac{2\pi}{k}, & \text{if i} = 0 \\ \dfrac{2\pi}{k} \le \alpha_2 \le \dfrac{4\pi}{k}, & \text{if i} = 1 \\ \dots \\ \dfrac{2\pi(k-1)}{k} \le \alpha_2 \le 2\pi, & \text{if i} = (\text{k}-1). \end{cases}$$

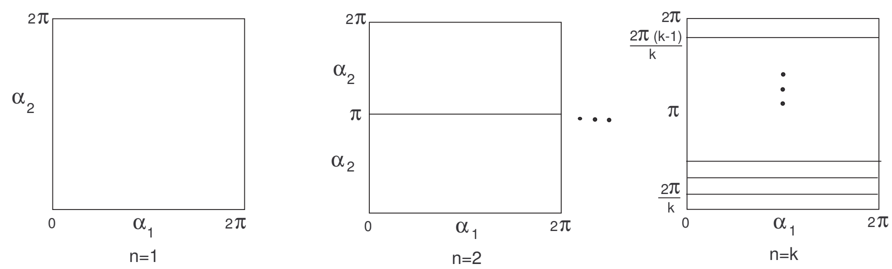

Figure 3:

Then, i counts the cells (in the vertical line) from 0 to 2π for each n, as we can observe in Figure 3.

Now, we define

$$\left\{\begin{array}{l} x_n^i(t,\alpha_1,\alpha_2) := x\left(t,\alpha_1,n\left(\alpha_2-\dfrac{2\pi i}{n}\right)\right) \\ y_n^i(t,\alpha_1,\alpha_2) := \dfrac{1}{n}y\left(t,\alpha_1,n\left(\alpha_2-\dfrac{2\pi i}{n}\right)\right)+\dfrac{2\pi i}{n}. \end{array}\right. \tag{3.2}$$

Note that, by the definition above

- for $n=1$, we have $i=0$, thus

$$\left\{\begin{array}{l} x_1^0(t,\alpha_1,\alpha_2) := x(t,\alpha_1,\alpha_2), \\ y_1^0(t,\alpha_1,\alpha_2) := y(t,\alpha_1,\alpha_2). \end{array}\right.$$

- for $n=2$, we have $0\leq i\leq 1$, thus

$$\left\{\begin{array}{l} x_2^0(t,\alpha_1,\alpha_2) := x(t,\alpha_1,2\alpha_2) \\ y_2^0(t,\alpha_1,\alpha_2) := \dfrac{1}{2}y(t,\alpha_1,2\alpha_2) \end{array}\right. ,\text{if } \ \text{i}=0$$

and

$$\left\{\begin{array}{l} x_2^1(t,\alpha_1,\alpha_2) := x(t,\alpha_1,2(\alpha_2-\pi)) \\ y_2^1(t,\alpha_1,\alpha_2) := \dfrac{1}{2}y(t,\alpha_1,2(\alpha_2-\pi))+\pi \end{array}\right. ,\text{if } \ \text{i}=1.$$

- for $n=k$, we have $0\leq i\leq k-1$, thus

$$\left\{\begin{array}{l} x_k^0(t,\alpha_1,\alpha_2) := x(t,\alpha_1,k\alpha_2) \\ y_k^0(t,\alpha_1,\alpha_2) := \dfrac{1}{k}y(t,\alpha_1,k\alpha_2) \end{array}\right. ,\text{if } \ \text{i}=0,$$

$$\vdots$$

$$\left\{\begin{array}{l} x_k^{k-1}(t,\alpha_1,\alpha_2) := x\left(t,\alpha_1,k\left(\alpha_2-\dfrac{2(k-1)\pi}{k}\right)\right) \\ y_2^{k-1}(t,\alpha_1,\alpha_2) \\ := \dfrac{1}{k}y\left(t,\alpha_1,k\left(\alpha_2\dfrac{2(k-1)\pi}{k}\right)\right)+\dfrac{2(k-1)\pi}{k} \end{array}\right. ,\text{if } \mathrm{i}=\mathrm{k}-1.$$

Then, we conclude that $0 \le x_n^i \le 2\pi$, $\dfrac{2\pi i}{n} \le y_n^i \le \dfrac{2\pi(i+1)}{n}$ and

$$\left\{\begin{array}{l} x_n^i(0,\alpha_1,\alpha_2) = x\left(0,\alpha_1,n\left(\alpha_2-\dfrac{2\pi i}{n}\right)\right) = \alpha_1 \\ y_n^i(0,\alpha_1,\alpha_2) = \dfrac{1}{n}y\left(0,\alpha_1,n\left(\alpha_2-\dfrac{2\pi i}{n}\right)\right)+\dfrac{2\pi i}{n} = \alpha_2. \end{array}\right.$$

Moreover, by (3.2) we have

$$\left\{\begin{array}{l} \dfrac{dx_n^i}{dt}(t,\alpha_1,\alpha_2) = \dfrac{dx}{dt}\left(t,\alpha_1,n\left(\alpha_2-\dfrac{2\pi i}{n}\right)\right) \\ \dfrac{dy_n^i}{dt}(t,\alpha_1,\alpha_2) = \dfrac{1}{n}\dfrac{dy}{dt}\left(t,\alpha_1,n\left(\alpha_2-\dfrac{2\pi i}{n}\right)\right) \end{array}\right. \tag{3.3}$$

Therefore, by (3.1) and (3.3)

$$\begin{aligned} \frac{dx_n^i}{dt} &= u_1^1\left(x\left(t,\alpha_1,n\left(\alpha_2-\frac{2\pi i}{n}\right)\right),y\left(t,\alpha_1,n\left(\alpha_2-\frac{2\pi i}{n}\right)\right)\right) \\ &= u_1^1\left(x_n^i(t,\alpha_1,\alpha_2),ny_n^i(t,\alpha_1,\alpha_2)-2\pi i\right) \\ &= u_1^1\left(x_n^i(t,\alpha_1,\alpha_2),ny_n^i(t,\alpha_1,\alpha_2)\right) \\ &= u_n^1\left(x_n^i(t,\alpha_1,\alpha_2),y_n^i(t,\alpha_1,\alpha_2)\right), \end{aligned}$$

and

$$\begin{aligned} \frac{dy_n^i}{dt} &= \frac{1}{n}u_1^2\left(x\left(t,\alpha_1,n\left(\alpha_2-\frac{2\pi i}{n}\right)\right),y\left(t,\alpha_1,n\left(\alpha_2-\frac{2\pi i}{n}\right)\right)\right) \\ &= \frac{1}{n}u_1^2\left(x_n^i(t,\alpha_1,\alpha_2),ny_n^i(t,\alpha_1,\alpha_2)-2\pi i\right) \\ &= \frac{1}{n}u_1^2\left(x_n^i(t,\alpha_1,\alpha_2),ny_n^i(t,\alpha_1,\alpha_2)\right) \\ &= u_n^2\left(x_n^i(t,\alpha_1,\alpha_2),y_n^i(t,\alpha_1,\alpha_2)\right). \end{aligned}$$

Now defining

$$\left\{\begin{array}{l} x_n(t,\alpha_1,\alpha_2) := x_n^i(t,\alpha_1,\alpha_2) \\ y_n(t,\alpha_1,\alpha_2) := y_n^i(t,\alpha_1,\alpha_2) \end{array}\right. \quad \text{if } \frac{2\pi \mathrm{i}}{\mathrm{n}} \le \alpha_2 \le \frac{2\pi(\mathrm{i}+1)}{\mathrm{n}},$$

we conclude that
$$\begin{cases} \dfrac{dx_n}{dt}(t,\alpha_1,\alpha_2) = u_n^1(x_n(t,\alpha_1,\alpha_2), y_n(t,\alpha_1,\alpha_2)) \\ \dfrac{dy_n}{dt}(t,\alpha_1,\alpha_2) = u_n^2(x_n(t,\alpha_1,\alpha_2), y_n(t,\alpha_1,\alpha_2)) \\ x_n(0,\alpha_1,\alpha_2) = \alpha_1 \\ y_n(0,\alpha_1,\alpha_2) = \alpha_2. \end{cases} \tag{3.4}$$

In the remain of the work, for simplicity, we will use the following notation: $D_i = (0,2\pi)^i = (0,2\pi)\times\cdots\times(0,2\pi)$, i times, where $i = 1,\cdots,4$. Now, we are ready to show the following result:

Theorem 3.1. *Consider (x_n, y_n) solution of (3.4). Let*

$$\begin{cases} c_n(t,x,y,\alpha_1,\alpha_2) = \delta((x,y)-(x_n(t,\alpha_1,\alpha_2), y_n(t,\alpha_1,\alpha_2))) \\ m_n(t,x,y,\alpha_1,\alpha_2) = u_n(x,y)\delta((x,y)-(x_n(t,\alpha_1,\alpha_2), y_n(t,\alpha_1,\alpha_2))). \end{cases}$$

Then,

$$\langle\varphi, c_n\rangle \to \frac{1}{2\pi}\int_{D_3}\int_0^T \varphi(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)\,dt\,d\alpha_1\,d\beta_2\,d\gamma$$

and $\langle\phi, m_n\rangle \to$

$$\frac{1}{2\pi}\int_{D_3}\int_0^T \phi^1(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\,dt\,d\alpha_1\,d\beta_2\,d\gamma,$$

whenever $n\to\infty$, for any $\varphi \in C_0^\infty(D_4\times(0,T))$ and $\phi \in \left(C_0^\infty(D_4\times(0,T))\right)^2$.

Proof. Let $\varphi \in C_0^\infty(D_4\times(0,T))$ and

$$c_n(t,x,y,\alpha_1,\alpha_2) = \delta((x,y)-(x_n,y_n)(t,\alpha_1,\alpha_2))$$

then, we have

$$\begin{aligned} \langle\varphi, c_n\rangle &= \int_{D_2}\int_0^T \varphi(x_n(t,\alpha_1,\alpha_2), y_n(t,\alpha_1,\alpha_2),\alpha_1,\alpha_2,t)\,dt\,d\alpha_1\,d\alpha_2 \\ &= \sum_{i=0}^{n-1}\int_{\frac{2\pi}{n}i}^{\frac{2\pi}{n}(i+1)}\int_0^{2\pi}\int_0^T \varphi\left(x_n^i(t,\alpha_1,\alpha_2), y_n^i(t,\alpha_1,\alpha_2),\alpha_1,\alpha_2,t\right)dt\,d\alpha_1\,d\alpha_2 \\ &= \sum_{i=0}^{n-1}\int_{\frac{2\pi}{n}i}^{\frac{2\pi}{n}(i+1)}\int_0^{2\pi}\int_0^T \varphi\left(x\left(t,\alpha_1,n\left(\alpha_2-\frac{2\pi i}{n}\right)\right),\right. \\ &\quad \left.\frac{1}{n}y\left(t,\alpha_1,n\left(\alpha_2-\frac{2\pi i}{n}\right)\right)+\frac{2\pi i}{n},\alpha_1,\alpha_2,t\right)dt\,d\alpha_1\,d\alpha_2 \end{aligned}$$

Now, make $\beta_2 = n\left(\alpha_2 - \dfrac{2\pi i}{n}\right) = n\alpha_2 - 2\pi i$, namely, $\alpha_2 = \dfrac{\beta_2}{n} + \dfrac{2\pi i}{n}$. Then, we have

$$\begin{aligned}
\langle \varphi, c_n \rangle &= \sum_{i=0}^{n-1} \int_{D_2} \int_0^T \varphi\left(x(t,\alpha_1,\beta_2), \frac{1}{n} y(t,\alpha_1,\beta_2) + \frac{2\pi i}{n},\right. \\
&\quad \left. \alpha_1, \frac{\beta_2}{n} + \frac{2\pi i}{n}, t\right) dt\, d\alpha_1 \frac{d\beta_2}{n} \\
&= A_n + B_n,
\end{aligned}$$

where

$$A_n = \frac{1}{2\pi} \sum_{i=0}^{n-1} \left(\int_{D_2} \int_0^T \varphi\left(x(t,\alpha_1,\beta_2), \frac{2\pi i}{n}, \alpha_1, \frac{2\pi i}{n}, t\right) dt\, d\alpha_1 d\beta_2 \right) \frac{2\pi}{n} \tag{3.5}$$

and

$$\begin{aligned}
B_n &= \sum_{i=0}^{n-1} \frac{1}{n} \int_{D_2} \int_0^T \left[\varphi\left(x(t,\alpha_1,\beta_2), \frac{1}{n} y(t,\alpha_1,\beta_2) + \frac{2\pi i}{n}, \alpha_1, \frac{\beta_2}{n} + \frac{2\pi i}{n}, t\right) \right. \\
&\quad \left. -\varphi\left(x(t,\alpha_1,\beta_2), \frac{2\pi i}{n}, \alpha_1, \frac{2\pi i}{n}, t\right)\right] dt\, d\alpha_1 d\beta_2.
\end{aligned}$$

Note that, by Mean Value Theorem, we have

$$\begin{aligned}
&\left|\varphi\left(x(t,\alpha_1,\beta_2), \frac{1}{n} y(t,\alpha_1,\beta_2) + \frac{2\pi i}{n}, \alpha_1, \frac{\beta_2}{n} + \frac{2\pi i}{n}, t\right) - \right. \\
&\left. -\varphi\left(x(t,\alpha_1,\beta_2), \frac{2\pi i}{n}, \alpha_1, \frac{2\pi i}{n}, t\right)\right| = \left| \frac{\partial \varphi}{\partial y} \frac{1}{n} y(t,\alpha_1,\alpha_2) + \frac{\partial \varphi}{\partial \beta_2} \frac{\beta_2}{n} \right| \leq \\
&\leq \left(\left\| \frac{\partial \varphi}{\partial y} \right\|_{L^\infty((0,2\pi)^4 \times (0,T))} \frac{|y(t,\alpha_1,\alpha_2)|}{n} + \left\| \frac{\partial \varphi}{\partial \beta_2} \right\|_{L^\infty((0,2\pi)^4 \times (0,T))} \frac{|\beta_2|}{n} \right) \leq \\
&\leq \frac{2\pi}{n} \left(\left\| \frac{\partial \varphi}{\partial y} \right\|_{L^\infty} + \left\| \frac{\partial \varphi}{\partial \beta_2} \right\|_{L^\infty} \right) \leq \frac{2\pi}{n} \parallel D\varphi \parallel_{L^\infty}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
|B_n| &\leq \sum_{i=0}^{n-1} \frac{1}{n} \int_{D_2} \int_0^T \left| \left[\varphi\left(x(t,\alpha_1,\beta_2), \frac{1}{n} y(t,\alpha_1,\beta_2) + \frac{2\pi i}{n}, \alpha_1, \frac{\beta_2}{n} + \frac{2\pi i}{n}, t\right) \right.\right. \\
&\quad \left.\left. -\varphi\left(x(t,\alpha_1,\beta_2), \frac{2\pi i}{n}, \alpha_1, \frac{2\pi i}{n}, t\right)\right] \right| dt\, d\alpha_1 d\beta_2 \\
&\leq \sum_{i=0}^{n-1} \frac{1}{n} \frac{2\pi}{n} \parallel D\varphi \parallel_{L^\infty} \int_0^{2\pi} \int_0^{2\pi} \int_0^T dt\, d\alpha_1 d\beta_2 \\
&= \frac{2\pi}{n} \parallel D\varphi \parallel_{L^\infty} 4\pi^2 T - \frac{2\pi}{n^2} \parallel D\varphi \parallel_{L^\infty} 4\pi^2 T \\
&\leq \frac{2\pi}{n} \parallel D\varphi \parallel_{L^\infty} 4\pi^2 T.
\end{aligned}$$

Therefore, $B_n \to 0$, when $n \to \infty$.

Now, define the function ψ by

$$\psi(\gamma) := \int_{D_2} \int_0^T \varphi(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)dtd\alpha_1 d\beta_2$$

thus, we rewrite (3.5) as

$$A_n = \frac{1}{2\pi}\sum_{i=0}^{n-1}\psi\left(\frac{2\pi i}{n}\right)\frac{2\pi}{n} = \frac{1}{2\pi}\sum_{i=0}^{n-1}\psi\left(\frac{2\pi i}{n}\right)\left(\frac{2\pi(i+1)}{n} - \frac{2\pi i}{n}\right).$$

By this form, if $\gamma_i = \dfrac{2\pi i}{n}$ then, $\{\gamma_0, \gamma_1, \cdots, \gamma_n\}$ is a partition of the $(0,2\pi)$ and $A_n = \dfrac{1}{2\pi}\sum_{i=0}^{n-1}\psi(\gamma_i)(\gamma_{i+1}-\gamma_i)$ is a Riemann sum. Therefore,

$$A_n \to \frac{1}{2\pi}\int_0^{2\pi}\psi(\gamma)d\gamma,$$

when $n \to \infty$. Then, we conclude that

$$\langle \varphi, c_n\rangle \to \frac{1}{2\pi}\int_{D_3}\int_0^T \varphi(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)dtd\alpha_1 d\beta_2 d\gamma,$$

when $n \to \infty$, for any $\varphi \in C_0^\infty(D_4 \times (0,T))$.

Now, consider $\phi \in \left(C_0^\infty(D_4 \times (0,T))\right)^2$ and let

$$\begin{aligned} m_n &= u_n(x,y)\delta((x,y)-(x_n(t,\alpha_1,\alpha_2),y_n(t,\alpha_1,\alpha_2))) \\ &= \left(u_1^1(x,ny), \frac{1}{n}u_1^2(x,ny)\right)\delta((x,y)-(x_n(t,\alpha_1,\alpha_2),y_n(t,\alpha_1,\alpha_2))). \end{aligned}$$

Then, we obtain

$$\begin{aligned} \langle \phi, m_n\rangle &= \int_{D_2}\int_0^T \phi(x_n(t,\alpha_1,\alpha_2),y_n(t,\alpha_1,\alpha_2),\alpha_1,\alpha_2,t)\Big[u_1^1(x_n(t,\alpha_1,\alpha_2), \\ &\quad ny_n(t,\alpha_1,\alpha_2)), \frac{1}{n}u_1^2(x_n(t,\alpha_1,\alpha_2),ny_n(t,\alpha_1,\alpha_2))\Big]dtd\alpha_1 d\alpha_2 \\ &= \sum_{i=0}^{n-1}\int_{\frac{2\pi}{n}i}^{\frac{2\pi}{n}(i+1)}\int_0^{2\pi}\int_0^T \phi(x_n^i(t,\alpha_1,\alpha_2),y_n^i(t,\alpha_1,\alpha_2),\alpha_1,\alpha_2,t) \\ &\quad \Big[u_1^1(x_n^i(t,\alpha_1,\alpha_2),ny_n^i(t,\alpha_1,\alpha_2)), \frac{1}{n}u_1^2(x_n^i(t,\alpha_1,\alpha_2), \\ &\quad ny_n^i(t,\alpha_1,\alpha_2))\Big]dtd\alpha_1 d\alpha_2, \end{aligned}$$

and therefore, making $\beta_2 = n\left(\alpha_2 - \dfrac{2\pi i}{n}\right)$ we have

$$\begin{aligned}
\langle \phi, m_n\rangle &= \sum_{i=0}^{n-1}\int_{D_2}\int_0^T \frac{1}{n}\phi^1\left(x(t,\alpha_1,\beta_2),\frac{1}{n}y(t,\alpha_1,\alpha_2)+\frac{2\pi i}{n},\alpha_1,\frac{\beta_2}{n}+\frac{2\pi i}{n},t\right)\\
&\quad u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))dt\,d\alpha_1 d\beta_2+\\
&\quad +\sum_{i=0}^{n-1}\int_{D_2}\int_0^T \frac{1}{n^2}\phi^2\left(x(t,\alpha_1,\beta_2),\frac{1}{n}y(t,\alpha_1,\alpha_2)+\frac{2\pi i}{n},\alpha_1,\frac{2\pi i}{n},t\right)\\
&\quad u_1^2(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))dt\,d\alpha_1 d\beta_2\\
&= A_n^1+B_n^1+A_n^2+B_n^2,
\end{aligned}$$

where,

$$\begin{aligned}
A_n^1 &= \frac{1}{2\pi}\sum_{i=0}^{n-1}\Big(\int_{D_2}\int_0^T \frac{1}{n}\phi^1\left(x(t,\alpha_1,\beta_2),\frac{2\pi i}{n},\alpha_1,\frac{2\pi i}{n},t\right)\\
&\quad u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))dt\,d\alpha_1 d\beta_2\Big)2\pi,
\end{aligned}$$

$$\begin{aligned}
A_n^2 &= \frac{1}{2\pi n}\sum_{i=0}^{n-1}\Big(\int_{D_2}\int_0^T \frac{1}{n}\phi^2\left(x(t,\alpha_1,\beta_2),\frac{2\pi i}{n},\alpha_1,\frac{2\pi i}{n},t\right)\\
&\quad u_1^2(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))dt\,d\alpha_1 d\beta_2\Big)2\pi,
\end{aligned}$$

$$\begin{aligned}
B_n^1 &= \sum_{i=0}^{n-1}\int_{D_2}\int_0^T \frac{1}{n}\left[\phi^1\left(x(t,\alpha_1,\beta_2),\frac{1}{n}y(t,\alpha_1,\beta_2)+\frac{2\pi i}{n},\right.\right.\\
&\quad \left.\left.\alpha_1,\frac{\beta_2}{n}+\frac{2\pi i}{n},t\right)-\phi^1\left(x(t,\alpha_1,\beta_2),\frac{2\pi i}{n},\alpha_1,\frac{2\pi i}{n},t\right)\right]\\
&\quad u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))dt\,d\alpha_1 d\beta_2,
\end{aligned}$$

$$\begin{aligned}
B_n^2 &= \sum_{i=0}^{n-1}\int_{D_2}\int_0^T \frac{1}{n^2}\left[\phi^2\left(x(t,\alpha_1,\beta_2),\frac{1}{n}y(t,\alpha_1,\beta_2)+\frac{2\pi i}{n},\alpha_1,\frac{\beta_2}{n}+\frac{2\pi i}{n},t\right)\right.\\
&\quad \left.-\phi^2\left(x(t,\alpha_1,\beta_2),\frac{2\pi i}{n},\alpha_1,\frac{2\pi i}{n},t\right)\right]u_1^2(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))dt\,d\alpha_1 d\beta_2,
\end{aligned}$$

As we seen before, we conclude that

$$\left|\phi^i\left(x(t,\alpha_1,\beta_2),\frac{1}{n}y(t,\alpha_1,\beta_2)+\frac{2\pi i}{n},\alpha_1,\frac{\beta_2}{n}+\frac{2\pi i}{n},t\right)-\right.$$

$$\left.-\phi^i\left(x(t,\alpha_1,\beta_2),\frac{2\pi i}{n},\alpha_1,\frac{2\pi i}{n},t\right)\right| \le \frac{2\pi}{n}\parallel D\phi\parallel_{L^\infty},\quad i=1,2.$$

Thus, we have the estimate

$$
\begin{aligned}
|B_n^1| &\leq \sum_{i=0}^{n-1}\frac{1}{n}\int_{D_2}\int_0^T \left|\phi^1\left(x(t,\alpha_1,\beta_2),\frac{1}{n}y(t,\alpha_1,\beta_2)+\frac{2\pi i}{n},\alpha_1,\frac{\beta_2}{n}+\frac{2\pi i}{n},t\right)\right.\\
&\left.-\phi^1\left(x(t,\alpha_1,\beta_2),\frac{2\pi i}{n},\alpha_1,\frac{2\pi i}{n},t\right)\right||u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))|dt d\alpha_1 d\beta_2,\\
&\leq \frac{2\pi}{n}\parallel D\phi\parallel_{L^\infty} 4\pi^2 TC.
\end{aligned}
$$

Therefore, $B_n^1 \to 0$, when $n\to\infty$. Now, we go to study the term B_n^2.

$$
\begin{aligned}
|B_n^2| &\leq \sum_{i=0}^{n-1}\frac{1}{n^2}\int_{D_2}\int_0^T \left|\phi^2\left(x(t,\alpha_1,\beta_2),\frac{1}{n}y(t,\alpha_1,\beta_2)+\frac{2\pi i}{n},\alpha_1,\frac{\beta_2}{n}+\frac{2\pi i}{n},t\right)\right.\\
&\left.-\phi^2\left(x(t,\alpha_1,\beta_2),\frac{2\pi i}{n},\alpha_1,\frac{2\pi i}{n},t\right)\right||u_1^2(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))|dt d\alpha_1 d\beta_2\\
&\leq \frac{2\pi}{n^2}\parallel D\phi\parallel_{L^\infty} 4\pi^2 TC,
\end{aligned}
$$

and, therefore, also $B_n^2 \to 0$, when $n\to\infty$.

Defining the function ψ by

$$
\psi^i(\gamma) := \int_{D_2}\int_0^T \phi^i(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)u_1^i(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))dt d\alpha_1 d\beta_2,
$$

$i=1,2$, we obtain,

$$
\begin{aligned}
A_n^1 &= \frac{1}{2\pi}\sum_{i=0}^{n-1}\psi^1\left(\frac{2\pi i}{n}\right)\frac{2\pi}{n}\\
&= \frac{1}{2\pi}\sum_{i=0}^{n-1}\psi^1\left(\frac{2\pi i}{n}\right)\left(\frac{2\pi(i+1)}{n}-\frac{2\pi i}{n}\right).
\end{aligned}
$$

By this form, if $\gamma_i = \dfrac{2\pi i}{n}$ then, $\{\gamma_0,\gamma_1,\cdots,\gamma_n\}$ is a partition of the $(0,2\pi)$ and $A_n^1 = \dfrac{1}{2\pi}\sum_{i=0}^{n-1}\psi^1(\gamma_i)(\gamma_{i+1}-\gamma_i)$ is a Riemann sum. Therefore,

$$
A_n^1 \to \frac{1}{2\pi}\int_0^{2\pi}\psi^1(\gamma)d\gamma,\ \text{ when n}\to\infty.
$$

For the last term, we have that

$$
A_n^2 = \frac{1}{2\pi n}\sum_{i=0}^{n-1}\psi^2\left(\frac{2\pi i}{n}\right)\frac{2\pi}{n}
$$

$$= \quad \frac{1}{2\pi n}\sum_{i=0}^{n-1}\psi^2\left(\frac{2\pi i}{n}\right)\left(\frac{2\pi(i+1)}{n}-\frac{2\pi i}{n}\right),$$

and therefore, $A_n^2 \to 0$, when $n \to \infty$.

Then, we conclude that

$\langle \phi, m_n\rangle \to$

$$\frac{1}{2\pi}\int_{D_3}\int_0^T \phi^1(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))dtd\alpha_1 d\beta_2 d\gamma,$$

when $n \to \infty$, for any $\phi \in \left(C_0^\infty(D_4 \times (0,T))\right)^2$. □

By the last theorem we can conclude that

$$\langle \varphi, c_n\rangle \to \frac{1}{2\pi}\int_{D_3}\int_0^T \varphi(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)dtd\alpha_1 d\beta_2 d\gamma$$

and $\langle \phi, m_n\rangle \to$

$$\frac{1}{2\pi}\int_{D_3}\int_0^T \phi^1(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))dtd\alpha_1 d\beta_2 d\gamma,$$

whenever $n \to \infty$, for any

$$\varphi \in C_0^\infty(D_4 \times (0,T)) \text{ and } \phi \in \left(C_0^\infty(D_4 \times (0,T))\right)^2.$$

Now, note that

$$2\pi\,\langle \varphi, c_n\rangle \to \int_{D_3}\int_0^T \varphi(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)dtd\alpha_1 d\beta_2 d\gamma$$

is equivalent to

$$\int_0^{2\pi}\langle \varphi, c_n\rangle\, d\beta_2 \to \int_{D_3}\int_0^T \varphi(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)dtd\alpha_1 d\gamma d\beta_2$$

and, therefore,

$$\int_0^{2\pi}\left[\langle \varphi, c_n\rangle - \int_{D_2}\int_0^T \varphi(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)dtd\alpha_1 d\gamma\right]d\beta_2 \to 0.$$

Then, we conclude that

$$\langle \varphi, c_n\rangle \to \int_{D_2}\int_0^T \varphi(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)dtd\alpha_1 d\gamma.$$

Of completely analogous way, we also conclude that

$\langle \phi, m_n \rangle \to$

$$\int_{D_2}\int_0^T \phi^1(x(t,\alpha_1,\beta_2),\gamma,\alpha_1,\gamma,t)u_1^1\left(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2)\right)dtd\alpha_1 d\gamma.$$

Thus, the limite (c,m) is given by

$$\begin{cases} c(x,y,\alpha_1,\alpha_2,t) = \delta\left((x,\alpha_2)-(x(t,\alpha_1,\beta_2),y)\right) \\ m(x,y,\alpha_1,\alpha_2,t) = \delta\left((x,\alpha_2)-(x(t,\alpha_1,\beta_2),y)\right)\left(u_1^1(x,y(t,\alpha_1,\beta_2)),0\right). \end{cases} \tag{3.6}$$

4 Solution to the Relaxed Euler Equations

In this section we will conclude our work showing that the pair (c,m), build in the before section, satisfy the relaxed Euler equations.

Theorem 4.1. *The pair of measures (c,m) defined in (3.6) satisfy the following equations*

$$\int_{D_2} c(t,x,y,\alpha_1,\alpha_2)d\alpha_1 d\alpha_2 = 1, \quad \partial_t c + \nabla\cdot m = 0, \quad \partial_t(cv)+\nabla\cdot(cv\otimes v)+c\nabla p = 0.$$

in the sense of distributions.

Proof. Note that

$$\langle 1, c\rangle = \int_{D_2}\int_0^T dtd\alpha_1 d\alpha_2 = \int_{D_2}\int_0^T dtdxdy.$$

Then, we obtain

$$\int_{D_2}\int_0^T \left(\int_{D_2} c(t,x,y,\alpha_1,\alpha_2)d\alpha_1 d\alpha_2 - 1\right)dtdxdy = 0$$

and, therefore, $\int_{D_2} c(t,x,y,\alpha_1,\alpha_2)d\alpha_1 d\alpha_2 = 1.$

Now, we will show that pair (c,m) satisfy the equation $\partial_t c + \nabla\cdot m = 0$. Consider $\varphi \in C_0^\infty(D_4\times(0,T))$, thus

$$\begin{aligned} &\langle \varphi(x,y,\alpha_1,\alpha_2,t), \partial_t c(t,x,y,\alpha_1,\alpha_2) + \nabla_{(x,y)}\cdot m(t,x,y,\alpha_1,\alpha_2)\rangle = \\ = \quad & -\int_{D_2}\int_0^T \partial_t\varphi(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy \\ & -\int_{D_2}\int_0^T \partial_x\varphi(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)u_1^1\left(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2)\right)dtd\alpha_1 dy \end{aligned}$$

$$
\begin{aligned}
&= -\int_{D_2}\int_0^T \partial_t\left(\varphi(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)\right)dtd\alpha_1 dy\\
&= -\int_{D_2}\left(\varphi(\alpha_1,y,\alpha_1,y,T)-\varphi(\beta_2,y,\alpha_1,y,0)\right)d\alpha_1 dy = 0.
\end{aligned}
$$

Finally, we will show that the pair (c,m) also satisfy the equation $\partial_t(cv)+\nabla\cdot(cv\otimes v)+c\nabla p = 0$. First note that $\left[div_{(x,y)}(u\otimes u)\right]^i = u^i div_{(x,y)}u + u\cdot\nabla_{(x,y)}u^i = u\cdot\nabla_{(x,y)}u^i$, because $div_{(x,y)}u = 0$. Let $\varphi\in\left(C_0^\infty(D_4\times(0,T))\right)^2$, then

$$
\begin{gathered}
\langle\varphi^1(x,y,\alpha_1,\alpha_2,t),\partial_t\left(c(t,x,y,\alpha_1,\alpha_2)v^1(x,y,\alpha_1,\alpha_2,t)\right\rangle +\\
+\langle\varphi^1(x,y,\alpha_1,\alpha_2,t),\nabla_{(x,y)}\cdot(c(t,x,y,\alpha_1,\alpha_2)v(x,y,\alpha_1,\alpha_2,t)\otimes v(x,y,\alpha_1,\alpha_2,t))^1\rangle\\
+\langle\varphi^1(x,y,\alpha_1,\alpha_2,t),c(x,y,\alpha_1,\alpha_2,t)\partial_x p(x,y)\rangle =
\end{gathered}
$$

$$
\begin{aligned}
&= -\int_{D_2}\int_0^T u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\partial_t\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy\\
&\quad -\int_{D_2}\int_0^T \left[u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\right]^2\partial_x\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy\\
&\quad +\int_{D_2}\int_0^T \partial_x p(x(t,\alpha_1,\beta_2),y)\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy\\
&= -\int_{D_2}\int_0^T u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\partial_t\left[\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)\right]dtd\alpha_1 dy\\
&\quad +\int_{D_2}\int_0^T \left[u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\right]^2\partial_x\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy\\
&\quad -\int_{D_2}\int_0^T \left[u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\right]^2\partial_x\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy\\
&\quad +\int_{D_2}\int_0^T \partial_x p(x(t,\alpha_1,\beta_2),y)\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy\\
&= \int_{D_2}\int_0^T \partial_t\left[u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\right]\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy\\
&\quad +\int_{D_2}\int_0^T \partial_x p(x(t,\alpha_1,\beta_2),y)\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy\\
&= \int_{D_2}\int_0^T \left[\partial_x u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\partial_t x(t,\alpha_1,\beta_2)\right.\\
&\quad \left.+\partial_y u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\partial_t y(t,\alpha_1,\beta_2)\right]\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy\\
&\quad +\int_{D_2}\int_0^T \partial_x p(x(t,\alpha_1,\beta_2),y)\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy\\
&= \int_{D_2}\int_0^T \nabla_{(x,y)}u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))
\end{aligned}
$$

$$u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy$$
$$+\int_{D_2}\int_0^T \partial_x p(x(t,\alpha_1,\beta_2),y)\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy.$$

Since that $p(x)=-\dfrac{1}{4}\cos(2x)$ we have that $\partial_x p=\dfrac{1}{2}\sin(2x)$ and then,

$$\partial_x p(x(t,\alpha_1,\beta_2),y)=\partial_x p(x(t,\alpha_1,\beta_2),y(x(t,\alpha_1,\beta_2)).$$

Therefore, we conclude that

$$\begin{gathered}\langle\varphi^1(x,y,\alpha_1,\alpha_2,t),\partial_t\left(c(t,x,y,\alpha_1,\alpha_2)v^1(x,y,\alpha_1,\alpha_2,t)\right\rangle+\\ +\langle\varphi^1(x,y,\alpha_1,\alpha_2,t),\nabla_{(x,y)}\cdot(c(t,x,y,\alpha_1,\alpha_2)v(x,y,\alpha_1,\alpha_2,t)\otimes v(x,y,\alpha_1,\alpha_2,t))^1\rangle+\\ +\langle\varphi^1(x,y,\alpha_1,\alpha_2,t),c(x,y,\alpha_1,\alpha_2,t)\partial_x p(x,y)\rangle=\end{gathered}$$

$$\begin{gathered}=\int_{D_2}\int_0^T\left[\nabla_{(x,y)}u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))u_1^1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\right.\\ \left.+\partial_x p(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))\right]\varphi^1(x(t,\alpha_1,\beta_2),y,\alpha_1,y,t)dtd\alpha_1 dy.\end{gathered}$$

Now, note that $u_1^1(x,ny)=-\cos(x)\sin(ny)$ and $u_1^2(x,ny)=\sin(x)\cos(ny)$, namely, $u_1^1(x,z)=-\cos(x)\sin(z)$ and $u_1^2(x,z)=\sin(x)\cos(z)$, where $z=ny$. Then,

$$\nabla u_1^1\cdot u_1=-\sin(x)\cos(x)=-\frac{1}{2}\sin(2x),$$

and, therefore,

$$\nabla u_1^1\cdot u_1+\partial_x p=-\frac{1}{2}\sin(2x)+\frac{1}{2}\sin(2x)=0.$$

Then,

$$\begin{gathered}\nabla_{(x,y)}u_1^1\left(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2)\right)\cdot u_1(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))+\\ +\partial_x p(x(t,\alpha_1,\beta_2),y(t,\alpha_1,\beta_2))=0\end{gathered}$$

and we can conclude that the pair of measures $(c,m=cv)$ satisfy

$$\partial_t(cv)+\nabla\cdot(cv\otimes v)+c\nabla p=0.$$

□

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Measure of Noncompactness and Nondensely Defined Semilinear Functional Differential Equations with Fractional Order

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ABSTRACT

This paper is devoted to study the existence of integral solutions for a nondensely defined semilinear functional differential equations involving the Riemann-Liouville derivative in

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a Banach space. The arguments are based upon Mönch's fixed point theorem and the technique of measures of noncompactness.

RESUMEN

Este artículo es dedicado al estudio de existencia de soluciones integrales para ecuaciones diferenciales funcionales semilineales envolviendo la derivada de Riemann-Liouville en un espacio de Banach. Los argumentos se basan en un teorema de punto fijo de Mönch y la técnica de no compacidad.

Key words and phrases: *Partial differential equations, fractional derivative, fractional integral, fixed point, semigroups, integral solutions, finite delay, measure of noncompactness, fixed point, Banach space.*

Math. Subj. Class.: *34G20, 34G25, 26A33, 26A42.*

1 Introduction

The theory of functional differential equations has emerged as an important branch of nonlinear analysis. It is worthwhile mentioning that several important problems of the theory of ordinary and delay differential equations lead to investigations of functional differential equations of various types, see the books of Hale and Verduyn Lunel [22], Kolmanovskii and Myshkis [26], Wu [42], and the references therein. On the other hand the theory of fractional differential equations is also intensively studied and finds numerous applications in describing real world problems (see for instance the monographs of Lakshmikantham *et al.* [27], Kilbas *et al.* [25], Miller and Ross [31], Podlubny [39], Samko *et al.* [40], and the papers of Agarwal *et al.* [1], Benchohra *et al.* [11, 12], Chang and Nieto [14], Diethelm *et al.* [16], Furati and Tatar [17, 18], Gaul *et al.* [19], Glockle and Nonnenmacher [20], Lakshmikantham and Devi [28], Mainardi [29], Metzler *et al.* [30], N'Guérékata et al [33, 34, 35], and the references therein). Jaradat *et al.* [23], studied the existence and uniqueness of mild solutions for a class of initial value problem for a semilinear integrodifferential equation involving the Caputo's fractional derivative.

In this paper we will examine the following semilinear functional differential equation of fractional order

$$D^r y(t) = Ay(t) + f(t, y_t), \quad t \in J = [0, b], \quad r > 0 \tag{1}$$

$$y(t) = \phi(t), \quad t \in [-\rho, 0], \tag{2}$$

where D^r is the standard Riemann-Liouville fractional derivative, $f : J \times C([-\rho, 0], E) \to E$ is a given function, $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator on E.

$\phi : [-\rho, 0] \to E$ a given continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a Banach space. For any function y defined on $[-\rho, b]$ and any $t \in J$ we denote by y_t the element of $C([-\rho, 0], E)$ defined by

$$y_t(\theta) = y(t+\theta),\ \theta \in [-\rho, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t-\rho$, up to the present time t.

Let us mention that the functional differential equation of the type (1) was investigated, in the case when A generates a C_0−semigroup, in a lot of papers and developed with the help of various tools of fixed-point theory see, for instance Belmekki *et al.* [8, 9, 10].

The principal goal of this paper is to extend such results to the case when the operator A is nondensely defined and satisfies the Hille-Yosida condition, and to initiate the application of the technique of measures of noncompactness to investigate the problem of the existence of integral solutions for (1)–(2). Especially that technique combined with an appropriate fixed point theorem has proved to be a very useful tool in the study of the existence of solutions for several types of integral and differential equations; see for example Alvàrez [3], Banaś *et al.* [5, 6, 7], Benchohra *et al.* [13], Guo *et al.* [21], Mönch [32], Mönch and Von Harten [37], and Szufla [41].

2 Preliminaries

In this section we collect some definitions, notations and results needed in the sequel. At first, we recall the definition of Riemann-Liouville fractional primitive and fractional derivative.

Denote by $C(J,E)$ the Banach space of continuous functions $J \to E$, with the usual supremum norm

$$\|y\|_\infty = \sup\{|y(t)|, \quad t \in J\}.$$

For $\psi \in C([-\rho, 0], E)$ the norm of ψ is defined by

$$\|\psi\|_{\mathscr{C}} = \sup\{|\psi(\theta)|,\ \theta \in [-\rho, 0]\}.$$

$B(E)$ denotes the Banach space of all bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = sup\{|N(y)| : |y| = 1\}.$$

Let $L^1(J,E)$ be the Banach space of measurable functions $y : J \to E$ which are Bochner integrable, equipped with the norm

$$\|y\|_{L^1} = \int_J |y(t)|dt.$$

Let $L^\infty(J,E)$ be the Banach space of measurable functions $y : J \to E$ which are bounded, equipped with the norm

$$\|y\|_{L^\infty} = \inf\{c > 0 : \|y(t)\| \le c,\ a.e.\ t \in J\}.$$

For a given set V of functions $v:[-\rho,b]\to E$, let us denote by

$$V(t)=\{v(t):v\in V\},\ t\in[-\rho,b]$$

and

$$V(J)=\{v(t):v\in V,\ t\in[-\rho,b]\}.$$

Definition 2.1. *([25, 39]). The Riemann-Liouville fractional primitive of order $r>0$ of a function $h:(0,b]\to E$ is defined by*

$$I_0^r h(t)=\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}h(s)ds,$$

provided the right side is pointwise defined on $(0,b]$, and where Γ is the gamma function.

Definition 2.2. *([25, 39]). The Riemann-Liouville fractional derivative of order $r\in(0,1]$ of a continuous function $h:(0,b]\to E$ is defined by*

$$\begin{aligned}\frac{d^r h(t)}{dt^r} &= \frac{1}{\Gamma(1-r)}\frac{d}{dt}\int_0^t (t-s)^{-r}h(s)ds\\ &= \frac{d}{dt}I_0^{1-r}h(t).\end{aligned}$$

Definition 2.3. *A map $f:J\times C([-\rho,0],E)\to E$ is said to be Carathéodory if*

(i) $t\longmapsto f(t,u)$ *is measurable for each* $u\in C([-\rho,0],E)$;

(ii) $u\longmapsto F(t,u)$ *is continuous for almost each* $t\in J$.

For completeness we gather some definitions and basic facts of integrated semigroups, and operators satisfying Hille-Yosida condition.

Definition 2.4. *[4]. Let E be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on E with the following properties:*

(i) $S(0)=0$;

(ii) $t\to S(t)$ *is strongly continuous;*

(iii) $S(s)S(t)=\displaystyle\int_0^s (S(t+\tau)-S(\tau))d\tau$, *for all* $t,s\geq 0$.

Definition 2.5. *[24]. An operator A is called a generator of an integrated semigroup if there exists $\omega\in\mathbb{R}$ such that $(\omega,\infty)\subset\rho_0(A)$ ($\rho_0(A)$, is the resolvent set of A) and there exists a strongly continuous exponentially bounded family $(S(t))_{t\geq 0}$ of bounded operators such that $S(0)=0$ and $R(\lambda,A):=(\lambda I-A)^{-1}=\lambda\int_0^\infty e^{-\lambda t}S(t)dt$ exists for all λ with $\lambda>\omega$.*

Proposition 2.1. *[4]. Let A be the generator of an integrated semigroup $(S(t))_{t\geq 0}$. Then for all $x \in E$ and $t \geq 0$,*

$$\int_0^t S(s)x\,ds \in D(A) \ \textit{and} \ S(t)x = A\int_0^t S(s)x\,ds + tx.$$

Definition 2.6. *We say that the linear operator A satisfies the Hille-Yosida condition if there exists $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho_0(A)$ and*

$$\sup\{(\lambda-\omega)^n |(\lambda I - A)^{-n}| : n \in \mathbb{N},\ \lambda > \omega\} \leq M.$$

Definition 2.7. *[24].*

(i) An integrated semigroup $(S(t))_{t\geq 0}$ is called locally Lipschitz continuous if, for all $\tau > 0$, there exists a constant L such that

$$|S(t) - S(s)| \leq L|t-s|, \quad t, s \in [0, \tau].$$

(ii) An integrated semigroup $(S(t))_{t\geq 0}$ is called non degenerate if $S(t)x = 0$, for all $t \geq 0$, implies that $x = 0$.

Theorem 2.1. *[24]. The following assertions are equivalent:*

(i) A is the generator of a non degenerate, locally Lipschitz continuous integrated semigroup;

(ii) A satisfies the Hille-Yosida condition.

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [4], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$ and $(S'(t))_{t\geq 0}$ is a C_0−semigroup on $\overline{D(A)}$.

Let $(S(t))_{t\geq 0}$ be the integrated semigroup generated by A. We note that, if A satisfies the Hille-Yosida condition, then $\|S'(t)\|_{B(E)} \leq Me^{\omega t},\ t \geq 0$, where M and ω are the constants considered in the Hille-Yosida condition.

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.8. *([6]) Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0, \infty]$ defined by*

$$\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \ \textit{and}\ diam(B_i) \leq \epsilon\};\ \textit{here}\ B \in \Omega_E.$$

Properties: The Kuratowski measure of noncompactness satisfies the following properties (for more details see [6]).

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact).

(b) $\alpha(B) = \alpha(\overline{B})$.

(c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.

(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$

(e) $\alpha(cB) = |c|\alpha(B)$; $c \in \mathbb{R}$.

(f) $\alpha(convB) = \alpha(B)$.

Theorem 2.2. *([2, 32]) Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If the implication*

$$V = \overline{conv}N(V) \quad or \quad V = N(V) \cup \{0\} \Rightarrow \alpha(V) = 0$$

holds for every subset V of D, then N has a fixed point.

Lemma 2.1. *([41]) Let D be a bounded, closed and convex subset of the Banach space $C(J, E)$, G a continuous function on $J \times J$ and f a function from $J \times C([-\rho, 0], E) \to E$ which satisfies the Carathéodory conditions and there exists $p \in L^1(J, \mathbb{R}_+)$ such that for each $t \in J$ and each bounded set $B \subset C([-\rho, 0], E)$ we have*

$$\lim_{k \to 0^+} \alpha(f(J_{t,k} \times B)) \leq p(t)\alpha(B); \text{ here } J_{t,k} = [t - k, t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\alpha\left(\left\{\int_J G(s,t) f(s, y_s) ds : y \in V\right\}\right) \leq \int_J \|G(t,s)\| p(s)\alpha(V(s)) ds.$$

3 Main Results

We start with the following principal assumption and the definition of integral solutions to the problem (1)-(2).

(H1) A satisfies the Hille-Yosida condition.

Definition 3.1. *We say that a continuous function $y : [-\rho, b] \to E$ is an integral solution of problem (1)-(2) if*

(i) $\int_0^t (t-s)^{r-1} y(s)ds \in D(A)$ *for* $t \in J$,

(ii) $y(t) = \phi(t)$, $t \in [-\rho, 0]$, *and*

(iii) $y(t) = \dfrac{1}{\Gamma(r)} A \int_0^t (t-s)^{r-1} y(s)ds + \dfrac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} f(s, y_s)ds,\ t \in J.$

From the definition it follows that $y(t) \in \overline{D(A)}, \forall\ t \geq 0$. Moreover, y satisfies the following variation of constants formula:

$$y(t) = \frac{1}{\Gamma(r)} \frac{d}{dt} \int_0^t S(t-s)(t-s)^{r-1} f(s, y_s)ds, \quad t \geq 0. \tag{3}$$

Let $B_\lambda = \lambda R(\lambda, A)$, then for all $x \in \overline{D(A)}$, $B_\lambda x \mapsto x\ as\ \lambda \mapsto \infty$.
We notice also that, if y satisfies (3), then

$$y(t) = \lim_{\lambda \to \infty} \frac{1}{\Gamma(r)} \int_0^t S'(t-s)(t-s)^{r-1} B_\lambda f(s, y_s)ds, \quad t \geq 0.$$

Without lost of generality, we will assume that $w > 0$.

Let us list some conditions on the functions involved in the problem (1)-(2).

(H2) The operator $S'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$ and

$$\|S'(t)\|_{B(E)} \leq M e^{\omega t},\ t \in J.$$

(H3) $f : J \times C([-\rho, 0], E) \to E$ is of Carathéodory.

(H4) There exists a function $p \in L^\infty(J, \mathbb{R}_+)$ such that

$$|f(t,u)| \leq p(t)(\|u\|_C + 1), \text{ for a.e. } t \in J, \text{ and each } u \in C([-\rho, 0], E).$$

(H5) For almost each $t \in J$ and each bounded set $B \subset C([-\rho, 0], E)$ we have

$$\lim_{h \to 0^+} \alpha(f(J_{t,h} \times B)) \leq p(t)\alpha(B); \text{ here } J_{t,h} = [t-h, t] \cap J.$$

(H6) Assume

$$\frac{M b^r p^* e^{\omega b}}{\Gamma(r+1)} < 1.$$

Let $p^* = \|p\|_{L^\infty}$. Our main result reads as follows

Theorem 3.1. *Assume that assumptions* $(H1)-(H6)$ *hold. Then the the problem (1)-(2) has at least one integral solution.*

Proof. We shall reduce the existence of solutions of (1)-(2) to a fixed point problem. Consider the operator $N : C([-\rho,b],E) \to C([-\rho,b],E)$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & t \in [-\rho,0], \\ \dfrac{1}{\Gamma(r)}\dfrac{d}{dt}\displaystyle\int_0^t (t-s)^{r-1}S(t-s)f(s,y_s)ds, & t \in [0,b]. \end{cases}$$

Let $r_0 > 0$ be such that

$$r_0 \geq \frac{Mp^* b^r e^{\omega b}}{\Gamma(r+1) - Mb^r p^* e^{\omega b}},$$

and consider the set

$$D_{r_0} = \{y \in C([-\rho,b],E) : \|y\|_\infty \leq r_0\}.$$

Clearly, the subset D_{r_0} is closed, bounded and convex. We shall show that N satisfies the assumptions of Theorem 2.2. The proof will be given in three steps.

Step 1: *N is continuous.*

Let us consider a sequence $\{y_n\}$ such that $y_n \to y$ in $C([-\rho,b],E)$. Then for each $t \in J$

$$\begin{aligned} |N(y_n)(t) - N(y)(t)| &= \left|\frac{1}{\Gamma(r)}\frac{d}{dt}\int_0^t (t-s)^{r-1}S(t-s)[f(s,y_{n_s}) - f(s,y_s)]ds\right| \\ &\leq \frac{Me^{\omega t}}{\Gamma(r)}\int_0^t e^{-\omega s}(t-s)^{r-1}|f(s,y_{n_s}) - f(s,y_s)|ds \\ &\leq \frac{Me^{\omega b}}{\Gamma(r)}\int_0^t (t-s)^{r-1}|f(s,y_{n_s}) - f(s,y_s)|ds. \end{aligned}$$

Let $\mu > 0$ be such that

$$\|y_n\|_\infty \leq \mu, \ \|y\|_\infty \leq \mu.$$

By (H4) we have

$$|(t-s)^{r-1}[f(s,y_{n_s}) - f(s,y_s)]| \leq 2p^*(\mu+1)(t-s)^{r-1} \in L^1(J,\mathbb{R}_+).$$

Since f is a Carathéodory function, the Lebesgue dominated convergence theorem implies that

$$\|N(y_n) - N(y)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Step 2: *N maps D_{r_0} into itself.*

For each $y \in D_{r_0}$, by $(H4)$ and $(H6)$ we have for each $t \in J$

$$|N(y)(t)| = \left|\frac{1}{\Gamma(r)}\frac{d}{dt}\int_0^t (t-s)^{r-1}S(t-s)f(s,y_s)ds\right|$$

$$
\begin{aligned}
&\leq \frac{Mb^r p^* e^{\omega b}(r_0+1)}{\Gamma(r+1)} \\
&\leq r_0
\end{aligned}
$$

Step 3: *$N(D_{r_0})$ is bounded and equicontinuous.*

By Step 2, it is obvious that $N(D_{r_0}) \subset D_{r_0}$ is bounded.

For the equicontinuity of $N(D_{r_0})$. Let $\tau_1,\ \tau_2 \in J,\ \tau_1 < \tau_2$, thus if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have for any $y \in D_{r_0}$;

$$
\begin{aligned}
|N(y)(\tau_2) - N(y)(\tau_1)| &= \Bigg| \lim_{\lambda\mapsto\infty} \frac{1}{\Gamma(r)} \int_0^{\tau_2} (\tau_2 - s)^{r-1} S'(\tau_2 - s) B_\lambda f(s, y_s) ds \\
&- \lim_{\lambda\mapsto\infty} \frac{1}{\Gamma(r)} \int_0^{\tau_1} (\tau_1 - s)^{r-1} S'(\tau_1 - s) B_\lambda f(s, y_s) ds \Bigg| \\
&\leq Mp^*(r_0+1) \Bigg(\frac{1}{\Gamma(r)} \int_0^{\tau_1-\epsilon} [(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}] ds \\
&+ \|S'(\tau_2 - \tau_1 + \epsilon) - S'(\epsilon)\|_{B(E)} \left\{ \frac{1}{\Gamma(r)} \int_0^{\tau_1-\epsilon} (\tau_2 - s)^{r-1} ds \right\} \\
&+ \frac{1}{\Gamma(r)} \int_{\tau_1-\epsilon}^{\tau_1} [(\tau_2 - s)^{r-1} - (\tau_1 - s)^{r-1}] ds \\
&+ \|S'(\tau_2 - \tau_1) - I\|_{B(E)} \left\{ \frac{1}{\Gamma(r)} \int_{\tau_1-\epsilon}^{\tau_1} (\tau_2 - s)^{r-1} ds \right\} \\
&\quad + \frac{1}{\Gamma(r)} \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{r-1} ds \Bigg).
\end{aligned}
$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero, since $S'(t)$ is a strongly continuous operator and the compactness of $S'(t)$ for $t > 0$ implies the continuity in the uniform operator topology (see [38]).

Now let V be a subset of D_{r_0} such that $V \subset \overline{conv}(N(V) \cup \{0\})$.
V is bounded and equicontinuous and therefore the function $v \to v(t) = \alpha(V(t))$ is continuous on $[-\rho, b]$. By (H5), Lemmas 2.1 and the properties of the measure α we have for each $t \in [-\rho, b]$

$$
\begin{aligned}
v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \\
&\leq \alpha(N(V)(t)) \\
&\leq \lim_{\lambda\mapsto\infty} \frac{1}{\Gamma(r)} \int_0^t (t-s)^{r-1} S'(t-s) B_\lambda p(s) \alpha(V(s)) ds
\end{aligned}
$$

$$\begin{aligned} &\leq \frac{Mp^* e^{\omega t}}{\Gamma(r)} \int_0^t (t-s)^{r-1} v(s) ds \\ &\leq \|v\|_\infty \frac{Mb^r p^* e^{\omega b}}{\Gamma(r+1)}. \end{aligned}$$

This means that

$$\|v\|_\infty \left(1 - \frac{Mb^r p^* e^{\omega b}}{\Gamma(r+1)}\right) \leq 0.$$

By (H6) it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in [-\rho, b]$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzelà theorem, V is relatively compact in D_{r_0}. Applying now Theorem 2.2 we conclude that N has a fixed point which is an integral solution for the problem (1)-(2). □

4 An Example

As an application of our results we consider the following fractional time partial functional differential equation of the form

$$\frac{\partial^\alpha}{\partial t^\alpha} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) + Q(t, z(t-r,x)),\ x \in [0,\pi],\ t \in [0,1],\ \alpha \in (0,1], \tag{4}$$

$$z(t,0) = z(t,\pi) = 0,\ t \in [0,1] \tag{5}$$

$$z(t,x) = \varphi(t,x),\ t \in [-r,0],\ x \in [0,\pi], \tag{6}$$

where $r > 0$, $\varphi : [-r,0] \times [0,\pi] \to \mathbb{R}$ is continuous and $Q : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given function.

Let

$$y(t)(x) = z(t,x),\ t \in J,\ x \in [0,\pi],$$

$$\phi(\theta)(x) = \varphi(\theta,x),\ \theta \in [-r,0],\ x \in [0,\pi],$$

$$F(t,\phi)(x) = Q(t, \varphi(\theta,x)),\ \theta \in [-r,0],\ x \in [0,\pi].$$

We choose $E = C([0,\pi];\mathbb{R})$ endowed with the uniform topology and consider the operator $A : D(A) \subset E \to E$ defined by:

$$D(A) = \{y \in C^2([0,\pi],\mathbb{R}) : y(0) = y(\pi) = 0\} \quad Ay = y''.$$

It is well known (see [15]) that the operator A Satisfies the Hille-Yosida condition with $(0,+\infty) \subset \rho(A)$, $\|(\lambda I - A)^{-1}\| \leq \dfrac{1}{\lambda}$ for $\lambda > 0$, and

$$\overline{D(A)} = \{y \in E; y(0) = y(\pi) = 0\} \neq E.$$

It follows that A generates an integrated semigroup $(S(t))_{t\geq 0}$ and $\|S'(t)\| \leq e^{-\mu t}$ for $t \geq 0$ and for some constant $\mu > 0$. We can show that problem (1)-(2) is an abstract formulation of problem (4)-(6).

Assume that the function Q satisfies the following conditions

(i) The function $Q : J \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory.

(ii) $|Q(t,z)| \leq \dfrac{1}{e^{t+2}}(|z|+1)$ for each $(t,z) \in J \times \mathbb{R}$.

It is clear that conditions (H1)-(H4) are satisfied. We shall show that (H6) holds with

$$p(t) = \frac{1}{e^{t+2}},\ t \in [0,1],$$

$$M = 1,\ b = 1,\ p^* = \frac{1}{e^2}.$$

Indeed, we have

$$\frac{Mb^r p^* e^{\omega b}}{\Gamma(r+1)} \leq \frac{1}{e^2\Gamma(r+1)} < 1, \text{ for each } r \in (0,1].$$

Hence, Theorem 3.1 implies that problem (4)-(6) has an integral solution z on $[-r,1] \times [0,\pi]$.

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On The Group of Strong Symplectic Homeomorphisms

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ABSTRACT

We generalize the "hamiltonian topology" on hamiltonian isotopies to an intrinsic "symplectic topology" on the space of symplectic isotopies. We use it to define the group $SSympeo$ (M,ω) of strong symplectic homeomorphisms, which generalizes the group $Hameo(M,\omega)$ of hamiltonian homeomorphisms introduced by Oh and Müller. The group $SSympeo(M,\omega)$ is arcwise connected, is contained in the identity component of $Sympeo(M,\omega)$; it contains $Hameo(M,\omega)$ as a normal subgroup and coincides with it when M is simply connected. Finally its commutator subgroup $[SSympeo(M,\omega), SSympeo(M,\omega)]$ is contained in $Hameo(M,\omega)$.

RESUMEN

Generalizamos la "topología hamiltoniano" sobre isotopias hamiltonianas para una "topología simpléctica" intrinseca en el espacio de isotopias simplécticas. Nosotros usamos esto para definir el grupo $SSympeo(M,\omega)$ de homeomorfismos simplécticos fuertes, el qual generaliza el grupo $Hameo(M,\omega)$ de homeomorfismos hamiltonianos introducido por Oh y Müller. El grupo $SSympeo(M,\omega)$ es conexo por arcos, es contenido en la componente identidad de $Sympeo(H,\omega)$; este contiene $Hameo(M,\omega)$ como un subgrupo normal y coincide con este cuando M es simplemente conexa. Finalmente su subgrupo conmutador $[SSympeo(M,\omega), SSympeo(M,\omega)]$ es contenido en $Hameo(M,\omega)$.

Key words and phrases: *Hamiltonian homeomorphisms, hamiltonian topology, symplectic topology, stromg symplectic homeomorphisms, C^0 symplectic topology.*

Math. Subj. Class.: *MSC2000:53D05; 53D35.*

1 Introduction

No natural metric on the group $Symp(M,\omega)$ of symplectic diffeomorphisms of a symplectic manifold (M,ω) is known. In this paper we construct a "Hofer-like" metric, depending on several ingredients. However, we prove that all these metrics are equivalent and hence define a natural metric topology on $Symp(M,\omega)$ (theorem 1'). We use this natural topology on $Symp(M,\omega)$ to define a new group of symplectic homeomorphisms, herein called the group of strong symplectic homeomorphisms (Theorem 2). This group may carry a Calabi invariant.

The Eliashberg-Gromov symplectic rigidity theorem says that the group $Symp(M,\omega)$ of symplectomorphisms of a closed symplectic manifold (M,ω) is C^0 closed in the group $Diff^\infty(M)$ of C^∞ diffeomorphisms of M [7],[9]. This means that the "symplectic" nature of a sequence of symplectomorphisms survives topological limits. Also Lalonde-McDuff-Polterovich have shown in [11] that for a symplectomorphism, being "hamiltonian" is topological in nature. These phenomenons attest that there is a C^0 *symplectic topology* underlying the symplectic geometry of a closed symplectic manifold (M,ω).

According to Oh-Müller ([13]), the automorphism group of the C^0 symplectic topology is the closure of the group $Symp(M,\omega)$ in the group $Homeo(M)$ of homeomorphisms of M endowed with the C^0 topology. That group, denoted $Sympeo(M,\omega)$ has been called the group of symplectic homeomorphisms:

$$Sympeo(M,\omega) =: \overline{Symp(M,\omega)}.$$

The C^0 topology on $Homeo(M)$ coincides with the metric topology coming from the metric

$$\overline{d}(g,h) = max(sup_{x\in M} d_0(g(x),h(x)), sup_{x\in M} d_0(g^{-1}(x),h^{-1}(x))$$

where d_0 is a distance on M induced by some riemannian metric [3].

On the space $PHomeo(M)$ of continuous paths $\gamma : [0,1] \to Homeo(M)$, one has the distance

$$\overline{d}(\gamma,\mu) = sup_{t\in[0,1]} \overline{d}(\gamma(t),\mu(t)).$$

Consider the space $PHam(M)$ of all isotopies $\Phi_H = [t \mapsto \Phi_H^t]$ where Φ_H^t is the family of hamiltonian diffeomorphisms obtained by integration of the family of vector fields X_H for a smooth family $H(x,t)$ of real functions on M, i.e.

$$\frac{d}{dt}\Phi_H^t(x) = X_H(\Phi_H^t(x))$$

and $\Phi_H^0 = id$.

Recall that X_H is uniquely defined by the equation

$$i(X_H)\omega = dH$$

where $i(.)$ is the interior product.

The set of time one maps of all hamiltonian isotopies $\{\Phi_H^t\}$ form a group, denoted $Ham(M,\omega)$ and called the group of hamiltonian diffeomorphisms.

Definition: *The hamiltonian topology* [13] on $PHam(M)$ is the metric topology defined by the distance

$$d_{ham}(\Phi_H, \Phi_{H'}) = ||H - H'|| + \overline{d}(\Phi_H, \Phi_{H'})$$

where

$$||H - H'|| = \int_0^1 osc(H - H')dt.$$

and the oscillation of a function u is

$$osc(u) = max_{x\in M} u(x) - min_{x\in M} u(x).$$

Let $Hameo(M,\omega)$ denote the space of all homeomorphisms h such that there exists a continuous path $\lambda \in PHomeo(M)$ such that $\lambda(0) = id$, $\lambda(1) = h$ and there exists a Cauchy sequence (for the d_{ham} distance) of hamiltonian isotopies Φ_{H^n}, which C^0 converges to λ (in the $\overline{d}$ metric).

The following is the first important theorem in the C^0 symplectic topology [13]:

Theorem (Oh-M$\ddot{u}$ller): *The set $Hameo(M,\omega)$ is a topological group. It is a normal subgroup of the identity component $Sympeo_0(M,\omega)$ in $Sympeo(M,\omega)$. If $H^1(M,\mathbb{R}) \neq 0$, then $Hameo(M,\omega)$ is strictly contained in $Sympeo_0(M,\omega)$.*

Remark: It is still unknown in general if the inclusion

$$Hameo(M,\omega) \subset Sympeo_0(M,\omega)$$

is strict.

The group $Hameo(M,\omega)$ is the topological analogue of the group $Ham(M,\omega)$ of hamiltonian diffeomorphisms.

The goal of this paper is to construct a subgroup of $Sympeo_0(M,\omega)$, denoted $SSympeo(M,\omega)$ and nicknamed the group of strong symplectic homeomorphisms, containing $Hameo(M,\omega)$, that is:

$$Hameo(M,\omega) \subset SSympeo(M,\omega) \subset Sympeo_0(M,\omega).$$

Like $Hameo(M,\omega)$, the group $SSympeo(M,\omega)$ is defined using a blend of the C^0 topology and the Hofer topology on the space $Iso(M,\omega)$ of symplectic isotopies of (M,ω).

We believe that $SSympeo(M,\omega)$ is "more right" than the group $Sympeo(M,\omega)$ for the C^0 symplectic topology. In particular the flux homomorphism seems to exist on $SSympeo(M,\omega)$. This will be the object of a future paper.

The results of this paper have been announced in [1].

The C^0 counter part of the C^∞ contact topology is been worked out in [5], [6].

2 The Symplectic Topology on $Iso(M,\omega)$

Let $Iso(M,\omega)$ denote the space of symplectic isotopies of a closed symplectic manifold (M,ω). Recall that a symplectic isotopy is a smooth map $\Phi : M \times [0,1] \to M$ such that for all $t \in [0,1]$, $\phi_t : M \to M, \quad x \mapsto \Phi(x,t)$ is a symplectic diffeomorphism and $\phi_0 = id$.

The "Lie algebra" of $Symp(M,\omega)$ is the space $symp(M,\omega)$ of symplectic vector fields, i.e the set of vector fields X such that $i_X\omega$ is a closed form.

Let ϕ_t be a symplectic isotopy, then

$$\dot{\phi}_t(x) = \frac{d\phi_t}{dt}(\phi_t^{-1}(x))$$

is a smooth family of symplectic vector fields.

By the theorem of existence and uniqueness of solutions of ODE's,

$$\Phi \in Iso(M,\omega) \mapsto \dot{\phi}_t$$

is a 1-1 correspondence between $Iso(M,\omega)$ and the space $C^\infty([0,1], symp(M,\omega))$ of smooth families of symplectic vector fields. Hence any distance on $C^\infty([0,1], symp(M,\omega))$ gives rise to a distance on $Iso(M,\omega)$.

An intrinsic topology on the space of symplectic vector fields.

We define a norm $||.||$ on $symp(M,\omega)$ as follows: first we fix a riemannian metric g (which may be the one we used to define d_0 above, or any other riemannian metric), and a basis $\mathcal{B} = \{h_1,..,h_k\}$ of harmonic 1-forms. For Hodge theory, we refer to [14].

Recall that the space $harm^1(M,g)$ of harmonic 1-forms is a finite dimensional vector space over $\mathbb{R}$ and its dimension is the first Betti number of M.

On $harm^1(M,g)$, we put the following "Euclidean" norm:

for $\mathscr{H} \in harm^1(M,g)$, $\mathscr{H} = \sum \lambda_i h_i$, define:

$$|\mathscr{H}|_{\mathscr{B}} := \sum |\lambda_i|.$$

This norm is equivalent to any other norm since $harm^1(M,g)$ is a finite dimensional vector space. Here we choose this one for convenience in the calculations and estimates to come later.

Given $X \in symp(M,\omega)$, we consider the Hodge decomposition of $i_X\omega$ [14] : there is a unique harmonic 1-form $\mathscr{H}_X$ and a unique function u_X such that

$$i_X\omega = \mathscr{H}_X + du_X$$

Recall that the function u_X is given by the following formula: $u_X = \delta G(i(X)\omega)$, where δ is the codifferential and G is the Green operator (see [14]).

This defines a decompsition of $X \in symp(M,\omega)$ as : $X = \#\mathscr{H}_X + X_{u_X}$,where $\#\mathscr{H}_X$ is defined by the equation $i(\#\mathscr{H}_X)\omega = \mathscr{H}_X$ and X_{u_X} is the hamiltonian vector field with u_X as hamiltonnian.

We now define a norm $||.||$ on the the vector space $symp(M,\omega)$ by:

$$||X|| = |\mathscr{H}_X|_{\mathscr{B}} + osc(u_X). \tag{1}$$

It is easy to see that this is a norm. Let us just verify that $||X|| = 0$ implies that $X = 0$. Indeed $|\mathscr{H}_X|_{\mathscr{B}} = 0$ implies that $i_X\omega = du_X$, and $osc(u_X) = 0$ implies that u_X is a constant, therefore $du_X = 0$.

Remark: This norm is not invariant by $Symp(M,\omega)$. Hence it does not define a Finsler metric on $Symp(M,\omega)$.

The norm $||.||$ defined above depends of course on the riemannian metric g and the basis $\mathscr{B}$ of harmonic 1-forms. However, we have the following:

Theorem 1: *All the norms $||.||$ defined by equation (1) using different riemannian metrics and different basis of harmonic 1-forms are equivalent.*

Hence the topology on the space $symp(M,\omega)$ of symplectic vector fields defined by the norm (1) is intrinsic : it is independent of the choice of the riemannian metric g and of the basis $\mathscr{B}$ of harmonic 1-forms.

For each symplectic isotopy $\Phi = (\phi_t)$, consider the Hodge decomposition of $i_{(\dot\phi_t)}\omega$

$$i_{(\dot\phi_t)}\omega = \mathcal{H}_t^\Phi + du_t^\Phi$$

where $\mathcal{H}_t^\Phi$ is a harmonic 1-form.

We define the length $l(\Phi)$ of the isotopy $\Phi = (\phi_t)$ by:

$$l(\Phi) = \int_0^1 (|\mathcal{H}_t^\Phi| + osc(u_t^\Phi))dt = \int_0^1 ||\dot\phi_t||dt.$$

One also writes

$$\int_0^1 ||\dot\phi_t||dt = |||\dot\phi_t|||.$$

In the expressions above, we have written $|\mathcal{H}_t^\Phi|$ for $|\mathcal{H}_t^\Phi|_{\mathcal{B}}$, where $\mathcal{B}$ is a fixed basis of $harm^1(M,g)$, for a fixed riemannian metric g.

We define the distance $D_0(\Phi,\Psi)$ between two symplectic isotopies $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ by:

$$D_0(\Phi,\Psi) = |||\dot\phi_t - \dot\psi_t||| := \int_0^1 (|\mathcal{H}_t^\Phi - \mathcal{H}_t^\Psi| + osc(u^{\Phi_t} - u^{\Psi_t}))dt.$$

Denote by $\Phi^{-1} = (\phi_t^{-1})$ and by $\Psi^{-1} = (\psi_t^{-1})$ the Inverse isotopies.

Remarks:

1. The distance $D_0(\Phi,\Psi) \neq l(\Psi^{-1}\Phi)$ unless Ψ and Φ are hamiltonian isotopies (see proposition 1).
2. $l(\Phi) \neq l(\Phi^{-1})$ unless Φ is hamiltonian. Indeed, $\mathcal{H}_t^{\Phi^{-1}} = -\mathcal{H}_t^\Phi$ but u_t^Φ is very differerent from $u_t^{\Phi^{-1}}$. The formula of the difference $u_t^\Phi - u_t^{\Phi^{-1}}$ follows from propositions 3, 4 and 5.

In view of the remarks above, we define a more "symmetrical" distance D by:

$$D(\Phi,\Psi) = (D_0(\Phi,\Psi) + D_0(\Phi^{-1},\Psi^{-1}))/2$$

Following [13], we define the *symplectic distance* on $Iso(M,\omega)$ by:

$$d_{symp}(\Phi,\Psi) = \overline{d}(\Phi,\Psi) + D(\Phi,\Psi).$$

Definition: The *symplectic topology* on $Iso(M,\omega)$ is the metric topology defined by the distance d_{symp}.

Theorem 1': *The symplectic topology on $Iso(M,\omega)$ is canonical: it is independent of all choices involved in its definition.*

We may also define another distance D^{∞} on $Iso(M,\omega)$:

$$D_0^{\infty}(\Phi,\Psi) = sup_{t\in[0,1]}(|\mathcal{H}_t^{\Phi} - \mathcal{H}_t^{\Psi}|) + sup_{t\in[0,1]} osc(u^{\Phi_t} - u^{\Psi_t}))$$

$$D^{\infty}(\Phi,\Psi) = (D_0^{\infty}(\Phi,\Psi) + D_0^{\infty}(\Phi^{-1},\Psi^{-1}))/2$$

and

$$d_{symp}^{\infty}(\Phi,\Psi) = \overline{d}(\Phi,\Psi) + D^{\infty}(\Phi,\Psi)$$

Proposition 1: *Let $\Phi = (\phi_t), \Psi = (\psi_t)$ be two hamiltonian isotopies and $\sigma_t = (\psi_t)^{-1}\phi_t$ then*

$$|||\dot{\sigma}_t||| = |||\dot{\phi}_t - \dot{\psi}_t||| = \int_0^1 osc(u_t^{\Phi} - u^{\Psi_t})dt$$

Proof: This follows immediately from the equation

$$\dot{\sigma}_t = ({\psi_t}^{-1})_*(\dot{\phi}_t - \dot{\psi}_t),$$

which is a consequence of proposition 4 stated in section 4. □

Corollary: *The distance d_{sym} reduces to the hamiltonian distance d_{ham} when Φ and Ψ are hamiltonian isotopies.*

The *symplectic topology* reduces to the "hamiltonian topology" of [13] on paths in $Ham(M,\omega)$.

A Hofer-like metric on $Symp(M,\omega)_0$

For any $\phi \in Symp(M,\omega)$, define:

$$e_0(\phi) = inf(l(\Phi))$$

where the infimum is taken over all symplectic isotopies Φ from ϕ to the identity. The following result was proved in [2].

Theorem: *The map $e : Symp(M,\omega)_0 \to \mathbb{R} \cup \{\infty\}$:*

$$e(\phi) =: (e_0(\phi) + e_0(\phi^{-1}))/2$$

is a metric on the identity component $Symp(M,\omega)_0$ in the group $Symp(M,\omega)$, i.e. it satisfies

(i) $e(\phi) \geq 0$ and $e(\phi) = 0$ iff ϕ is the identity.

(ii) $e(\phi) = e((\phi)^{-1})$

(iii) $e(\phi.\psi) \leq (e\phi) + e(\psi)$.

The restriction to $Ham(M,\omega)$ is bounded from above by the Hofer norm.

Recall that the Hofer norm [10] of a hamiltonian diffeomorphism ϕ is

$$||\phi||_H = inf(l(\Phi_H))$$

where the infimum is taken over all hamiltonian isotopies from ϕ to the identity.

The Hofer-like metric above depends on the choice of a riemannian metric g and a basis $\mathscr{B}$ of harmonic 1-forms. Hence it is not “natural”. However, by theorem 1, all the metrics constructed that way are equivalent; so they define a natural topology on $Symp(M,\omega)_0$.

3 Strong Symplectic Homeomorphisms

Definition: *A homeomorphism h is said to be a strong symplectic homeomorphism if there exists a continuous path $\lambda : [0,1] \to Homeo(M)$ such that $\lambda(0) = id$; $\lambda(1) = h$ and a sequence $\Phi^n = (\phi_t^n)$ of symplectic isotopies, which converges to λ in the C^0 topology (induced by the norm $\overline{d}$) and such that Φ^n is Cauchy for the metric d_{symp}.*

We will denote by $SSympeo(M,\omega)$ the set of all strong symplectic homeomorphisms. This set is well defined independently of any riemannian metric or any basis of harmonic 1-forms.

Clearly, if M is simply connected, the set $SSympeo(M,\omega)$ coincides with the group $Hameo(M,\omega)$.

We denote by $SSympeo(M,\omega)^\infty$ the set defined like in $SSympeo(M,\omega)$ but replacing the norm d_{symp} by the norm d^∞_{symp}.

Let $\mathscr{P}Homeo(M)$ be the set of continuous paths $\gamma : [0,1] \to Homeo(M)$ such that $\gamma(0) = id$, and let $\mathscr{P}^\infty(Harm^1(M)$ be the space of smooth paths of harmonic 1-forms.

We have the following maps:

$A_1 : Iso(M,\omega) \to \mathscr{P}Homeo(M), \Phi \mapsto \Phi(t)$

$A_2 : Iso(M,\omega) \to \mathscr{P}^\infty(Harm^1(M), \Phi \mapsto \mathscr{H}_t^\Phi$

$A_3 : Iso(M,\omega) \to C^\infty(M \times [0,1], \mathbb{R}), \Phi \mapsto u^\Phi$

Let $\mathscr{Q}$ be the image of the mapping $A = A_1 \times A_2 \times A_3$ and $\overline{\mathscr{Q}}$ the closure of $\mathscr{Q}$ inside $\mathscr{I}(M,\omega) =: \mathscr{P}Homeo(M) \times \mathscr{P}^\infty(Harm^1(M) \times C^\infty(M \times [0,1], \mathbb{R})$, with the symplectic topology, which is the C^0 topology on the first factor and the metric topology from D on the second and third factor.

Then $SSympeo(M,\omega)$ is just the image of the evaluation map of the path at t= 1 of the image of the projection of $\overline{\mathscr{Q}}$ on the first factor. This defines a surjective map:

$$a : \overline{\mathscr{Q}} \to SSympeo(M,\omega)$$

The symplectic topology on $SSympeo(M,\omega)$ is the quotient topology induced by a.

Our main results are :

Theorem 2: *The set $\overline{\mathcal{Q}}$ is a topological group.*

Theorem 3: *Let (M,ω) be a closed symplectic manifold. Then $SSympeo(M,\omega)$ is an arc-wise connected topological group (with the sympectic topology), containing $Hameo(M,\omega)$ as a normal subgroup, and contained in the path component of the identity $Sympeo_0(M,\omega)$ of $Sympeo(M,\omega)$.*

If M is simply connected, $SSympeo(M,\omega) = Hameo(M,\omega)$. Finally, the commutator sub-group $[SSympeo(M,\omega), SSympeo(M,\omega)]$ of $SSympeo(M,\omega)$ is contained in $Hameo(M,\omega)$.

Conjectures:

1. Let (M,ω) be a closed symplectic manifold, then $[SSympeo(M,\omega), SSympeo(M,\omega)] = Hameo(M,\omega)$.
2. The inclusion $SSympeo(M,\omega) \subset Sympeo_0(M,\omega)$ is strict.
3. The results in theorem 3 hold for $SSympeo(M,\omega)^{\infty}$.

Conjecture 3 is supported by a result of Muller asserting that $Hameo(M,\omega)$ coincides with $Hameo(M,\omega)^{\infty}$ which is defined by replacing the $L^{(1,\infty)}$ Hofer norm by the L^{∞} norm [12].

Measure preserving homeomorphisms

On a symplectic $2n$ dimensional manifold (M,ω), we consider the measure μ_ω defined by the Liouville volume ω^n. Let $Homeo_0^{\mu_\omega}(M)$ be the identity component in the group of homeomorphisms preserving μ_ω. We have:

$$Sympeo_0(M,\omega) \subset Homeo_0^{\mu_\omega}(M).$$

Oh and M$\ddot{u}$ller [13] have observed that $Hameo(M,\omega)$ is a sub-group of the kernel of Fathi's mass-flow homomorphism [8]. This is a homomorphism $\theta : Homeo_0^{\mu_\omega}(M) \to H_1(M,\mathbb{R})/\Gamma$, where Γ is some sub-group of $H_1(M,\mathbb{R})$. Fathi proved that if the dimension of M is bigger than 2, then $Ker\theta$ is a simple group. This leaves open the following question [13]:

$$\textit{Is } Homeo_0^{\mu_\omega}(S^2) = Sympeo_0(S^2,\omega) \textit{ a simple group?}$$

But $Sympeo_0(S^2,\omega)$ contains $Hameo(S^2,\omega)$ as a normal subgroup. The question is to decide if the inclusion

$$Hameo(S^2,\omega) \subset Sympeo_0(S^2,\omega)$$

is strict. Since $SSympeo(S^2,\omega) = Hameo(S^2,\omega)$, our conjecture 2 implies that $Homeo_0^{\mu_\omega}(S^2) = Sympeo_0(S^2,\omega)$ is not a simple group, a conjecture of [13].

Questions

1. Is $SSympeo(M,\omega)$ a normal subgroup of $Sympeo_0(M,\omega)$?

2. Is $[Sympeo_0(M,\omega), Sympeo_0(M,\omega)]$ contained in $Hameo(M,\omega)$?

4 Proofs of the Results

4.1 Proof of theorem 1

If $\mathscr{B}$ and $\mathscr{B}'$ are two basis of $harm^1(M,g)$, then elementary linear algebra shows that $|.|_{\mathscr{B}}$ and $|.|_{\mathscr{B}'}$ are equivalent. This implies that the corresponding norms on $symp(M,\omega)$ are also equivalent.

Let us now start our construction with a riemannian metric g and a basis $\mathscr{B} = (h_1, ..h_k)$ of $harm^1(M,g)$. We saw that for any $X \in symp(M,\omega)$,

$$i_X\omega = \mathscr{H}_X + du_X$$

and we wrote $\mathscr{H}_X = \sum \lambda_i h_i$.

Let g' be another riemannian metric. The g'-Hodge decomposition of $i_X\omega$ is:

$$i_X\omega = \mathscr{H}'_X + du'_X$$

where $\mathscr{H}'_X$ is g'-harmonic.

Consider the g'-Hodge decompositions of the members h_i of the basis $\mathscr{B}$ i.e.

$$h_i = h'_i + dv_i$$

where h'_i is g' harmonic. $\mathscr{B}' = (h'_1, ..h'_k)$ is a basis of $harm^1(M,g')$. Indeed suppose that $\sum r_i h'_i = 0$. The 1-form $\sum r_i h_i = d(\sum r_i v_i)$ is g-harmonic and exact :$\sum r_i h_i = d(\sum r_i v_i)$. But an exact harmonic form must be identically zero. Therefore all r_i are zero since $\{h_i\}$ form a basis. Hence $\{h'_i\}$ are linearly independent.

The 1-form

$$\mathscr{H}''_X =: \sum \lambda_i h'_i$$

is a g'- harmonic form representing the cohomology class of $i_X\omega$. By uniqueness, $\mathscr{H}'_X = \mathscr{H}''_X$.

Hence

$$|\mathscr{H}'_X|_{\mathscr{B}'} = \sum |\lambda_i| = |H_X|_{\mathscr{B}}$$

Furthermore $\mathcal{H}'_X = \sum \lambda_i (h_i - dv_i) = \mathcal{H}_X + dv$ where $v = -\sum \lambda_i v_i$. Hence

$$i_X\omega = \mathcal{H}'_X + du'_X = \mathcal{H}_X + d(v + u'_X)$$

By uniqueness in the g-Hodge decomposition of $i_X\omega$,

$$u_X = v + u'_X.$$

Denote by $||X||_{g'}$, resp. $||X||_g$, the norm of X using the riemannian metric g' and the basis $\mathcal{B}'$, resp. using the riemannian metric g and the basis $\mathcal{B}$. Then:

$$\begin{aligned} ||X||_{g'} = |\mathcal{H}'_X|_{\mathcal{B}'} + osc(u'_X) &= |\mathcal{H}'_X|_{\mathcal{B}'} + osc(u_X - v) \\ &\leq |\mathcal{H}'_X|_{\mathcal{B}'} + osc(u_X) + osc(-v) \\ &= |\mathcal{H}_X|_{\mathcal{B}} + osc(u_X) + osc(v) = ||X||_g + osc(v). \end{aligned}$$

Let $c = 2max_i|v_i|$, since $v = -\sum \lambda_i v_i$, we get the following inequality:

$$osc(v) \leq 2max(|v|) \leq c|\mathcal{H}_X|_{\mathcal{B}} = c|\mathcal{H}'_X|_{\mathcal{B}'}$$

Therefore

$$||X||_{g'} \leq ||X||_g + osc(v) \leq ||X||_g + c|\mathcal{H}_X|_{\mathcal{B}} \leq ||X||_g + c(|\mathcal{H}_X|_{\mathcal{B}} + osc(u_X)) = (c+1)||X||_g$$

Similarly,

$$\begin{aligned} ||X||_g &= |\mathcal{H}_X|_{\mathcal{B}} + osc(u_X) = |\mathcal{H}_X|_{\mathcal{B}} + osc(u'_X + v) \leq |\mathcal{H}_X|_{\mathcal{B}} + osc(u'_X) + osc(v) \\ &= |\mathcal{H}'_X|_{\mathcal{B}'} + osc(u'_X) + osc(v) = ||X||_{g'} + osc(v) \leq ||X||_{g'} + c|\mathcal{H}'_X|_{\mathcal{B}'} \\ &\leq ||X||_{g'} + c(|\mathcal{H}'_X|_{\mathcal{B}'} + osc(u'_X) = (c+1)||X||_{g'} \end{aligned}$$

Hence the metrics $||X||_g$ and $||X||_g$ are equivalent □

For the purpose of the proof of the main theorem, we fix a riemannian metric g and a basis $\mathcal{B} = (h_1, .., h_k)$ of $harm^1(M, g)$. The norm of a harmonic 1-form $\mathcal{H}$ will be simply denoted $|\mathcal{H}|$ and the norm of a symplectic vector field X will be simply denoted $||X||$.

4.2 Proof of theorem 3

We prove first that the set $SSympeo(M, \omega) subset Sympeo(M, \omega)$ is closed under composition and inverse maps.

Let $h_i \in SSympeo(M, \omega)$ $i = 1, 2$ and let λ_i be continuous paths in $Homeo(M)$ with $\lambda_i(0) = id$, $\lambda_i(1) = h_i$ and let Φ^n_i be d_{symp} - Cauchy sequences of symplectic isotopies converging C^0 to λ_i. Then $\Phi^n_1.(\Phi^n_2)^{-1}$ converges C^0 to the path $\lambda_1(t)(\lambda_2(t))^{-1}$. Here $\Phi^n_1.(\Phi^n_2)^{-1}(t) = \phi^n_1(t).(\phi^n_2(t))^{-1}$.

By definition of the distance d_{symp}, Φ^n is a d_{symp} - Cauchy sequence if and only if both Φ^n and $(\Phi^n)^{-1}$ are D_0 - Cauchy and $\overline{d}$- Cauchy sequences.

Main Lemma: *If $\Phi^n = (\phi_t^n)$ and $\Psi_t^n = (\psi_t^n)$ are d_{symp} - Cauchy sequences in $Iso(M)$, so is $\rho_t^n = \phi_t^n \psi_t^n$.*

The proof of the main lemma is very delicate; it will take most of the remaining part of this paper. The estimates are much more involved than in the hamiltonian case, due to the fact that the decomposition of a symplectic isotopy into a hamiltonian one and a harmonic one does not behave nicely with respect to the product of isotopies.

It will be enough to prove that ρ_t^n is a D_0 - Cauchy sequence. Indeed since $(\Phi^n)^{-1}$ and $(\Psi^n)^{-1}$ are D_0 - Cauchy by assumption, the main lemma applied to their product implies that their product is also D_0 Cauchy.

Hence $(\Psi^n)^{-1}(\Phi^n)^{-1} = (\Phi^n\Psi^n)^{-1} = (\rho_t^n)^{-1}$ is a D_0 - Cauchy sequence. This will conclude the proof that $SSympeo(M,\omega)$ is a group. We leave the details to the reader.

We will use the following estimate:

Proposition 2: *There exists a constant E such that for any $X \in symp(M,\omega)$, and $\mathcal{H} \in harm^1(M,g)$*

$$|\mathcal{H}(X)| =: sup_{x\in M}|\mathcal{H}(x)(X(x))| \leq E||X||.|\mathcal{H}|$$

Proof: Let $(h_1,..,h_r)$ be the chosen basis for harmonic 1-forms and let $E = max_i E_i$ and $E_i = sup_V(sup_{x\in M}|h_i(x)(V(x))|$ where V runs over all symplectic vector fields V such that $||V|| = 1$.

Without loss of generality, we may suppose $X \neq 0$ and set $V = X/||X||$. Let $\mathcal{H} = \sum \lambda_i h_i$. Then $\mathcal{H}(X) = ||X||\sum \lambda_i h_i(V)$. Hence

$$|\mathcal{H}(X)| \leq ||X||\sum|\lambda_i|sup_x(|h_i(x)(V)(x)|) \leq ||X||\sum|\lambda_i|E = E||X||.|\mathcal{H}|.$$

□

We will also need the following standard facts:

Proposition 3: *Let ϕ be a diffeomorphism, X a vector field and θ a differential form on a smooth manifold M. Then*

$$(\phi^{-1})^*[i_X\phi^*\theta] = i_{\phi_* X}\theta$$

Proposition 4: *If ϕ_t, ψ_t are any isotopies, and if we denote by $\rho_t = \phi_t\psi_t$, and by $\underline{\phi}_t = (\phi)_t^{-1}$ then*

$$\dot{\rho}_t = \dot{\phi}_t + (\phi_t)_*\dot{\psi}_t$$

and

$$\dot{\underline{\phi}}_t = -((\phi)_t^{-1})_*(\dot{\phi}_t)$$

Proposition 5: *Let θ_t be a smooth family of closed 1-forms and ϕ_t an isotopy, then*

$$\phi_t^*\theta_t - \theta_t = dv_t$$

where

$$v_t = \int_0^t (\theta_t(\dot{\phi}_s)\circ\phi_s)ds$$

Proof of the Main Lemma: If ϕ_t, ψ_t are symplectic isotopies, and if $\rho_t = \phi_t\psi_t$, propositions 3, 4 and 5 give:

$$i(\dot{\rho}_t)\omega = \mathcal{H}_t^{\Phi} + \mathcal{H}_t^{\Psi} + dK(\Phi,\Psi) \qquad (I)$$

where $K = K(\Phi,\Psi) = u_t^{\Phi} + (u_t^{\Psi})\circ(\phi_t)^{-1} + v_t(\Phi,\Psi)$, and

$$v_t(\Phi,\Psi) = \int_0^t (\mathcal{H}_t^{\Psi}(\underline{\dot{\phi}}_s)\circ\phi_s^{-1})ds. \qquad (II)$$

Let now ϕ_t^n, ψ_t^n be Cauchy sequences of symplectic isotopies, and consider the sequence $\rho_t^n = \phi_t^n\psi_t^n$.

We have:

$$\begin{aligned}
|||\dot{\rho}_t^n - \dot{\rho}_t^m||| &= \int_0^1 |\mathcal{H}_t^{\Phi^n} - \mathcal{H}_t^{\Phi^m} + \mathcal{H}_t^{\Psi^n} - \mathcal{H}_t^{\Psi^m}| + osc(K(\Phi^n,\Psi^n) - K(\Phi^m,\Psi^m))dt \\
&\le \int_0^1 |\mathcal{H}_t^{\Phi^n} - \mathcal{H}_t^{\Phi^m})|dt + \int_0^1 |\mathcal{H}_t^{\Psi^n} - \mathcal{H}_t^{\Psi^m})|dt \\
&+ \int_0^1 osc(u_t^{\Phi^n} - u_t^{\Phi^m})dt + \int_0^1 osc(u_t^{\Psi^n})\circ(\phi_t^n)^{-1} - u_t^{\Psi^m}\circ(\phi_t^m)^{-1})dt \\
&+ \int_0^1 osc(v_t(\Phi^n,\Psi^n) - v_t(\Phi^m,\Psi^m)dt \\
&= |||\dot{\phi^n}_t - \dot{\phi^m}_t||| + \int_0^1 |\mathcal{H}_t^{\Psi^n} - \mathcal{H}_t^{\Psi^m})|dt + A + B
\end{aligned}$$

where

$$A = \int_0^1 osc(u_t^{\Psi^n})\circ(\phi_t^n)^{-1} - u_t^{\Psi^m}\circ(\phi_t^m)^{-1})dt$$

and

$$B = \int_0^1 osc(v_t(\Phi^n,\Psi^n) - v_t(\Phi^m,\Psi^m)dt. \qquad (III)$$

We have:

$$A \leq \int_0^1 osc(u_t^{\Psi^n}) \circ (\phi_t^n)^{-1} - u_t^{\Psi^m} \circ (\phi_t^n)^{-1})dt + \int_0^1 osc(u_t^{\Psi^m}) \circ (\phi_t^n)^{-1} - (u_t^{\Psi^m}) \circ (\phi_t^m)^{-1})dt$$

$$= \int_0^1 osc(u_t^{\Psi^n} - u_t^{\Psi^m})dt + C$$

where

$$C = \int_0^1 osc(u_t^{\Psi^m} \circ (\phi_t^n)^{-1} - u_t^{\Psi^m} \circ (\phi_t^m)^{-1})dt.$$

Hence

$$|||\dot{\rho}_t^n - \dot{\rho}_t^m||| \leq |||\dot{\phi}_t^n - \dot{\phi}_t^m|||$$

$$+ \int_0^1 |\mathscr{H}_t^{\Psi^n} - \mathscr{H}_t^{\Psi^m})|dt + \int_0^t osc(u_t^{\Psi^n} - u_t^{\Psi^m})dt + B + C$$

$$= |||\dot{\phi}_t^n - \dot{\phi}_t^m||| + |||\dot{\psi}_t^n - \dot{\psi}_t^m||| + B + C$$

We now show that $C \to 0$ when $m, n \to \infty$.

Sub-Lemma 1 (Reparametrization Lemma [13]): $\forall \epsilon \geq 0, \exists m_0$ *such that*

$$C = \int_0^1 osc(u_t^{\Psi^m} \circ (\phi_t^n)^{-1} - u_t^{\Psi^m} \circ (\phi_t^m)^{-1})dt =: ||u_t^{\Psi^m} \circ (\phi_t^n)^{-1} - u_t^{\Psi^m} \circ (\phi_t^m)^{-1})|| \leq \epsilon$$

if $m \geq m_0$ *and* n *large enough*

Remark: This is the "reparametrization lemma" of Oh-M*ü*ller [13] (lemma 3.21. (2)). For the convenience of the reader and further references, we include their proof.

Proof: For short, we write u_m for $u_t^{\Psi^m}$ and μ_t^n for $(\phi_t^n)^{-1}$.

First, there exists m_0 large such that $||u_m - u_{m_0}|| \leq \epsilon/3$ for $m \geq m_0$, since (u_m) is a Cauchy sequence for the distance $d(u_n, u_m) = \int_0^1 osc(u_n - u_m)dt$.

Therefore

$$||u_m \circ \mu_t^n - u_m \circ \mu_t^m))|| \leq ||u_m \circ \mu_t^n - u_{m_0} \circ \mu_t^n))|| + ||u_{m_0} \circ \mu_t^n - u_{m_0} \circ \mu_t^m))|| + ||u_{m_0} \circ \mu_t^m - u_m \circ \mu_t^m))||$$

$$= ||u_m - u_{m_0}|| + ||u_{m_0} \circ \mu_t^n - u_{m_0} \circ \mu_t^m))|| + ||u_{m_0} - u_m||$$

$$\leq (2/3)\epsilon + ||u_{m_0} \circ \mu_t^n - u_{m_0} \circ \mu_t^m))|.$$

By uniform continuity of u_{m_0}, there exists a positive δ such that if $\overline{d}(\mu_t^m, \mu_t^n) \leq \delta$, then max $osc((u_{m_0} \circ \mu_t^n - u_{m_0} \circ \mu_t^m)) \leq \epsilon/3$. Hence $||u_{m_0} \circ \mu_t^n - u_{m_0} \circ \mu_t^m))|| \leq \epsilon/3$ for n, m large. Recall that μ_t^n is a $\overline{d}$- Cauchy sequence. □

To show that $\dot{\rho}_t^n$ is a Cauchy sequence, the only thing which is left is to show that $B \to 0$ when $n,m \to \infty$.

Let us denote $v_t(\Phi^n, \Psi^n)$ by v_t^n , $\mathcal{H}_t^{\Psi^n}$ by $\mathcal{H}_n^t$ or $\mathcal{H}_n$ and $(\phi_t^n)^{-1}$ by μ_t^n.

For a function on M, we consider the norm

$$|f| = sup_{x\in M}|f(x)|$$

We have:

$$|v_t^n - v_t^m| = |\int_0^t (\mathcal{H}_n^t(\dot{\mu}_s^n)\circ\mu_s^n - \mathcal{H}_m^t(\dot{\mu}_s^m)\circ\mu_s^m)ds|$$

$$\leq \int_0^1 |((\mathcal{H}_n^t - \mathcal{H}_m^t)(\dot{\mu}_s^n))\circ\mu_s^n|ds$$

$$+ \int_0^1 |\mathcal{H}_m^t(\dot{\mu}_s^n - \dot{\mu}_s^m))\circ\mu_s^m|ds$$

$$+ \int_0^1 |\mathcal{H}_m^t(\dot{\mu}_s^n)\circ\mu_s^n - \mathcal{H}_m^t(\dot{\mu}_s^n)\circ\mu_s^m|ds$$

The last integral can be estimated as follows:

$$\int_0^1 |\mathcal{H}_m^t(\dot{\mu}_s^n)\circ\mu_s^n - \mathcal{H}_m^t(\dot{\mu}_s^n)\circ\mu_s^m|ds$$

$$\leq \int_0^1 |\mathcal{H}_m^t(\dot{\mu}_s^n)\circ\mu_s^n - \mathcal{H}_m^t(\dot{\mu}_s^{n_0})\circ\mu_s^n|ds \tag{1}$$

$$+ \int_0^1 |\mathcal{H}_m^t(\dot{\mu}_s^{n_0})\circ\mu_s^n - \mathcal{H}_m^t(\dot{\mu}_s^{n_0})\circ\mu_s^m|ds \tag{2}$$

$$+ \int_0^1 |\mathcal{H}_m^t(\dot{\mu}_s^{n_0})\circ\mu_s^m - \mathcal{H}_m^t(\dot{\mu}_s^n)\circ\mu_s^m|ds \tag{3}$$

for some integer n_0.

Proposition 2 gives $E|\mathcal{H}_m|D_0((\Phi^n)^{-1},(\Phi^{n_0})^{-1}) \leq 2E|\mathcal{H}_m|D((\Phi^n),(\Phi^{n_0})^{-1})$ as an upper bound for (1) and (3).

It also gives the following estimates:

$$\int_0^1 |((\mathcal{H}_n^t - \mathcal{H}_m^t)(\dot{\mu}_s^n))\circ\mu_s^n|ds \leq E|\mathcal{H}_n^t - \mathcal{H}_m^t|\int_0^1 ||\dot{\mu}_s^n)||ds$$

$$= E.|\mathcal{H}_n^t - \mathcal{H}_m^t|.l((\Phi^n)^{-1})$$

and

$$\int_0^1 |(\mathcal{H}_m^t(\dot{\mu}_s^n - \dot{\mu}_s^m)) \circ \mu_s^m| ds \leq E.|\mathcal{H}_m^t| \int_0^1 ||(\dot{\mu}_s^n - \dot{\mu}_s^m)|| ds$$
$$= E|\mathcal{H}_m^t| D_0((\Phi^n)^{-1}, (\Phi^m))^{-1}) \leq 2E|\mathcal{H}_m^t| D(\Phi^n, \Phi^m).$$

Therefore, we get the following estimate:

$$|v_t^n - v_t^m| \leq E.|\mathcal{H}_n^t - \mathcal{H}_m^t| l(\Phi^n)^{-1}) + E|\mathcal{H}_m^t| 2(D(\Phi^n, \Phi^m) + 2D(\Phi^n, \Phi^{n_0})) + G$$

where

$$G = \int_0^1 |\mathcal{H}_m^t(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathcal{H}_m^t(\dot{\mu}_s^{n_0}) \circ \mu_s^m| ds$$

Since $osc(v_t^n - v_t^m) \leq 2|v_t^n - v_t^m|$, we see that

$$\int_0^1 osc(v_t^n - v_t^m) dt \leq 2E(l(\Phi^n)^{-1}) \int_0^1 |\mathcal{H}_n^t - \mathcal{H}_m^t| dt$$

$$+ E2(D(\Phi^m, \Phi^n) + 2ED(\Phi^n, \Phi^{n_0}) \int_0^1 |\mathcal{H}_m^t| dt) + \int_0^1 G dt$$

We need the following facts:

Sub-Lemma 2 (Reparametrization Lemma): $\forall \epsilon \geq 0, \exists n_0$ *such that*

$$L = \int_0^1 G dt = \int_0^1 \left(\int_0^1 |\mathcal{H}_m^t(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathcal{H}_m^t(\dot{\mu}_s^{n_0}) \circ \mu_s^m| ds \right) dt \leq \epsilon$$

for $n \geq n_0$ and m sufficiently large.

Proposition 6: $l((\Phi^n))^{-1}$ *and* $\int_0^1 |\mathcal{H}_m^t| dt$ *are bounded for every n, m.*

We finish first the estimate for $\int_0^1 osc(v_t^n - v_t^m) dt$ using sub-lemma 2 and proposition 6.

Putting together all the information we gathered, we see that:

$$\int_0^1 osc(v_t^n - v_t^m) dt \leq 2E(l(\Phi^n)^{-1}) \int_0^1 |\mathcal{H}_n^t - \mathcal{H}_m^t| dt$$

$$+ E(2D(\Phi^m, \Phi^n)) + 2ED(\Phi^n, \Phi^{n_0}) (\int_0^1 |\mathcal{H}_m^t| dt) + L$$

$$\leq 2El((\Phi^n)^{-1}) D(\Phi^n, \Phi^m) + E(2D(\Phi^m, \Phi^n) + 2ED(\Phi^n, \Phi^{n_0}) \int_0^1 |\mathcal{H}_m^t| dt + L$$

Therefore:

$$\int_0^1 osc(v_t^n - v_t^m) dt \to 0$$

when $n,m \to \infty$, and n_0 is chosen sufficiently large Now let $n_0 \to \infty$ as well.. This finishes the proof of the main lemma.

Proof of Sub-Lemma 2:

$$G = \int_0^1 |\mathscr{H}_m^t(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathscr{H}_m^t(\dot{\mu}_s^{n_0}) \circ \mu_s^m| ds$$

$$\leq \int_0^1 |\mathscr{H}_m^t(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathscr{H}_{m_0}^t(\dot{\mu}_s^{n_0}) \circ \mu_s^n| ds$$

$$+ \int_0^1 |\mathscr{H}_{m_0}^t(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathscr{H}_{m_0}^t(\dot{\mu}_s^{n_0}) \circ \mu_s^m| ds$$

$$+ \int_0^1 |\mathscr{H}_{m_0}^t(\dot{\mu}_s^{n_0}) \circ \mu_s^m - \mathscr{H}_m^t(\dot{\mu}_s^{n_0}) \circ \mu_s^m| ds$$

for some m_0.

Exactly like in the proof of sub-lemma 1

$$G(t,n,m) \leq 2E|\mathscr{H}_m^t - \mathscr{H}_{m_0}^t|.(l((\Psi^{n_0})^{-1}) + F$$

where

$$F = \int_0^1 |\mathscr{H}_{m_0}^t(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathscr{H}_{m_0}^t(\dot{\mu}_s^{n_0}) \circ \mu_s^m| ds$$

By uniform continuity of $\mathscr{H}_{m_0}^t(\dot{\mu}_s^{n_0})$, $F \to 0$ when $n,m \to \infty$ since μ_t^n is Cauchy.

By similar arguments as in the sub-lemma 1, $G \to 0$ and hence $L \to 0$ when $m,n \to \infty$ and $m_0 \to \infty$. □

We have just proved that the subset $SSymp(M,\omega)$ of $Symeo(M,\omega)$ is closed under composition and inversion. This concludes the proof that $SSympeo(M,\omega)$ is a group.

The fact that it is arcwise connected in the ambiant topology of $Homeo(M)$ is obvious from the definition.

$Hameo(M,\omega)$ is a normal subgroup of $SSympeo(M,\omega)$ since it is normal in $Sympeo(M,\omega)$ [13].

Let $h,g \in SSympeo(M,\omega)$ and let Φ^n, Ψ^n be symplectic isotopies which form Cauchy sequences and C^0 converge to h,g. By the main lemma the sequence $\Phi^n.\Psi^n.(\Phi^n)^{-1}(\Phi^n)^{-1}$ is a Cauchy sequence. It obviously converges C^0 to the commutator $hgh^{-1}g^{-1} \in SSympeo(M,\omega)$.

It is a standard fact that $\Phi^n.\Psi^n.(\Phi^n)^{-1}(\Psi^n)^{-1}$ is a hamiltonian isotopy.

Indeed let ϕ_t and ψ_t be symplectic isotopies, and let $\sigma_t = \phi_t\psi_t\phi_t^{-1}\psi_t^{-1}$, then

$$\dot{\sigma}_t = X_t + Y_t + Z_t + U_t$$

with $X_t = \dot{\phi}_t$, $Y_t = (\phi_t)_* \dot{\psi}_t$, $Z_t = -(\phi_t \psi_t \phi_t^{-1})_* \dot{\phi}_t$, and $U_t = -(\sigma_t)_* \dot{\psi}_t$.

By proposition 5, $i(X_t + Z_t)\omega$ and $i(Y_t + U_t)\omega$ are exact 1-forms. Hence σ_t is a hamiltonnian isotopy.

By Proposition 1, the metric D coincides with the one for hamiltonian isotopies. Hence $\Phi^n.\Psi^n.(\Phi^n)^{-1}(\Psi^n)^{-1}$ is a Cauchy sequence for d_{ham}. Therefore: $[SSympeo(M,\omega), SSympeo(M,\omega)] \subset Hameo(M,\omega)]$. This concludes the proof of theorem 3 □

Proof of Theorem 2: We now prove that $SSympeo(M,\omega)$, with the symplectic topology, is a topological group.

In fact, we prove that $\overline{\mathcal{Q}}$ (see section 3) is a topological group. Recall that an element of $\overline{\mathcal{Q}}$ is a couple $(\gamma, V = (\mathcal{H}, u)$

where $\gamma \in \mathcal{P}Homeo(M)$, $\mathcal{H} \in L^{(1,\infty)}([0,1], harm^1(M,\omega)$, $u \in L^{(0,1)}([0,1]xM,\mathbb{R})$, and there exists a d_{symp} - Cauchy sequence of symplectic isotopies $\Phi_n(t)$ such that $\Phi_n(1) \to \gamma$, in the C^0 topology and $\lim_{n\to\infty} (\mathcal{H}_n, u_n) = (\mathcal{H}, u)$. Here we wrote $\mathcal{H}_n$ for $\mathcal{H}^{\Phi_n}$ and u_n for $u_n^{\Phi_n}$.

The product and the inverse in $\overline{\mathcal{Q}}$ are given by:

$$(\gamma,(\mathcal{H},u)).(\gamma',(\mathcal{H}',u')) = (\gamma\gamma',(\mathcal{H}+\mathcal{H}', u + u'\circ\gamma + v))$$

$$(\gamma,(\mathcal{H},u))^{-1} = (\gamma^{-1},(-\mathcal{H}, -(u\circ\gamma + w))$$

where v is the limit of the Cauchy sequence $v_n(t)$ given by formula (II):

$$v_n(t) = \int_0^1 (\mathcal{H}_n'(\dot{\sigma}_n(s))\circ\sigma_n(s))ds,$$

with $\sigma_n(s) = (\Phi_n'(s))^{-1}$. and w the limit of a similar sequence in which σ_n is replaced by Φ_n.

Part I. Let us first show that the inversion is continuous: let $(\gamma_k,(\mathcal{H}_k,u_k))$ be a sequence converging to $(\gamma,(\mathcal{H},u))$, For each k, there is a Cauchy sequence Φ_n^k of symplectic isotopies such that $\Phi_n^k \to \gamma_k$ as $n \to \infty$ in the C^0 topology, $\mathcal{H}_n^k \to \mathcal{H}_k, u_n^k \to u_k$.

We need only to show that $w_k \to w$, that is (*)

$$lim_{n,k\to\infty} \int_0^1 |\mathcal{H}_n^k(\dot{\Phi}_n^k(s))\circ\Phi_n^k(s) - \mathcal{H}_n(\dot{\Phi}_n(s))\circ\Phi_n(s)|ds = 0.$$

We have the following inequalities :

$$||\dot{\Phi}_n^k - \dot{\Phi}_n|| \le ||\dot{\Phi}_n^k - V^k|| + ||V^k - V|| + ||V - \dot{\Phi}_n||$$

and each term in the right hand of this inequality $\to 0$ as $n,k \to \infty$.

Similarly,

$$|\mathcal{H}_n^k - \mathcal{H}_n| \le |\mathcal{H}_n^k - \mathcal{H}^k| + |\mathcal{H}^k - \mathcal{H}| + |\mathcal{H} - \mathcal{H}_n|$$

and each term in the right hand of this inequality $\to 0$ as $n,k \to \infty$.

Formula (*) follows from these inequalities and the techniques developped in this paper (including the reparametrisation lemma). We leave the details to the reader.

Part II. Now we prove that the composition is continuous: let $(\gamma^k, V^k = (\mathcal{H}^k, u^k))$ and $(\gamma'^k, V'^k = (\mathcal{H}'^k, u'^k))$ converging to $(\gamma, (\mathcal{H}, u))$ and $(\gamma', (\mathcal{H}', u'))$.

By part I, if $\dot{\sigma}_n^k \to U^k$ and $\dot{\sigma'}_n^k \to U'^k$,then by part I, $U^k \to U$. Here we denoted by σ_n^k, and σ'^k_n respectively $(\Phi_n^k)^{-1}, (\Phi'^k_n)^{-1}$.

We only need to prove:

1) $u^k \circ \gamma_k \to u \circ \gamma$

2) $v^k \to v$.

The proof of (1) goes along the lines explained in this paper (including the reparamareization lemma) and the details are left tothe reader.

The proof of (2) follows from part I and uses the inequalities:

$$||\dot{\sigma}_n^k - \dot{\rho}_n|| \leq ||\dot{\sigma}_n^k - U^k|| + ||U_-^k U|| + ||U - \dot{\rho}_n||$$

Each of the three parts of the second member of the inequality $\to 0$as $n,k \to \infty$. The details are left tothe reader.

This concludes the proof of theorem 2. □

Appendix: For the convenience of the reader, we give here the proofs of propositons 3, 4, and 5.

Proof of Proposition 3: Let θ be a p-form, X a vector field and ϕ a diffeomorphism. For any $x \in M$ and any vector fields $Y_1, ..Y_{p-1}$, we have:

$(\phi^{-1})^*[i_X \phi^* \theta](x)(Y_1, ..., Y_{p-1}) = (i_X \phi^* \theta)(\phi^{-1}(x))(D_x \phi^{-1}(Y_1(x), ...(D_x \phi^{-1}(Y_{p-1}(x))$

$= (\phi^* \theta)(\phi^{-1}(x))(X_{\phi^{-1}(x)}, D_x \phi^{-1}(Y_1(x)), ...(D_x \phi^{-1}(Y_{p-1}(x))$

$= \theta(\phi(\phi^{-1}(x))(D_{\phi^{-1}(x)} \phi(X_{\phi^{-1}(x)}), D_{\phi^{-1}(x)} \phi D_x \phi^{-1}(Y_1(x)), ...D_{\phi^{-1}(x)} \phi D_x \phi^{-1}(Y_{p-1}(x)$

$= \theta(x)((\phi_* X)_x, Y_1(x), ..Y_{p-1}(x))$

$= (i(\phi_* X)\theta)(x)(Y_1, .., Y_{p-1})$

since $D_{\phi^{-1}(x)} \phi D_x \phi^{-1} = D_x(\phi \phi^{-1}) = id$.

Therefore $(\phi^{-1})^*[i_X \phi^* \theta] = i(\phi_* X))\theta$ □

Proof of Proposition 4: This is just the chain rule. See [10] page 145. □

Proof of proposition 5: For a fixed t, we have

$$\frac{d}{ds}\phi_s^*\theta_t = \phi_s^*(L_{\dot{\phi}_s}\theta_t),$$

where L_X is the Lie derivative in the direction X. Since θ is closed, we have:

$$\frac{d}{ds}\phi_s^*\theta_t = \phi_s^*(di_{\dot{\phi}_s}\theta_t) = d(\phi_s^*(\theta_t(\dot{\phi}_s)) = d(\theta_t(\dot{\phi}_s)\circ\phi_s).$$

Hence for every u

$$\phi_u^*\theta_t - \theta_t = \int_0^u \frac{d}{ds}\phi_s^*\theta_t ds = d(\int_0^u (\theta_t(\dot{\phi}_s)\circ\phi_s)ds)$$

Now set $u = t$. □

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$\mathscr{L}$-Random and Fuzzy Normed Spaces and Classical Theory

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ABSTRACT

In this paper we study $\mathscr{L}$-random and $\mathscr{L}$-fuzzy normed spaces and prove open mapping and closed graph theorems for these spaces.

RESUMEN

En este artículo estudiamos espacios normados $\mathscr{L}$-random and $\mathscr{L}$-fuzzy. Probamos el teorema de la aplicación abierta y el teorema del gráfico cerrado.

Key words and phrases: *$\mathscr{L}$-random normed space, $\mathscr{L}$-fuzzy normed space, completeness, quotient space, open mapping and closed graph.*

Math. Subj. Class.: *PLEASE INFORM.*

1 Introduction and Preliminaries

In this paper we study $\mathscr{L}$-random and $\mathscr{L}$-fuzzy normed spaces and study completeness for these spaces. Further we prove open mapping and closed graph theorems in this setting. The ideas here are motivated from the functional analysis literature. The plan in sections 1-3 is to present in detail the $\mathscr{L}$-random normed space setting. In section 4 we see from the definition how easily the theory extends to the $\mathscr{L}$-fuzzy normed space situation.

Let $\mathscr{L} = (L, \geq_L)$ be a complete lattice, i.e. a partially ordered set in which every nonempty subset admits a supremum and infimum, and $0_{\mathscr{L}} = \inf L$, $1_{\mathscr{L}} = \sup L$. The space of lattice random distribution functions, denoted by Δ_L^+, is defined as the set of all mappings $F : \mathbb{R} \cup \{-\infty, +\infty\} \to L$ such that F is continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0_{\mathscr{L}}, F(+\infty) = 1_{\mathscr{L}}$.

Now $D_L^+ \subseteq \Delta_L^+$ is defined as $D_L^+ = \{F \in \Delta_L^+ : l^- F(+\infty) = 1_{\mathscr{L}}\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x. The space Δ_L^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \geq G$ if and only if $F(t) \geq_L G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ_L^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0_{\mathscr{L}}, & \text{if } t \leq 0, \\ 1_{\mathscr{L}}, & \text{if } t > 0. \end{cases}$$

Define the mapping $\mathscr{T}_\wedge$ from L^2 to L by:

$$\mathscr{T}_\wedge(x, y) = \begin{cases} x, \text{ if } y \geq_L x, \\ y, \text{ if } x \geq_L y. \end{cases}$$

Recall (see [4], [5]) that if $\{x_n\}$ is a given sequence in L, $(\mathscr{T}_\wedge)_{i=1}^n x_i$ is defined recurrently by $(\mathscr{T}_\wedge)_{i=1}^1 x_i = x_1$ and $(\mathscr{T}_\wedge)_{i=1}^n x_i = \mathscr{T}_\wedge((\mathscr{T}_\wedge)_{i=1}^{n-1} x_i, x_n)$ for $n \geq 2$.

A negation on $\mathscr{L}$ is any decreasing mapping $\mathscr{N} : L \to L$ satisfying $\mathscr{N}(0_{\mathscr{L}}) = 1_{\mathscr{L}}$ and $\mathscr{N}(1_{\mathscr{L}}) = 0_{\mathscr{L}}$. If $\mathscr{N}(\mathscr{N}(x)) = x$, for all $x \in L$, then $\mathscr{N}$ is called an involutive negation. In the following $\mathscr{L}$ is endowed with a (fixed) negation $\mathscr{N}$.

Definition 1.1. A *lattice random normed space* (briefly, $\mathscr{L}$-random normed space) is a triple $(X, \mathscr{P}, \mathscr{T})$, where X is a vector space, $\mathscr{T}$ is a t–norm on the lattice $\mathscr{L}$ and $\mathscr{P}$ is a mapping from $X \times [0, \infty)$ into D_L^+ such that the following conditions hold:

(LRN1) $\mathscr{P}(x, t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = 0$;
(LRN2) $\mathscr{P}(\alpha x, t) = \mathscr{P}\left(x, \frac{t}{|\alpha|}\right)$ for all x in X, $\alpha \neq 0$ and $t \geq 0$;
(LRN3) $\mathscr{P}(x + y, t + s) \geq_L \mathscr{T}(\mathscr{P}(x, t), \mathscr{P}(y, s))$ for all $x, y \in X$ and $t, s \geq 0$.

We note from (LPN2) that $\mathscr{P}(-x,t)=\mathscr{P}(x,t) \quad (x\in X, t\geq 0)$.

Example 1.2. Let $L=[0,1]\times[0,1]$ and operation $\leq_L$ defined by:

$$L=\{(a_1,a_2):(a_1,a_2)\in[0,1]\times[0,1] \text{and } a_1+a_2\leq 1\},$$

$$(a_1,a_2)\leq_L (b_1,b_2) \Longleftrightarrow a_1\leq b_1,\ a_2\geq b_2, \quad \forall a=(a_1,a_2), b=(b_1,b_2)\in L.$$

Then $(L,\leq_L)$ is a complete lattice (see [2]). In this complete lattice, we denote its units by $0_L=(0,1)$ and $1_L=(1,0)$. Let $(X,\|\cdot\|)$ be a normed space. Let $\mathscr{T}(a,b)=(\min\{a_1,b_1\},\max\{a_2,b_2\})$ for all $a=(a_1,a_2)$, $b=(b_1,b_2)\in[0,1]\times[0,1]$ and μ be a mapping defined by

$$\mathscr{P}(x,t)=\left(\frac{t}{t+\|x\|},\frac{\|x\|}{t+\|x\|}\right), \quad \forall t\in\mathbb{R}^+.$$

Then $(X,\mathscr{P},\mathscr{T})$ is a $\mathscr{L}$-random normed space.

Definition 1.3. Let $(X,\mathscr{P},\mathscr{T})$ be a $\mathscr{L}$-random normed space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $t>0$ and $\varepsilon\in L\setminus\{0_{\mathscr{L}}\}$, there exists a positive integer N such that $\mathscr{P}(x_n-x,t)>_L \mathscr{N}(\varepsilon)$ whenever $n\geq N$.

(2) A sequence $\{x_n\}$ in X is called *Cauchy sequence* if, for every $t>0$ and $\varepsilon\in L\setminus\{0_{\mathscr{L}}\}$, there exists a positive integer N such that $\mathscr{P}(x_n-x_m,t)>_L \mathscr{N}(\varepsilon)$ whenever $n\geq m\geq N$.

(3) A $\mathscr{L}$-random normed space $(X,\mathscr{P},\mathscr{T})$ is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Theorem 1.4. *If $(X,\mathscr{P},\mathscr{T})$ is a $\mathscr{L}$-random normed space and $\{x_n\}$ is a sequence such that $x_n\to x$, then $\lim_{n\to\infty}\mathscr{P}(x_n,t)=\mathscr{P}(x,t)$.*

Proof. The proof is the same as in [9]. □

Let $(X,\mathscr{P},\mathscr{T})$ be a $\mathscr{L}$-random normed space. For $t>0$ we define the open ball $B(x,r,t)$ with center x and radius $r\in L\setminus\{0_{\mathscr{L}},1_{\mathscr{L}}\}$ as

$$B(x,r,t)=\{y\in X:\ \mathscr{P}(x-y,t)>_L \mathscr{N}(r)\}.$$

Henceforth we assume that $\mathscr{T}$ is a continuous t–norm on the lattice $\mathscr{L}$ such that for every $\mu\in L\setminus\{0_{\mathscr{L}},1_{\mathscr{L}}\}$, there is a $\lambda\in L\setminus\{0_{\mathscr{L}},1_{\mathscr{L}}\}$ such that

$$\mathscr{T}^{n-1}(\mathscr{N}(\lambda),\ldots,\mathscr{N}(\lambda))>_L \mathscr{N}(\mu).$$

Lemma 1.5. *Let $(X,\mathscr{P},\mathscr{T})$ be a $\mathscr{L}$-random normed space. Let $\mathscr{N}$ be a continuous negator on $\mathscr{L}$. Define $E_{\lambda,\mathscr{P}}:V\to\mathbf{R}^+\cup\{0\}$ by*

$$E_{\lambda,\mathscr{P}}(x)=\inf\{t>0:\mathscr{P}(x,t)>_L \mathscr{N}(\lambda)\}$$

for each $\lambda\in L\setminus\{0_{\mathscr{L}},1_{\mathscr{L}}\}$ and $x\in V$. Then we have the following properties.

(i) *For any* $\mu \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ *there exists* $\lambda \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ *such that*

$$E_{\mu,\mathscr{P}}(x+y) \le E_{\lambda,\mathscr{P}}(x) + E_{\lambda,\mathscr{P}}(y)$$

for any $x, y \in V$.

(ii) *The sequence* $(x_n)_{n\in\mathbf{N}}$ *is convergent w.r.t. a* $\mathscr{L}$*-random norm* $\mathscr{P}$ *if and only if* $E_{\lambda,\mathscr{P}}(x_n - x) \to 0$. *Also the sequence* $(x_n)_{n\in\mathbf{N}}$ *is Cauchy w.r.t. a* $\mathscr{L}$*-random norm* $\mathscr{P}$ *if and only if it is Cauchy w.r.t.* $E_{\lambda,\mathscr{P}}$.

Proof. For *(i)*, by the continuity of the t-norm $\mathscr{T}$ and the negator $\mathscr{N}$, for every $\mu \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ we can find a $\lambda \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ such that

$$\mathscr{T}(\mathscr{N}(\lambda), \mathscr{N}(\lambda)) \ge_L \mathscr{N}(\mu).$$

By Definition 1.1 we have

$$\begin{aligned}\mathscr{P}(x+y, E_{\lambda,\mathscr{P}}(x) + E_{\lambda,\mathscr{P}}(y) + 2\delta) &\ge_L \mathscr{T}(\mathscr{P}(x, E_{\lambda,\mathscr{M}}(x)+\delta), \mathscr{P}(y, E_{\lambda,\mathscr{P}}(y)+\delta))\\ &\ge_L \mathscr{T}(\mathscr{N}(\lambda), \mathscr{N}(\lambda))\\ &\ge_L \mathscr{N}(\mu),\end{aligned}$$

for every $\delta > 0$, which implies that

$$E_{\mu,\mathscr{P}}(x+y) \le E_{\lambda,\mathscr{P}}(x) + E_{\lambda,\mathscr{P}}(y) + 2\delta.$$

Since $\delta > 0$ was arbitrary, we have

$$E_{\mu,\mathscr{P}}(x+y) \le E_{\lambda,\mathscr{P}}(x) + E_{\lambda,\mathscr{P}}(y).$$

For *(ii)*, we have

$$\mathscr{P}(x_n - x, \eta) >_L \mathscr{N}(\lambda) \iff E_{\lambda,\mathscr{P}}(x_n - x) < \eta$$

for every $\eta > 0$. □

2 Quotient Spaces

Definition 2.1. Let $(V, \mathscr{P}, \mathscr{T})$ be a $\mathscr{L}$-random normed space, W a linear manifold in V and let $Q : V \longrightarrow V/W$ be the natural map, $Qx = x + W$. For $t > 0$, we define:

$$\bar{\mathscr{P}}(x+W, t) = \sup\{\mathscr{P}(x+y, t) : y \in W\}.$$

Theorem 2.2. *Let W be a closed subspace of a $\mathscr{L}$-random normed space $(V,\mathscr{P},\mathscr{T})$. If $x \in V$ and $\epsilon > 0$, then there is an x' in V such that $x' + W = x + W$, $E_{\lambda,\mathscr{P}}(x') < E_{\lambda,\bar{\mathscr{P}}}(x+W)+\epsilon$.*

Proof. By the properties of sup, there always exists $y \in W$ such that $E_{\lambda,\mathscr{P}}(x+y) < E_{\lambda,\bar{\mathscr{P}}}(x+W)+\epsilon$. Now it is enough to put $x' = x+y$. □

Theorem 2.3. *Let W be a closed subspace of a $\mathscr{L}$-random normed space $(V,\mathscr{P},\mathscr{T})$ and $\bar{\mathscr{P}}$ be given in the above definition. Then:*

(1) *$\bar{\mathscr{P}}$ is a $\mathscr{L}$-random normed space, on V/W.*

(2) *$\bar{\mathscr{P}}(Qx,t) \geq_L \mathscr{P}(x,t)$.*

(3) *If $(V,\mathscr{P},\mathscr{T})$ is a complete $\mathscr{L}$-random normed space, then so is $(V/W,\bar{\mathscr{P}},\mathscr{T})$.*

Proof. It is clear that $\bar{\mathscr{P}}(x+W,t) >_L 0_{\mathscr{L}}$. Let $\bar{\mathscr{P}}(x+W,t) = 1_{\mathscr{L}}$. By definition there is a sequence $\{x_n\}$ in W such that $\mathscr{P}(x+x_n,t) \longrightarrow 1_{\mathscr{L}}$. Thus, $x+x_n \longrightarrow 0$ or equivalently $x_n \longrightarrow (-x)$ and since W is closed, $x \in W$ and $x+W = W$, the zero element of V/W. Then we have

$$\begin{aligned}\bar{\mathscr{P}}((x+W)+(y+W),t) &= \bar{\mathscr{P}}((x+y)+W,t)\\ &\geq_L \mathscr{P}((x+m)+(y+n),t)\\ &\geq_L \mathscr{T}(\mathscr{P}(x+m,t_1),\mathscr{P}(y+n,t_2))\end{aligned}$$

for $m,n \in W$, $x,y \in V$ and $t_1+t_2 = t$. Now if we take the sup, then we have

$$\bar{\mathscr{P}}((x+W)+(y+W),t) \geq_L \mathscr{T}(\bar{\mathscr{P}}(x+W,t_1),\bar{\mathscr{P}}(y+W,t_2)).$$

Therefore $\bar{\mathscr{P}}$ is a $\mathscr{L}$-random norm on V/W.

(2) By Definition 2.1, we have

$$\bar{\mathscr{P}}(Qx,t) = \bar{\mathscr{P}}(x+W,t) = \sup\{\mathscr{P}(x+y,t) : y \in W\} \geq_L \mathscr{P}(x,t).$$

Note that, by Lemma 1.5,

$$\begin{aligned}E_{\lambda,\bar{\mathscr{P}}}(Qx) &= \inf\{t>0 : \bar{\mathscr{P}}(Qx,t) >_L \mathscr{N}(\lambda)\} \leq \inf\{t>0 : \mathscr{P}(x,t) >_L \mathscr{N}(\lambda)\}\\ &= E_{\lambda,\mathscr{P}}(x).\end{aligned}$$

(3) Let $\{x_n+W\}$ be a Cauchy sequence in V/W. Then there exists $n_0 \in \mathbf{N}$ such that for every $n \geq n_0$, $E_{\lambda,\bar{\mathscr{P}}}((x_n+W)-(x_{n+1}+W)) \leq 2^{-n}$. Let $y_1 = 0$. Choose $y_2 \in W$ such that

$$E_{\lambda,\mathscr{P}}(x_1-(x_2-y_2),t) \leq E_{\lambda,\bar{\mathscr{P}}}((x_1-x_2)+W)+1/2.$$

However $E_{\lambda,\bar{\mathscr{P}}}((x_1-x_2)+W) \leq 1/2$ and so $E_{\lambda,\mathscr{P}}(x_1-(x_2-y_2)) \leq 1/2^2$.

Now suppose y_{n-1} has been chosen, so choose $y_n \in W$ such that

$$E_{\lambda,\mathscr{P}}((x_{n-1}+y_{n-1})-(x_n+y_n)) \leq E_{\lambda,\bar{\mathscr{P}}}((x_{n-1}-x_n)+W)+2^{-n+1}.$$

Hence we have

$$E_{\lambda,\mathscr{P}}((x_{n-1}+y_{n-1})-(x_n+y_n)) \leq 2^{-n+2}.$$

However for every positive integer $m > n$ and by Lemma 1.5 for $\lambda \in L$ there exists $\gamma \in L$, such that

$$\begin{aligned} E_{\lambda,\mathscr{P}}((x_m+y_m)-(x_n+y_n)) &\leq E_{\gamma,\mathscr{P}}((x_{n+1}+y_{n+1})-(x_n+y_n))+ \\ &\quad \cdots + E_{\gamma,\mathscr{P}}((x_m+y_m)-(x_{m-1}+y_{m-1})) \\ &\leq \sum_{i=n}^{m} 2^{-i}. \end{aligned}$$

By Lemma 1.5, $\{x_n+y_n\}$ is a Cauchy sequence in V. Since V is complete, there is an x_0 in V such that $x_n+y_n \longrightarrow x_0$ in V. On the other hand,

$$x_n+W = Q(x_n+y_n) \longrightarrow Q(x_0) = x_0+W.$$

Therefore, every Cauchy sequence $\{x_n+W\}$ is convergent in V/W and so V/W is complete. Thus $(V/W,\bar{\mathscr{P}},\mathscr{T})$ is a complete $\mathscr{L}$-random normed space. □

Theorem 2.4. *Let W be a closed subspace of a $\mathscr{L}$-random normed space $(V,\mathscr{P},\mathscr{T})$. If two of the spaces V, W and V/W are complete, then so is the third one.*

Proof. If V is a complete $\mathscr{L}$-random normed space, then so are V/W and W. Hence all that needs to be checked is that V is complete whenever both W and V/W are complete. Suppose that W and V/W are complete $\mathscr{L}$-random normed spaces and let $\{x_n\}$ be a Cauchy sequence in V. Since $E_{\lambda,\bar{\mathscr{P}}}((x_n-x_m)+W) \leq E_{\lambda,\mathscr{P}}(x_n-x_m)$ for each $m,n \in \mathbf{N}$, the sequence $\{x_n+W\}$ is Cauchy in V/W and so converges to $y+W$ for some $y \in W$. Thus there is a $n_0 \in \mathbf{N}$ such that for every $n \geq n_0$, we have $E_{\lambda,\bar{\mathscr{P}}}((x_n-y)+W) < 2^{-n}$. Now by the last theorem there exist a sequence $\{y_n\}$ in V such that $y_n+W = (x_n-y)+W$, $E_{\lambda,\mathscr{P}}(y_n) < E_{\lambda,\bar{\mathscr{P}}}((x_n-y)+W)+2^{-n}$. Thus we have $\lim_n E_{\lambda,\mathscr{P}}(y_n) \leq 0$ by Lemma 1.5, $\mathscr{P}(y_n,t) \to 1_{\mathscr{L}}$ for every $t > 0$, i.e. $\lim_n y_n = 0$. Therefore, $\{x_n-y_n-y\}$ is a Cauchy sequence in W and thus is convergent to a point $z \in W$. This implies that $\{x_n\}$ converges to $z+y$ and hence V is complete. □

3 Open Mapping and Closed Graph Theorems

Definition 3.1. A linear operator $T:(V,\mathscr{P},\mathscr{T}) \longrightarrow (V',\mathscr{P}',\mathscr{T}')$ is said to be *$\mathscr{L}$-random bounded* if there exist constants $h \in \mathbf{R}^+$ such that for every $x \in V$ and for every $t > 0$,

$$\mathscr{P}'(Tx,t) \geq_L \mathscr{P}(x,t/h). \tag{3.1}$$

Note that, by (3.1) we have

$$\begin{aligned} E_{\lambda,\mathscr{P}'}(Tx) &= \inf\{t>0:\mathscr{P}'(Tx,t)>_L \mathscr{N}(\lambda)\} \le \inf\{t>0:\mathscr{P}(x,t/h)>_L \mathscr{N}(\lambda)\} = \\ &= h\inf\{t>0:\mathscr{P}(x,t)>_L \mathscr{N}(\lambda)\} \\ &= hE_{\lambda,\mathscr{P}}(x). \end{aligned}$$

Theorem 3.2. *Every linear operator $T:(V,\mathscr{P},\mathscr{T})\longrightarrow(V',\mathscr{P}',\mathscr{T}')$ is $\mathscr{L}$-random bounded if and only if it is continuous.*

Proof. By (3.1) every $\mathscr{L}$-random bounded linear operator is continuous. Now, we prove the converse. Let the linear operator T be continuous but not $\mathscr{L}$-random bounded. Then, for each n in $\mathbf{N}$ there is a x_n in V such that $E_{\lambda,\mathscr{P}'}(Tx_n)\ge nE_{\lambda,\mathscr{P}}(p_n)$. If we let $y_n=\frac{x_n}{nE_{\lambda,\mathscr{P}}(x_n)}$ then it is easy to see $y_n\to 0$ but Ty_n do not tend to 0.

Theorem 3.3. (Open mapping theorem) *If T is a $\mathscr{L}$-random bounded linear operator from a complete $\mathscr{L}$-random normed space $(V,\mathscr{P},\mathscr{T})$ onto a complete $\mathscr{L}$-random normed space $(V',\mathscr{P}',\mathscr{T})$ then T is an open mapping.*

Proof. The theorem will be proved in several steps.

Step1: Let E be a neighborhood of the 0 in V. We show $0\in\left(\overline{T(E)}\right)^o$. Let W be a balanced neighborhood of 0 such that $W+W\subset E$. Since $T(V)=V'$ and W is absorbing, it follows that $V'=\cup_n T(nW)$, so by Theorem 3.17 in [6], there exists a $n_0\in\mathbf{N}$ such that $\overline{T(n_0W)}$ has nonempty interior. Therefore, $0\in\left(\overline{T(W)}\right)^o-\left(\overline{T(W)}\right)^o$. On the other hand,

$$\left(\overline{T(W)}\right)^o-\left(\overline{T(W)}\right)^o\subset\overline{T(W)}-\overline{T(W)}=\overline{T(W)+T(W)}\subset\overline{T(E)}.$$

Thus the set $\overline{T(E)}$ includes the neighborhood $\left(\overline{T(W)}\right)^o-\left(\overline{T(W)}\right)^o$ of 0.

Step 2: We show $0\in(T(E))^o$. Since $0\in E$ and E is an open set, there exists $0_{\mathscr{L}}<_L\alpha<_L 1_{\mathscr{L}}$ and $t_0\in(0,\infty)$ such that $B(0,\alpha,t_0)\subset E$. However $0_{\mathscr{L}}<_L\alpha<_L 1_{\mathscr{L}}$ so a sequence $\{\epsilon_n\}$ can be found such that $\mathscr{T}^{m-n}(\mathscr{N}(\epsilon_{n+1}),\mathscr{N}(\epsilon_m))\to 1_{\mathscr{L}}$, $\mathscr{N}(\alpha)<_L\lim_n\mathscr{T}^{n-1}(\mathscr{N}(\epsilon_1),\mathscr{N}(\epsilon_n))$ in which $m>n$. On the other hand, $0\in\overline{T(B(0,\epsilon_n,t'_n))}$, where $t'_n=\frac{1}{2^n}t_0$, so by step 1, there exist $0_{\mathscr{L}}<_L\sigma_n<_L 1_{\mathscr{L}}$ and $t_n>0$ such that $B(0,\sigma_n,t_n)\subset\overline{T(B(0,\epsilon_n,t'_n))}$. Since the set $\{B(0,r,1/n)\}$ is a countable local base at zero and $t'_n\longrightarrow 0$ as $n\longrightarrow\infty$, so t_n and σ_n can be chosen such that $t_n\longrightarrow 0$ and $\sigma_n\longrightarrow 0_{\mathscr{L}}$ as $n\to\infty$.

Now we show $B(0,\sigma_1,t_1)\subset(T(E))^o$. Suppose $y_0\in B(0,\sigma_1,t_1)$. Then $y_0\in\overline{T(B(0,\epsilon_1,t'_1))}$ and so for $0_{\mathscr{L}}<_L\sigma_2$ and $t_2>0$ the ball $B(y_0,\sigma_2,t_2)$ intersects $T(B(0,\epsilon_1,t'_1))$. Therefore there exists $x_1\in B(0,\epsilon_1,t'_1)$ such that $Tx_1\in B(y_0,\sigma_2,t_2)$, i.e. $\mathscr{P}'(y_0-Tx_1,t_2)>_L\mathscr{N}(\sigma_2)$ or equivalently $y_0-Tx_1\in B(0,\sigma_2,t_2)\subset\overline{T(B(0,\epsilon_1,t'_1))}$. By the similar argument there exist x_2 in $B(0,\epsilon_2,t'_2)$ such that

$$\mathscr{P}'(y_0-(Tx_1+Tx_2),t_3)=\mathscr{P}'((y_0-Tx_1)-Tx_2,t_3)>_L\mathscr{N}(\sigma_3).$$

If this process is continued, it leads to a sequence $\{x_n\}$ such that $x_n \in B(0,\epsilon_n,t'_n)$, $\mathscr{P}'\left(y_0-\sum_{j=1}^{n-1}Tx_j,t_n\right)>_L \mathscr{N}(\sigma_n)$. Now if $n,m\in\mathbf{N}$ and $m>n$, then

$$\mathscr{P}\left(\sum_{j=1}^{n}x_j-\sum_{j=n+1}^{m}x_j,t\right)=\mu\left(\sum_{j=n+1}^{m}x_j,t\right)\geq_L \mathscr{T}^{m-n}(\mathscr{P}(x_{n+1},t_{n+1}),\mathscr{P}(x_m,t_m))$$

where $t_{n+1}+t_{n+2}+\cdots+t_m=t$. Put $t'_0=\min\{t_{n+1},t_{n+2},\cdots,t_m\}$. Since $t'_n\longrightarrow 0$, there exists $n_0\in\mathbf{N}$ such that $0<t'_n\leq t'_0$ for $n>n_0$. Therefore, for $m>n$ we have

$$\begin{aligned}\mathscr{T}^{m-n}(\mathscr{P}(x_{n+1},t'_0),\mathscr{P}(x_m,t'_0)) &\geq_L \mathscr{T}^{m-n}(\mathscr{P}(x_{n+1},t'_{n+1}),\mathscr{P}(x_m,t'_m))\\ &\geq_L \mathscr{T}^{m-n}(\mathscr{N}(\epsilon_{n+1}),\mathscr{N}(\epsilon_m)).\end{aligned}$$

Hence,

$$\lim_{n\longrightarrow\infty}\mathscr{P}\left(\sum_{j=n+1}^{m}x_j,t\right)\geq_L \lim_{n\longrightarrow\infty}\mathscr{T}^{m-n}(\mathscr{N}(\epsilon_{n+1}),\mathscr{N}(\epsilon_m))=1_{\mathscr{L}}.$$

That is, $\mathscr{P}\left(\sum_{j=n+1}^{m}x_j,t\right)\longrightarrow 1_{\mathscr{L}}$, for all $t>0$. Thus the sequence $\left\{\sum_{j=1}^{n}x_j\right\}$ is a Cauchy sequence and consequently the series $\left\{\sum_{j=1}^{\infty}x_j\right\}$ converges to some point $x_0\in V$, because V is a complete space.

By fixing $t>0$, there exists $n_0\in\mathbf{N}$ such that $t>t_n$ for $n>n_0$, because $t_n\longrightarrow 0$. Thus

$$\mathscr{P}'\left(y_0-T\left(\sum_{j=1}^{n-1}x_j\right),t\right)\geq_L \mathscr{P}'\left(y_0-T\left(\sum_{j=1}^{n-1}x_j\right),t_n\right)\geq_L \mathscr{N}(\sigma_n)$$

and thus $\mathscr{P}'\left(y_0-T\left(\sum_{j=1}^{n-1}x_j\right),t\right)\longrightarrow 1_{\mathscr{L}}$. Therefore,

$$y_0=\lim_n T\left(\sum_{j=1}^{n-1}x_j\right)=T\left(\lim_n\sum_{j=1}^{n-1}x_j\right)=Tx_0.$$

But, by Proposition 1 of [7],

$$\begin{aligned}\mathscr{P}(x_0,t_0) &= \lim_n\mathscr{P}\left(\sum_{j=1}^{n}x_j,t_0\right)\geq_L \mathscr{T}^n(\lim_n(\mathscr{P}(x_1,t'_1),\mathscr{P}(x_n,t'_n))\\ &\geq_L \lim_n\mathscr{T}^{n-1}(\mathscr{N}(\epsilon_1),\ldots,\mathscr{N}(\epsilon_n))>_L\mathscr{N}(\alpha)\end{aligned}$$

Hence $x_0\in B(0,\alpha,t_0)$.

Step 3: Let G be an open subset of V and $x\in G$. Then we have

$$T(G)=Tx+T(-x+G)\supset Tx+(T(-x+G))^o.$$

Hence $T(G)$ is open, because it includes a neighborhood of each of its point. $\square$

Corollary 3.4. *Every one-to-one $\mathscr{L}$-random bounded linear operator from a complete $\mathscr{L}$-random normed space onto a complete $\mathscr{L}$-random normed space has a $\mathscr{L}$-random bounded inverse.*

Definition 3.5. Let $\mathscr{T}$ and $\mathscr{T}'$ be two continuous t-norms. Then $\mathscr{T}'$ *dominates* $\mathscr{T}$, denoted by $\mathscr{T}' \gg_L \mathscr{T}$, if for all $x_1, x_2, y_1, y_2 \in \mathscr{L}$,

$$\mathscr{T}[\mathscr{T}'(x_1,x_2),\mathscr{T}'(y_1,y_2)] \leq_L \mathscr{T}'[\mathscr{T}(x_1,y_1),\mathscr{T}(x_2,y_2)].$$

Theorem 3.6. (Closed graph theorem) *Let T be a linear operator from the complete $\mathscr{L}$-random normed space $(V,\mathscr{P},\mathscr{T})$ into the complete $\mathscr{L}$-random normed space $(V',\mathscr{P}',\mathscr{T})$. Suppose for every sequence $\{x_n\}$ in V such that $x_n \longrightarrow x$ and $Tx_n \longrightarrow y$ for some elements $x \in V$ and $y \in V'$ it follows that $Tx = y$. Then T is $\mathscr{L}$-random bounded.*

Proof. For any $t > 0$, $x \in V$ and $y \in V'$, define

$$\Phi((x,y),t) = \mathscr{T}'(\mathscr{P}(x,t),\mathscr{P}'(y,t)),$$

where $\mathscr{T}' \gg_L \mathscr{T}$. First we show that $(V \times V',\Phi,\mathscr{T})$ is a complete $\mathscr{L}$-random normed space. The properties of (LRN1) and (LRN2) are immediate from the definition. For the triangle inequality (LRN3), suppose that $x,z \in V$, $y,u \in V'$ and $t,s > 0$, then

$$\begin{aligned}
\mathscr{T}(\Phi((x,y),t),\Phi((z,u),s)) &= \mathscr{T}[\mathscr{T}'(\mathscr{P}(x,t),\mathscr{P}'(y,t)),\mathscr{T}'(\mathscr{P}(z,s),\mathscr{P}'(u,s))] \\
&\leq_L \mathscr{T}'[\mathscr{T}(\mathscr{P}(x,t),\mathscr{P}(z,s)),\mathscr{T}(\mathscr{P}'(y,t),\mathscr{P}'(u,s))] \\
&\leq_L \mathscr{T}'(\mathscr{P}(x+z,t+s),\mathscr{P}'(y+u,t+s)) \\
&= \Phi((x+z,y+u),t+s).
\end{aligned}$$

Now if $\{(x_n,y_n)\}$ is a Cauchy sequence in $V \times V'$, then for every $\epsilon \in L \setminus \{0_{\mathscr{L}}\}$ and $t > 0$ there exists $n_0 \in \mathbf{N}$ such that $\Phi((x_n,y_n)-(x_m,y_m),t) >_L \mathscr{N}(\epsilon)$ for $m,n > n_0$. Thus for $m,n > n_0$,

$$\begin{aligned}
\mathscr{T}'(\mathscr{P}(x_n-x_m,t),\mathscr{P}'(y_n-y_m,t)) &= \Phi((x_n-x_m,y_n-y_m),t) \\
&= \Phi((x_n,y_n)-(x_m,y_m),t) >_L \mathscr{N}(\epsilon).
\end{aligned}$$

Therefore $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in V and V', respectively, and there exist $x \in V$ and $y \in V'$ such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$ and consequently $(x_n,y_n) \longrightarrow (x,y)$. Hence $(V \times V',\Phi,\mathscr{T})$ is a complete $\mathscr{L}$-random normed space. The remainder of the proof is the same as the classical case. □

4 $\mathscr{L}$-fuzzy normed space

We conclude the paper with the setting of $\mathscr{L}$-fuzzy normed spaces. Consider the $\mathscr{L}$-fuzzy normed space $(X,\mathscr{F},\mathscr{T})$ in which $\mathscr{F}$ is a $\mathscr{L}$-fuzzy set on $X \times]0,+\infty[$ satisfying the following conditions for every x, y in X and t, s in $(0, +\infty)$:

(a) $0_{\mathscr{L}} <_L \mathscr{F}(x,t)$;

(b) $\mathscr{F}(x,t) = 1_{\mathscr{L}}$ if and only if $x = 0$;

(c) $\mathscr{F}(\alpha x, t) = \mathscr{F}(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$;

(d) $\mathscr{T}(\mathscr{F}(x,t), \mathscr{F}(y,s)) \leq_L \mathscr{F}(x+y, t+s)$;

(e) $\mathscr{F}(x,\cdot) :]0,\infty[\to L$ is continuous;

(f) $\lim_{t\to 0} \mathscr{F}(x,t) = 0_{\mathscr{L}}$ and $\lim_{t\to\infty} \mathscr{F}(x,t) = 1_{\mathscr{L}}$.

In this case $\mathscr{F}$ is called a *$\mathscr{L}$-fuzzy norm*. For some details on the $\mathscr{L}$-fuzzy normed spaces, please see [1]

It is clear that all the results in section 2 and 3 can be written for $\mathscr{L}$-fuzzy normed spaces.

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The Semigroup and the Inverse of the Laplacian on the Heisenberg Group[1]

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ABSTRACT

By decomposing the Laplacian on the Heisenberg group into a family of parametrized partial differential operators $\tilde{L}_\tau, \tau \in \mathbb{R} \setminus \{0\}$, and using parametrized Fourier-Wigner transforms, we give formulas and estimates for the strongly continuous one-parameter semigroup generated by $\tilde{L}_\tau$, and the inverse of $\tilde{L}_\tau$. Using these formulas and estimates, we obtain Sobolev estimates for the one-parameter semigroup and the inverse of the Laplacian.

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RESUMEN

Mediante descomposición del Laplaceano sobre el grupo de Heisenberg en una familia de operadores diferenciales parciales parametrizados $\tilde{L}_\tau, \tau \in \mathbb{R}\setminus\{0\}$, y usando transformada de Fourier-Wigner parametrizada, damos fórmulas y estimativas para la continuidad fuerte del semigrupo generado por $\tilde{L}_\tau$, y la inversa de $\tilde{L}_\tau$. Usando esas fórmulas y estimativas obtenemos estimativas de Sobolev para el semigrupo a un parámetro y la inversa del Laplaceano.

Key words and phrases: *Heisenberg group, Laplacian, parametrized partial differential operators, Hermite functions, Fourier-Wigner transforms, heat equation, one parameter semigroup, inverse of Laplacian, Sobolev spaces.*

Math. Subj. Class.: *47F05, 47G30, 35J70.*

1 The Laplacian on the Heisenberg Group

If we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ via

$$\mathbb{R}^2 \ni (x,y) \leftrightarrow z = x+iy \in \mathbb{C}$$

and let

$$\mathbb{H} = \mathbb{C}\times\mathbb{R},$$

then $\mathbb{H}$ becomes a non-commutative group when equipped with the multiplication $\cdot$ given by

$$(z,t)\cdot(w,s) = \left(z+w, t+s+\frac{1}{4}[z,w]\right), \quad (z,t),(w,s)\in\mathbb{H},$$

where $[z,w]$ is the symplectic form of z and w defined by

$$[z,w] = 2\,\mathrm{Im}(z\overline{w}).$$

In fact, $\mathbb{H}$ is a unimodular Lie group on which the Haar measure is just the ordinary Lebesgue measure $dz\,dt$.

Let $\mathfrak{h}$ be the Lie algebra of left-invariant vector fields on $\mathbb{H}$. A basis for $\mathfrak{h}$ is then given by X, Y and T, where

$$X = \frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial t},$$
$$Y = \frac{\partial}{\partial y} - \frac{1}{2}x\frac{\partial}{\partial t},$$

and

$$T = \frac{\partial}{\partial t}.$$

The Laplacian $\Delta_{\mathbb{H}}$ on $\mathbb{H}$ is defined by

$$\Delta_{\mathbb{H}} = -(X^2 + Y^2 + T^2).$$

A simple computation gives

$$\Delta_{\mathbb{H}} = -\Delta - \frac{1}{4}(x^2+y^2)\frac{\partial^2}{\partial t^2} + \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2},$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Let g be the Riemannian metric on $\mathbb{R}^3$ given by

$$g(x,y,t) = \begin{bmatrix} 1 & 0 & y/2 \\ 0 & 1 & -x/2 \\ y/2 & -x/2 & \frac{1}{4}(x^2+y^2) \end{bmatrix}$$

for all $(x,y,t) \in \mathbb{R}^3$. Then $\Delta_{\mathbb{H}}$ is also given by

$$-\Delta_{\mathbb{H}} = \frac{1}{\sqrt{\det g}} \sum_{1\le j,k\le 3} \partial_j(\sqrt{\det g}\, g_{j,k}\partial_k),$$

where $\partial_1 = \partial/\partial x$, $\partial_2 = \partial/\partial y$, $\partial_3 = \partial/\partial t$. Since the symbol $\sigma(\Delta_{\mathbb{H}})$ of $\Delta_{\mathbb{H}}$ is given by

$$\sigma(\Delta_{\mathbb{H}})(x,y,t;\xi,\eta,\tau) = \left(\xi + \frac{1}{2}y\tau\right)^2 + \left(\eta - \frac{1}{2}x\tau\right)^2 + \tau^2$$

for all (x,y,t) and (ξ,η,τ) in $\mathbb{R}^3$, it is easy to see that $\Delta_{\mathbb{H}}$ is an elliptic partial differential operator on $\mathbb{R}^3$ but not globally elliptic in the sense of Shubin [11]. Let us recall that $\Delta_{\mathbb{H}}$ is globally elliptic if there exist positive constants C and R such that

$$|\sigma(\Delta_{\mathbb{H}})(x,y,t;\xi,\eta,\tau)| \ge C\big(1+|x|+|y|+|t|+|\xi|+|\eta|+|\tau|\big)^2$$

whenever

$$|x|+|y|+|t|+|\xi|+|\eta|+|\tau| \ge R.$$

The aim of this paper is to give new estimates for the strongly continuous one-parameter semigroup $e^{-u\Delta_{\mathbb{H}}}$, $u>0$, generated by $\Delta_{\mathbb{H}}$ and the inverse $\Delta_{\mathbb{H}}^{-1}$ of $\Delta_{\mathbb{H}}$. More precisely, we use the Sobolev spaces $L^2_s(\mathbb{H})$, $s \in \mathbb{R}$, as in [1, 2] to estimate $\|e^{-u\Delta_{\mathbb{H}}}f\|_{L^2_s(\mathbb{H})}$, $u>0$, in terms of $\|f\|_{L^2(\mathbb{H})}$ for all f in $L^2(\mathbb{H})$, and to give an estimate for $\|e^{-u\Delta_{\mathbb{H}}}f\|_{L^2(\mathbb{H})}$ in terms of $\|f\|_{L^2_s(\mathbb{H})}$. These Sobolev spaces are also used to estimate $||\Delta_{\mathbb{H}}^{-1}f||_{L^2_{s+2}(\mathbb{H})}$ in terms of $||f||_{L^2_s(\mathbb{H})}$ for all f in $L^2_s(\mathbb{H})$.

The function F on $\mathbb{H}\times(0,\infty)$ given by

$$F(z,t,u)=(e^{-u\Delta_{\mathbb{H}}}f)(z,t),\quad (z,t)\in\mathbb{H},\,u>0,$$

is in fact the solution of the initial value problem

$$\begin{cases}\frac{\partial F}{\partial u}(z,t,u)=-(\Delta_{\mathbb{H}}F)(z,t,u), & (z,t)\in\mathbb{H},\,u>0,\\ F(z,t,0)=f(z,t), & (z,t)\in\mathbb{H},\end{cases}$$

for the Laplacian $\Delta_{\mathbb{H}}$.

Using the same techniques as in [1], we get for all $f\in L^2(\mathbb{H})$ and $u>0$,

$$(e^{-u\Delta_{\mathbb{H}}}f)(z,t)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{-it\tau}(e^{-u\tilde{L}_\tau}f^\tau)(z)\,d\tau,\quad (z,t)\in\mathbb{H},\tag{1.1}$$

where $\tilde{L}_\tau$, $\tau\in\mathbb{R}\setminus\{0\}$, is given by

$$\tilde{L}_\tau=-\Delta+\frac{1}{4}(x^2+y^2)\tau^2-i\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)\tau+\tau^2$$

and f^τ is the function on $\mathbb{C}$ given by

$$f^\tau(z)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{it\tau}f(z,t)\,dt,\quad z\in\mathbb{C},$$

provided that the integral exists. In fact, $f^\tau(z)$ is the inverse Fourier transform of $f(z,t)$ with respect to t evaluated at τ. In this paper, the nonzero parameter τ can be looked at as Planck's constant.

To obtain the estimates in this paper, we use formulas for $e^{-u\tilde{L}_\tau}$ and $\tilde{L}_\tau^{-1}$ in terms of the τ-Weyl transforms and the τ-Fourier–Wigner transforms of Hermite functions, $\tau\in\mathbb{R}\setminus\{0\}$, which we recall in, respectively, Section 2 and Section 3. The L^2-boundedness and the Hilbert–Schimdt property of τ-Weyl transforms are instrumental in obtaining the estimates.

Basic information on the classical Fourier–Wigner transforms, Wigner transforms and Weyl transforms can be found in [13] among others.

In Section 2, we introduce the τ-Weyl transforms and prove results on the L^2-boundedness and the Hilbert–Schmidt property of the τ-Weyl transforms. The τ-Fourier–Wigner transforms of Hermite functions are recalled in Section 3. A formula for $e^{-u\tilde{L}_\tau}f$, $u>0$, for every function f in $L^2(\mathbb{C})$ and an estimate for $\|e^{-u\tilde{L}_\tau}f\|_{L^2(\mathbb{C})}$, $u>0$, in terms of $\|f\|_{L^p(\mathbb{C})}$, $1\le p\le 2$, are given in Section 4. This formula gives a formula for $e^{-u\Delta_{\mathbb{H}}}$, $u>0$, immediately using the inverse Fourier transform as indicated by (1.1). In Section 5, we use the family $L_s^2(\mathbb{H})$, $s\in\mathbb{R}$, of Sobolev spaces with respect to the center of the Heisenberg group as in [1, 2] to obtain Sobolev estimates for $e^{-u\Delta_{\mathbb{H}}}f$, $u>0$, in terms of $\|f\|_{L^2(\mathbb{H})}$, and Sobolev estimates for

$\|e^{-u\Delta_{\mathbb{H}}}f\|_{L^2(\mathbb{H})}$, $u > 0$, in terms of the Sobolev norms $\|f\|_{L^2_s(\mathbb{H})}$ of f in $L^2_s(\mathbb{H})$. In Section 6, we obtain a formula for $\tilde{L}_\tau^{-1}$ and estimates for $\tilde{L}_\tau^{-1}$ which are then used to estimate $\Delta_{\mathbb{H}}^{-1}$. In Section 7, estimates for $\|\Delta_{\mathbb{H}}^{-1}f\|_{L^2_{s+2}(\mathbb{H})}$ in terms of $\|f\|_{L^2_s(\mathbb{H})}$ for all f in $L^2_s(\mathbb{H})$ are given.

We end this section by putting in perspectives the results in this paper. While the semigroup and the inverse can be studied in the framework of functional analysis as explained in [3, 4, 5, 8, 9, 16], the results and methods in this paper are based on explicit formulas in *hard* analysis and are related to the works in [1, 2, 6, 7, 10, 12, 14, 15].

2 τ-Weyl Transforms

Let f and g be functions in $L^2(\mathbb{R})$. Then for τ in $\mathbb{R}\setminus\{0\}$, the τ-Fourier–Wigner transform $V_\tau(f,g)$ is defined by

$$V_\tau(f,g)(q,p) = (2\pi)^{-1/2}|\tau|^{1/2}\int_{-\infty}^{\infty} e^{i\tau q y} f\left(y+\frac{p}{2}\right)\overline{g\left(y-\frac{p}{2}\right)}\,dy$$

for all q and p in $\mathbb{R}$. In fact,

$$V_\tau(f,g)(q,p) = |\tau|^{1/2}V(f,g)(\tau q,p), \quad q,p\in\mathbb{R},$$

where $V(f,g)$ is the classical Fourier–Wigner transform of f and g. A proof can be found in [1].

It can be proved that $V_\tau(f,g)$ is a function in $L^2(\mathbb{C})$ and we have the Moyal identity stating that

$$\|V_\tau(f,g)\|_{L^2(\mathbb{C})} = \|f\|_{L^2(\mathbb{R})}\|g\|_{L^2(\mathbb{R})}, \quad \tau\in\mathbb{R}\setminus\{0\}. \tag{2.1}$$

We define the τ-Wigner transform $W_\tau(f,g)$ of f and g by

$$W_\tau(f,g) = V_\tau(f,g)^\wedge. \tag{2.2}$$

Then we have the following connection of the τ-Wigner transform with the usual Wigner transform.

Theorem 2.1. *Let $\tau\in\mathbb{R}\setminus\{0\}$. Then for all functions f and g in $L^2(\mathbb{R})$,*

$$W_\tau(f,g)(x,\xi) = |\tau|^{-1/2}W(f,g)(x/\tau,\xi), \quad x,\xi\in\mathbb{R},$$

where $W(f,g)$ is the classical Wigner transform of f and g.

It is obvious that

$$W_\tau(f,g) = \overline{W_\tau(g,f)}, \quad f,g\in L^2(\mathbb{R}). \tag{2.3}$$

Let $\sigma \in L^p(\mathbb{C})$, $1 \le p \le \infty$. Then for all τ in $\mathbb{R}\setminus\{0\}$ and all functions f in the Schwartz space $\mathscr{S}(\mathbb{R})$ on $\mathbb{R}$, we define $W_\sigma^\tau f$ to be the tempered distribution on $\mathbb{R}$ by

$$(W_\sigma^\tau f, g) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sigma(x,\xi) W_\tau(f,g)(x,\xi)\, dx\, d\xi \tag{2.4}$$

for all g in $\mathscr{S}(\mathbb{R})$, where (F,G) is defined by

$$(F,G) = \int_{\mathbb{R}^n} F(z)\overline{G(z)}\, dz$$

for all measurable functions F and G on $\mathbb{R}^n$, provided that the integral exists. We call W_σ^τ the τ-Weyl transform associated to the symbol σ. It is easy to see that if σ is a symbol in the Schwartz space $\mathscr{S}(\mathbb{C})$ on $\mathbb{C}$, then $W_\sigma^\tau f$ is a function in $\mathscr{S}(\mathbb{R})$ for all f in $\mathscr{S}(\mathbb{R})$.

We have the following estimate for the norm of the Weyl transform $W_{\hat\sigma}^\tau$ in terms of the L^p norm of the symbol σ when $\sigma \in L^p(\mathbb{C})$, $1 \le p \le 2$.

Theorem 2.2. *Let $\sigma \in L^p(\mathbb{C})$, $1 \le p \le 2$. Then $W_{\hat\sigma}^\tau : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator and*

$$\|W_{\hat\sigma}^\tau\|_* \le (2\pi)^{-1/p} |\tau|^{-(1/2)+(1/p)} \|\sigma\|_{L^p(\mathbb{C})},$$

where $\|W_{\hat\sigma}^\tau\|_*$ is the operator norm of $W_{\hat\sigma}^\tau : L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Proof Let f and g be functions in $\mathscr{S}(\mathbb{R})$. Then

$$\begin{aligned}
(W_{\hat\sigma}^\tau f, g) &= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat\sigma(x,\xi) W_\tau(f,g)(x,\xi)\, dx\, d\xi \\
&= (2\pi)^{-1} |\tau|^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat\sigma(x,\xi) W(f,g)(x/\tau,\xi)\, dx\, d\xi \\
&= (2\pi)^{-1} |\tau|^{1/2} \int_{-\infty}^{\infty} \int_{\infty}^{\infty} \hat\sigma(\tau x,\xi) W(f,g)(x,\xi)\, dx\, d\xi.
\end{aligned}$$

But

$$\hat\sigma(\tau x, \xi) = |\tau|^{-1} \widehat{\sigma_{1/\tau}}(x,\xi), \quad x,\xi \in \mathbb{R},$$

where $\sigma_{1/\tau}$ is the dilation of σ with respect to the first variable by the amount $1/\tau$. More precisely,

$$\sigma_{1/\tau}(q,p) = \sigma(q/\tau, p), \quad q,p \in \mathbb{R}.$$

So,

$$\begin{aligned}
(W_{\hat\sigma}^\tau f, g) &= (2\pi)^{-1/2} |\tau|^{-1/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widehat{\sigma_{1/\tau}}(x,\xi) W(f,g)(x,\xi)\, dx\, d\xi \\
&= |\tau|^{-1/2} (W_{\widehat{\sigma_{1/\tau}}} f, g),
\end{aligned}$$

where $W_{\widehat{\sigma_{1/\tau}}}$ is the classical Weyl transform with symbol $\widehat{\sigma_{1/\tau}}$. Thus, it follows from Theorem 21.1 in [14] that $W_{\hat{\sigma}}^{\tau}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator and

$$\|W_{\hat{\sigma}}^{\tau}\|_* \le |\tau|^{-1/2}(2\pi)^{-1/p}\|\sigma_{1/\tau}\|_{L^p(\mathbb{C})} = (2\pi)^{-1/p}|\tau|^{-(1/2)+(1/p)}\|\sigma\|_{L^p(\mathbb{C})}.$$

$\square$

We have the following result for the Hilbert–Schmidt norm of the Weyl transform $W_{\hat{\sigma}}^{\tau}$ in terms of the L^2 norm of the symbol σ when $\sigma \in L^2(\mathbb{C})$.

Theorem 2.3. *Let $\sigma \in L^2(\mathbb{C})$. Then $W_{\hat{\sigma}}^{\tau}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Hilbert–Schmidt operator and*

$$\|W_{\hat{\sigma}}^{\tau}\|_{HS} = (2\pi)^{-1/2}\|\sigma\|_{L^2(\mathbb{C})},$$

where $\|W_{\hat{\sigma}}^{\tau}\|_{HS}$ is the Hilbert–Schmidt norm of $W_{\hat{\sigma}}^{\tau}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

Proof Let f and g be functions in $\mathscr{S}(\mathbb{R})$. Then

$$\begin{aligned}
(W_{\hat{\sigma}}^{\tau} f, g) &= (2\pi)^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\sigma}(x,\xi)W_{\tau}(f,g)(x,\xi)\,dx\,d\xi \\
&= (2\pi)^{-1/2}|\tau|^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\hat{\sigma}(x,\xi)W(f,g)(x/\tau,\xi)\,dx\,d\xi \\
&= (2\pi)^{-1/2}|\tau|^{1/2}\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\hat{\sigma}(\tau x,\xi)W(f,g)(x,\xi)\,dx\,d\xi.
\end{aligned}$$

But

$$\hat{\sigma}(\tau x,\xi) = |\tau|^{-1/2}\widehat{\sigma_{1/\tau}}(x,\xi), \quad x,\xi \in \mathbb{R},$$

where $\sigma_{1/\tau}$ is the dilation of σ with respect to the first variable by the amount $1/\tau$, *i.e.*,

$$\sigma_{1/\tau}(q,p) = \sigma(q/\tau,p), \quad q,p \in \mathbb{R}.$$

So,

$$\begin{aligned}
(W_{\hat{\sigma}}^{\tau} f, g) &= (2\pi)^{-1}|\tau|^{-1/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widehat{\sigma_{1/\tau}}(x,\xi)W(f,g)(x,\xi)\,dx\,d\xi \\
&= |\tau|^{-1/2}(W_{\widehat{\sigma_{1/\tau}}} f, g),
\end{aligned}$$

where $W_{\widehat{\sigma_{1/\tau}}}$ is the classical Weyl transform with symbol $\widehat{\sigma_{1/\tau}}$. Thus, it follows from Theorem 7.5 in [13] that $W_{\hat{\sigma}}^{\tau}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a Hilbert–Schmidt operator and

$$\begin{aligned}
\|W_{\hat{\sigma}}^{\tau}\|_{HS} &= |\tau|^{-1/2}\|W_{\widehat{\sigma_{1/\tau}}}\|_{HS} \\
&= (2\pi)^{-1/2}|\tau|^{-1/2}\|\sigma_{1/\tau}\|_{L^2(\mathbb{C})} \\
&= (2\pi)^{-1/2}\|\sigma\|_{L^2(\mathbb{C})}.
\end{aligned}$$

$\square$

3 Fourier–Wigner Transforms of Hermite Functions

For $\tau \in \mathbb{R}\setminus\{0\}$ and for $k = 0,1,2,\dots$, we define e_k^τ to be the function on $\mathbb{R}$ by

$$e_k^\tau(x) = |\tau|^{1/4} e_k(\sqrt{|\tau|}x), \quad x \in \mathbb{R}.$$

Here, e_k is the Hermite function of order k defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \quad x \in \mathbb{R},$$

where H_k is the Hermite polynomial of degree k given by

$$H_k(x) = (-1)^k e^{x^2/2} \left(\frac{d}{dx}\right)^k (e^{-x^2}), \quad x \in \mathbb{R}.$$

For $j,k = 0,1,2,\dots$, we define $e_{j,k}^\tau$ on $\mathbb{R}^2$ by

$$e_{j,k}^\tau = V_\tau(e_j^\tau, e_k^\tau).$$

The following theorem gives the connection of $\{e_{j,k}^\tau : j,k = 0,1,2,\dots\}$ with $\{e_{j,k} : j,k = 0,1,2,\dots\}$, where

$$e_{j,k} = V(e_j, e_k), \quad j,k = 0,1,2,\dots.$$

A proof can be found in [1].

Theorem 3.1. *For $\tau \in \mathbb{R}\setminus\{0\}$ and for $j,k = 0,1,2,\dots$,*

$$e_{j,k}^\tau(q,p) = |\tau|^{1/2} e_{j,k}\left(\frac{\tau}{\sqrt{|\tau|}} q, \sqrt{|\tau|}p\right), \quad q,p \in \mathbb{R}.$$

Theorem 3.2. *$\{e_{j,k}^\tau : j,k = 0,1,2,\dots\}$ forms an orthonormal basis for $L^2(\mathbb{R}^2)$.*

Theorem 3.2 follows from Theorem 3.1 and Theorem 21.2 in [13] to the effect that $\{e_{j,k} : j,k = 0,1,2,\dots\}$ is an orthonormal basis for $L^2(\mathbb{R}^2)$.

Theorem 3.3. *For $j,k = 0,1,2,\dots$,*

$$\tilde{L}_\tau e_{j,k}^\tau = (2k + 1 + |\tau|)|\tau| e_{j,k}^\tau.$$

Theorem 3.3 can be proved using Theorem 3.1, Theorem 3.3 in [2] and Theorem 22.2 in [13] telling us that for $j,k = 0,1,2,\dots$, $e_{j,k}$ is an eigenfunction of L_1 corresponding to the eigenvalue $2k+1$ and the fact that, $\tilde{L}_\tau = L_\tau + \tau^2$.

4 A Formula and an Estimate for $e^{-u\tilde{L}_\tau}$, $u > 0$

Let $\tau \in \mathbb{R}\setminus\{0\}$. Then a formula for $e^{-u\tilde{L}_\tau}$, $u > 0$, is given by the following theorem.

Theorem 4.1. *Let $f \in L^2(\mathbb{C})$. Then for $u > 0$,*

$$e^{-u\tilde{L}_\tau} f = (2\pi)^{1/2}\sum_{k=0}^{\infty} e^{-(2k+1+|\tau|)|\tau|u} V_\tau(W^\tau_{\hat{f}} e^\tau_k, e^\tau_k),$$

where the convergence of the series is understood to be in $L^2(\mathbb{C})$.

Proof Let $f \in L^2(\mathbb{C})$. Then from Theorem 3.3 we have for $u > 0$

$$e^{-u\tilde{L}_\tau} f = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} e^{-(2k+1+|\tau|)|\tau|u}(f, e^\tau_{j,k})e^\tau_{j,k} = e^{-|\tau|^2 u} e^{-uL_\tau} f, \tag{4.1}$$

where the series is convergent in $L^2(\mathbb{C})$. Now, using the formula for $e^{-uL_\tau} f$ in [2] and (4.1), we get

$$e^{-u\tilde{L}_\tau} f = (2\pi)^{1/2}\sum_{k=0}^{\infty} e^{-(2k+1+|\tau|)|\tau|u} V_\tau(W^\tau_{\hat{f}} e^\tau_k, e^\tau_k)$$

for all f in $L^2(\mathbb{C})$ and $u > 0$. □

For all τ in $\mathbb{R}\setminus\{0\}$, we have the following estimate for the L^2 norm of $e^{-u\tilde{L}_\tau} f$, $u > 0$, in terms of the L^p norm of f.

Theorem 4.2. *Let $\tau \in \mathbb{R}\setminus\{0\}$. Then for all functions f in $L^p(\mathbb{C})$, $1 \le p \le 2$,*

$$\|e^{-u\tilde{L}_\tau} f\|_{L^2(\mathbb{C})} \le (2\pi)^{-(1/p)+(1/2)}|\tau|^{-(1/2)+(1/p)} e^{-\tau^2 u}\frac{1}{2\sinh(|\tau|u)}\|f\|_{L^p(\mathbb{C})}.$$

Proof By Theorem 4.1, the Moyal identity (2.1) and the fact that

$$\|e^\tau_k\|_{L^2(\mathbb{R})} = 1, \quad k = 0, 1, 2, \dots,$$

we get

$$\|e^{-u\tilde{L}_\tau} f\|_{L^2(\mathbb{C})} \le (2\pi)^{1/2} e^{-(|\tau|+|\tau|^2)u}\sum_{k=0}^{\infty} e^{-2k|\tau|u}\|W^\tau_{\hat{f}} e^\tau_k\|_{L^2(\mathbb{R})}, \quad u > 0. \tag{4.2}$$

Applying Theorem 2.2 to (4.2), we get

$$\begin{aligned}
&\|e^{-u\tilde{L}_\tau} f\|_{L^2(\mathbb{C})} \\
\le\ & (2\pi)^{-(1/p)+(1/2)}|\tau|^{-(1/2)+(1/p)} e^{-(|\tau|+|\tau|^2)u}\left(\sum_{k=0}^{\infty} e^{-2k|\tau|u}\right)\|f\|_{L^p(\mathbb{C})} \\
=\ & (2\pi)^{-(1/p)+(1/2)}|\tau|^{-(1/2)+(1/p)} e^{-|\tau|^2 u}\frac{1}{2\sinh(|\tau|u)}\|f\|_{L^p(\mathbb{C})},
\end{aligned}$$

as asserted.

□

5 Sobolev Estimates for $e^{-\Delta_{\mathbb{H}}}$, $u > 0$

Let $s \in \mathbb{R}$. Then we define $L^2_s(\mathbb{H})$ to be the set of all tempered distributions f in $\mathscr{S}'(\mathbb{H})$ such that $f^\tau(z)$ is a measurable function and

$$\int_{\mathbb{C}}\int_{-\infty}^{\infty} |\tau|^{2s}|f^\tau(z)|^2 d\tau\, dz < \infty.$$

For every f in $L^2_s(\mathbb{H})$, we define the norm $\|f\|_{L^2_s(\mathbb{H})}$ by

$$\|f\|^2_{L^2_s(\mathbb{H})} = \int_{\mathbb{C}}\int_{-\infty}^{\infty} |\tau|^{2s}|f^\tau(z)|^2 d\tau\, dz.$$

Then it can be shown easily that $L^2_s(\mathbb{H})$ is an inner product space in which the inner product $(\,,\,)_{L^2_s(\mathbb{H})}$ is given by

$$(f,g)_{L^2_s(\mathbb{H})} = \int_{\mathbb{C}}\int_{-\infty}^{\infty} |\tau|^{2s} f^\tau(z)\overline{g^\tau(z)}\, d\tau\, dz$$

for all f and g in $L^2_s(\mathbb{H})$.

Theorem 5.1. *Let $s \geq 1$. Then for $u > 0$, $e^{-u\Delta_{\mathbb{H}}} : L^2(\mathbb{H}) \to L^2_s(\mathbb{H})$ is a bounded linear operator and*

$$\|e^{-u\Delta_{\mathbb{H}}} f\|_{L^2_s(\mathbb{H})} \leq \frac{c_s}{2u^s}\|f\|_{L^2(\mathbb{H})}, \quad f \in L^2(\mathbb{H}),$$

where

$$c_s = \sup_{\tau\in\mathbb{R}\setminus\{0\}} (|\tau|^s/\sinh|\tau|).$$

Proof Let $u > 0$ and $f \in L^2(\mathbb{H})$. Then by (1.1), Fubini's theorem, Plancherel's theorem and Theorem 4.2 with $p = 2$,

$$\begin{aligned}
\|e^{-u\Delta_{\mathbb{H}}} f\|^2_{L^2_s(\mathbb{H})} &= \int_{\mathbb{C}}\int_{-\infty}^{\infty} |\tau|^{2s}|(e^{-u\Delta_{\mathbb{H}}} f)^\tau(z)|^2 d\tau\, dz \\
&= \int_{-\infty}^{\infty} |\tau|^{2s}\left(\int_{\mathbb{C}} |(e^{-u\Delta_{\mathbb{H}}} f)^\tau(z)|^2 dz\right) d\tau \\
&= \int_{-\infty}^{\infty} |\tau|^{2s}\left(\int_{\mathbb{C}} |(e^{-u\tilde{L}_\tau} f^\tau)(z)|^2 dz\right) d\tau \\
&= \int_{-\infty}^{\infty} |\tau|^{2s}\|e^{-u\tilde{L}_\tau} f^\tau\|^2_{L^2(\mathbb{C})} d\tau \\
&\leq \frac{1}{4}\left(\int_{-\infty}^{\infty} \frac{e^{-2\tau^2 u}|\tau|^{2s}}{\sinh^2(|\tau|u)}\|f^\tau\|^2_{L^2(\mathbb{C})} d\tau\right) \\
&\leq \frac{1}{4}\int_{-\infty}^{\infty} \frac{|\tau|^{2s}}{\sinh^2(|\tau|u)}\left(\int_{\mathbb{C}} |f^\tau(z)|^2 dz\right) d\tau
\end{aligned}$$

$$= \frac{1}{4u^{2s+1}} \int_{-\infty}^{\infty} \frac{|\tau|^{2s}}{\sinh^2(|\tau|u)} \left(\int_{\mathbb{C}} |\check{f}(z,\tau/u)|^2 dz \right) d\tau,$$

where $\check{f}$ is the inverse Fourier transform of f with respect to t. So, using a simple change of variable and letting

$$C_s = \sup_{\tau \in \mathbb{R}\setminus\{0\}} (|\tau|^{2s}/\sinh^2|\tau|),$$

we get

$$\|e^{-u\Delta_{\mathbb{H}}} f\|^2_{L^2_s(\mathbb{H})} \leq \frac{C_s}{4u^{2s}} \int_{-\infty}^{\infty} \left(\int_{\mathbb{C}} |\check{f}(z,\tau)|^2 dz \right) d\tau = \frac{C_s}{4u^{2s}} \|f\|^2_{L^2(\mathbb{H})}$$

and this completes the proof. □

The following result complements Theorem 5.1.

Theorem 5.2. *Let $s \leq -1$. Then for $u > 0$, $e^{-u\Delta_{\mathbb{H}}} : L^2_s(\mathbb{H}) \to L^2(\mathbb{H})$ is a bounded linear operator and*

$$\|e^{-u\Delta_{\mathbb{H}}} f\|_{L^2(\mathbb{H})} \leq \frac{c_{-s}}{2u^{-s}} \|f\|_{L^2_s(\mathbb{H})}, \quad f \in L^2_s(\mathbb{H}),$$

where

$$c_{-s} = \sup_{\tau \in \{0\}} (|\tau|^{-s} \sinh|\tau|).$$

The proof of Theorem 5.2 is very similar to that of Theorem 5.1 and is hence omitted.

6 Two Formulas and an Estimate for $\tilde{L}_\tau^{-1}$

Let $\tau \in \mathbb{R} \setminus \{0\}$. Then a formula for L_τ^{-1} is given by the following theorem.

Theorem 6.1. *Let $f \in L^2(\mathbb{C})$. Then*

$$\tilde{L}_\tau^{-1} f = (2\pi)^{1/2} \sum_{k=0}^{\infty} \frac{1}{(2k+1+|\tau|)|\tau|} V_\tau(W^\tau_{\hat{f}} e^\tau_k, e^\tau_k),$$

where the convergence of the series is understood to be in $L^2(\mathbb{C})$.

Proof Let $f \in L^2(\mathbb{C})$. Then

$$\tilde{L}_\tau^{-1} f = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(2k+1+|\tau|)|\tau|} (f, e^\tau_{j,k}) e^\tau_{j,k}, \tag{6.1}$$

where the series is convergent in $L^2(\mathbb{C})$. Now, by Plancherel's theorem and (2.2)–(2.4),

$$(f, e^\tau_{j,k}) = \int_{\mathbb{C}} f(z) \overline{V_\tau(e^\tau_j, e^\tau_k)(z)} dz = \int_{\mathbb{C}} \hat{f}(\zeta) \overline{V_\tau(e^\tau_j, e^\tau_k)^\wedge(\zeta)} d\zeta$$

$$= \int_{\mathbb{C}} \hat{f}(\zeta)\overline{W_\tau(e_j^\tau, e_k^\tau)(\zeta)}\,d\zeta = (2\pi)^{1/2}(W_{\hat{f}} e_k^\tau, e_j^\tau) \tag{6.2}$$

for $j,k = 0,1,2,\ldots$. Similarly, for $j,k = 0,1,2,\ldots$, and g in $L^2(\mathbb{C})$, we get

$$(e_{j,k}^\tau, g) = \overline{(g, e_{j,k}^\tau)} = (2\pi)^{1/2}\overline{(W_{\hat{g}}^\tau e_k^\tau, e_j^\tau)} = (2\pi)^{1/2}(e_j^\tau, W_{\hat{g}}^\tau e_k^\tau). \tag{6.3}$$

So, by (6.1)–(6.3), Fubini's theorem and Parseval's identity,

$$\begin{aligned}(\tilde{L}_\tau^{-1} f, g) &= 2\pi \sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|} \sum_{j=0}^\infty (W_{\hat{f}}^\tau e_k^\tau, e_j^\tau)(e_j^\tau, W_{\hat{g}}^\tau e_k^\tau) \\ &= 2\pi \sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|}(W_{\hat{f}}^\tau e_k^\tau, W_{\hat{g}}^\tau e_k^\tau). \end{aligned} \tag{6.4}$$

By Plancherel's theorem and (2.2)–(2.4),

$$\begin{aligned}(W_{\hat{f}}^\tau e_k^\tau, W_{\hat{g}}^\tau e_k^\tau) &= (2\pi)^{-1/2}\overline{\int_{\mathbb{C}} \hat{g}(z) W_\tau(e_k^\tau, W_{\hat{f}}^\tau e_k^\tau)(z)\,dz} \\ &= (2\pi)^{-1/2}\int_{\mathbb{C}} W_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau)(z)\overline{\hat{g}(z)}\,dz \\ &= (2\pi)^{-1/2}\int_{\mathbb{C}} V_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau)(z)\overline{g(z)}\,dz \end{aligned} \tag{6.5}$$

for $k = 0,1,2,\ldots$. Thus, by (6.4), (6.5) and Fubini's theorem,

$$\begin{aligned}(\tilde{L}_\tau^{-1} f, g) &= (2\pi)^{1/2} \sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|}(V_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau), g) \\ &= (2\pi)^{1/2}\left(\sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|} V_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau), g\right) \end{aligned} \tag{6.6}$$

for all f and g in $L^2(\mathbb{C})$. Thus, by (6.6),

$$\tilde{L}_\tau^{-1} f = (2\pi)^{1/2} \sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|} V_\tau(W_{\hat{f}}^\tau e_k^\tau, e_k^\tau)$$

for all f in $L^2(\mathbb{C})$. □

The formula (6.4) is an important formula in its own right and we upgrade it to the status of a theorem.

Theorem 6.2. *For all $\tau \in \mathbb{R}\setminus\{0\}$, the inverse $\tilde{L}_\tau^{-1}$ of the parametrized partial differential operators $\tilde{L}_\tau$ is given by*

$$(\tilde{L}_\tau^{-1} f, g) = 2\pi \sum_{k=0}^\infty \frac{1}{(2k+1+|\tau|)|\tau|}(W_{\hat{f}}^\tau e_k^\tau, W_{\hat{g}}^\tau e_k^\tau), \quad f, g \in L^2(\mathbb{C}).$$

For all τ in $\mathbb{R}\setminus\{0\}$, we have the following estimate for the L^2 norm of $\tilde{L}_\tau^{-1}f$ in terms of the L^2 norm of f.

Theorem 6.3. *Let $\tau\in\mathbb{R}\setminus\{0\}$. Then for all functions f in $L^2(\mathbb{C})$,*

$$\|\tilde{L}_\tau^{-1}f\|_{L^2(\mathbb{C})}\le|\tau|^{-2}\|f\|_{L^2(\mathbb{C})}.$$

Proof Let f and g be functions in $L^2(\mathbb{R})$. Then by Theorems 2.3 and 6.2,

$$\begin{aligned}|(\tilde{L}_\tau^{-1}f,g)| &\le 2\pi\frac{1}{|\tau|^2}\sum_{k=0}^{\infty}|(W_{\hat{f}}^\tau e_k^\tau,W_{\hat{g}}^\tau e_k^\tau)|\\ &\le 2\pi\frac{1}{|\tau|^2}\|W_{\hat{f}}^\tau\|_{HS}\|W_{\hat{g}}^\tau\|_{HS}\\ &= \frac{1}{|\tau|^2}\|f\|_{L^2(\mathbb{C})}\|g\|_{L^2(\mathbb{C})}\end{aligned}$$

and this completes the proof. □

7 Sobolev Estimates for $\Delta_{\mathbb{H}}^{-1}$

We have the following simple result giving the connection of $\Delta_{\mathbb{H}}^{-1}$ with $\tilde{L}_\tau^{-1}$, $\tau\in\mathbb{R}\setminus\{0\}$, which can be proved easily using the elementary properties of the Fourier transform and the Fourier inversion formula.

Theorem 7.1. *Let $f\in L^2(\mathbb{H})$. Then*

$$(\Delta_{\mathbb{H}}^{-1}f)(z,t)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}e^{-it\tau}(\tilde{L}_\tau^{-1}f^\tau)(z)\,d\tau,\quad (z,t)\in\mathbb{H}.$$

We can now give the following theorem, which can be seen as another manifestation of the ellipticity of $\Delta_{\mathbb{H}}$.

Theorem 7.2. *Let $s\in\mathbb{R}$. Then $\Delta_{\mathbb{H}}^{-1}:L_s^2(\mathbb{H})\to L_{s+2}^2(\mathbb{H})$ and*

$$\|\Delta_{\mathbb{H}}^{-1}f\|_{L_{s+2}^2(\mathbb{H})}\le\|f\|_{L_s^2(\mathbb{H})},\quad f\in L_s^2(\mathbb{H}).$$

Proof By Fubini's theorem, Plancherel's theorem, Theorems 6.3 and 7.1,

$$\begin{aligned}\|\Delta_{\mathbb{H}}^{-1}f\|_{L_{s+2}^2(\mathbb{H})}^2 &= \int_{\mathbb{C}}\int_{-\infty}^{\infty}|\tau|^{2(s+2)}|(\Delta_{\mathbb{H}}^{-1}f)^\tau(z)|^2d\tau\,dz\\ &= \int_{-\infty}^{\infty}|\tau|^{2(s+2)}\left(\int_{\mathbb{C}}|(\Delta_{\mathbb{H}}^{-1}f)^\tau(z)|^2dz\right)d\tau\end{aligned}$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} |\tau|^{2(s+2)} \left(\int_{\mathbb{C}} |(\tilde{L}_\tau^{-1} f^\tau)(z)|^2 dz \right) d\tau \\
&= \int_{-\infty}^{\infty} |\tau|^{2(s+2)} \|\tilde{L}_\tau^{-1} f^\tau\|^2_{L^2(\mathbb{C})} d\tau \\
&\leq \int_{-\infty}^{\infty} |\tau|^{2s} \|f^\tau\|^2_{L^2(\mathbb{C})} d\tau \\
&= \int_{-\infty}^{\infty} |\tau|^{2s} \left(\int_{\mathbb{C}} |f^\tau(z)|^2 dz \right) d\tau \\
&= \int_{\mathbb{C}} \int_{-\infty}^{\infty} |\tau|^{2s} |f^\tau(z)|^2 d\tau\, dz \\
&= \|f\|^2_{L^2_s(\mathbb{H})},
\end{aligned}
$$

as asserted. □

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Self-Dual and Anti-Self-Dual Solutions of Discrete Yang-Mills Equations on a Double Complex

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ABSTRACT

We study a discrete model of the $SU(2)$ Yang-Mills equations on a combinatorial analog of $\mathbb{R}^4$. Self-dual and anti-self-dual solutions of discrete Yang-Mills equations are constructed. To obtain these solutions we use both techniques of a double complex and the quaternionic approach. Interesting analogies between instanton, anti-instanton solutions of discrete and continual self-dual, anti-self-dual equations are also discussed.

RESUMEN

Estudiamos el modelo discreto de las ecuaciones de Yang-Mills $SU(2)$ sobre un análogo combinatório de $\mathbb{R}^4$. Soluciones auto-dual y anti-auto-dual para las ecuaciones discretas de Yang-Mills son construidas. Para obtener estas soluciones usamos las técnicas de doble complejo y abordage cuaternionico. Interesantes analogías entre soluciones instantones y anti-instantones de ecuaciones discretas y continuas auto-dual y anti-auto-dual son discutidas.

Key words and phrases: *Yang-Mills equations, self-dual and anti-self-dual equations, instantons and anti-instantons, difference equations.*

Math. Subj. Class.: *81T13, 39A12.*

1 Introduction

It is well known that the self-dual and anti-self-dual connections are the absolute minima of the Lagrangian for a 4-dimensional non-abelian gauge theory. The first self-dual solution - the one instanton - to the $SU(2)$ Yang-Mills equations on $\mathbb{R}^4$ was obtained by Belavin et al [3]. Later other more general multi-instanton solutions were described in [5, 11]. Since then numerous extensions have been made. Classical references are the books by Atiyah [1], Freed and Uhlenbeck [8].

In this paper we study a discrete analog of the $SU(2)$ Yang-Mills equations on a combinatorial analog of $\mathbb{R}^4$. The ideas presented here are strongly influenced by book of Dezin [6]. We develop discrete models of some objects in differential geometry, including the Hodge star operator, the differential and the covariant exterior differential operator, in such a way that they preserve the geometric structure of their continual analogs. We continue the investigations which were originated in [7, 19, 20, 21]. The purpose of this paper is to construct the self-dual and anti-self-dual solutions of discrete $SU(2)$ Yang-Mills equations which imitate the corresponding solutions of continual theory. The geometrical discretisation techniques used here extend those introduced in [6] and [19]. A combinatorial model of $\mathbb{R}^4$ based on the use of the double complex construction is taken from [21].

There are many other approaches to the discretisation of Yang-Mills theories. Numerous papers have been written on this subject. See, for example, [2, 4, 9, 10, 12, 13, 15, 18, 16] and the references therein. Most of them are based on the lattice discretisation scheme. However, in the case of the lattice formulation there are difficulties in keeping geometrical properties of an origin gauge theory. An alternative geometrical discretisation scheme of a field theory can be found in [17].

The paper is organized as follows. In Section 2 we review some basic facts of the $SU(2)$ Yang-Mills theory on $\mathbb{R}^4$. We begin by recalling the connection between the Lie group $SU(2)$ and the space of quaternions. Finally, we write down the basic instanton and anti-instanton solutions in quaternionic form. The notations here are compiled from [1] and [14].

Section 3 contains a brief summary of definitions and properties due to the double complex construction. We repeat here the relevant material from [21]. This article is also the main reference for this section. In particular, we introduce discrete matrix-valued forms (analog of differential forms) and define analogs of the main continual operations on them.

In Section 4 using the quaternionic approach we present the discrete Yang-Mills equations. We write out components of the discrete curvature 2-form in quaternionic form. The discrete self-dual and anti-self-dual equations are described. We try to be as close to continual $SU(2)$ Yang-Mills theory as possible. Hence we discuss conditions when the discrete curvature will be $su(2)$-valued.

Finally, Section 5 is devoted to self-dual and anti-self-dual solutions of the discrete Yang-Mills equations. We construct these solutions as discrete quaternionic 1-forms and discuss some analogies with continual instanton and anti-instanton solutions.

2 Quaternions and $SU(2)$-Connection

In this section we briefly recall some well known settings of the smooth Yang-Mills theory in Euclidean 4-dimensional space (see, for example, [14]).

We begin with a brief review of some preliminaries about quaternions. The quaternions are formed from real numbers by adjoining three symbols $\mathbf{i},\mathbf{j},\mathbf{k}$ and an arbitrary quaternion x can be written as

$$x = x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}, \tag{2.1}$$

where $x_1, x_2, x_3, x_4 \in \mathbb{R}$. The symbols $\mathbf{i},\mathbf{j},\mathbf{k}$ satisfy the following identities

$$\begin{aligned} &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \\ &\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \end{aligned} \tag{2.2}$$

It is clear that the space of quaternions is isomorphic to $\mathbb{R}^4$. By analogy with the complex numbers x_1 is called the real part of x and $x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$ is called the imaginary part. In further we will write

$$\operatorname{Im} x = x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}.$$

The conjugate quaternion of x is defined by

$$\bar{x} = x_1 - x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k}.$$

Then the norm $|x|$ of a quaternion can be introduced as follows

$$|x|^2 = x\bar{x} = x_1^2 + x_2^2 + x_3^2 + x_4^2. \tag{2.3}$$

If $x \neq 0$, then it has a unique inverse x^{-1} given by

$$x^{-1} = \bar{x}/|x|^2. \tag{2.4}$$

The algebra of quaternions can be represented as a sub-algebra of the 2×2 complex matrices $M(2,\mathbb{C})$. We identify the quaternion (2.1) with a matrix $f(x) \in M(2,\mathbb{C})$ by setting

$$f(x) = \begin{pmatrix} x_1 + x_2 i & x_3 + x_4 i \\ -x_3 + x_4 i & x_1 - x_2 i \end{pmatrix}. \tag{2.5}$$

Here i is the imaginary unit.

It is well known that the unit quaternions, i.e., they have norm $|x| = 1$, form a group and this group is isomorphic to $SU(2)$. The following 2×2 complex matrices

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2.6}$$

realize a representation of the Lie algebra $su(2)$ of the group $SU(2)$. Note that multiplying by $-i$ these tree matrices we obtain the standard Pauli matrices. Matrices (2.6) correspond to the units $\mathbf{i},\mathbf{j},\mathbf{k}$ given by (2.2). Thus the Lie algebra $su(2)$ can be viewed as the pure imaginary quaternions with basis $\mathbf{i},\mathbf{j},\mathbf{k}$.

Let now A be an $SU(2)$-connection. This means that A is an $su(2)$-valued 1-form and we can write

$$A = \sum_{\mu} A_{\mu}(x) dx^{\mu}, \tag{2.7}$$

where $A_{\mu}(x) \in su(2)$ and $x = (x_1, ..., x_4)$ is a point of $\mathbb{R}^4$. The connection A is also called a gauge potential. Define a gauge transformation by a function $g(x)$ taking value in $SU(2)$. Then the gauge potential A must transform like

$$A \to g^{-1} A g + g^{-1} dg. \tag{2.8}$$

Let us define the curvature 2-form F by

$$F = dA + A \wedge A, \tag{2.9}$$

where $\wedge$ denotes the exterior multiplication.

Consider also the covariant exterior differential operator d_A given by

$$d_A \Omega = d\Omega + A \wedge \Omega + (-1)^{p+1} \Omega \wedge A, \tag{2.10}$$

where Ω is a $su(2)$-valued p-form.

The Yang-Mills action S can be expressed in terms of the 2-forms F and $*F$ as

$$S = -tr \int_{\mathbb{R}^4} F \wedge *F, \tag{2.11}$$

where $*$ is the Hodge star operator. In $\mathbb{R}^4$ the operator $*^2$ is either an involution or anti-involution, i.e., $*^2 = \pm 1$. The Yang-Mills Lagrangian $L = -tr(F \wedge *F)$ is invariant under the gauge transformation (2.8). By the physical requirement it is clear that the action S should be finite. Hence the curvature F should be square integrable. This means that $F \to 0$ as $|x| \to \infty$.

Consequently, we must describe the boundary condition at infinity for the connection A. By virtue of gauge freedom (2.8) we have

$$A \sim g^{-1}dg \qquad \text{as} \qquad |x| \to \infty, \tag{2.12}$$

where $\sim$ implies asymptotic behaviour. Here and subsequently we do not specify the rate of decay.

Written in terms of the covariant exterior differential operator d_A the Euler-Lagrange equations for the extrema of (2.11) have the form

$$d_A F = 0, \qquad d_A * F = 0. \tag{2.13}$$

These equations are the Yang-Mills equations. The first equation of (2.13) is known also as the Bianchi identity. In 4-dimensional Yang-Mills theories the following equations

$$F = *F, \qquad F = - * F \tag{2.14}$$

are called self-dual and anti-self-dual respectively. These equations are first-order non-linear equations for the potential A which imply the second-order Yang-Mills equations (2.13). Solutions of (2.14) – the self-dual and anti-self-dual connections – are called also instantons and anti-instantons [8]. It is known that the self-dual and anti-self-dual connections are the absolute minima of the action S.

The connection 1-form A can be defined also as taking values in the space of pure imaginary quaternions. To express A in quaternion form we consider the quaternion differential

$$dx = dx_1 + dx_2\mathbf{i} + dx_3\mathbf{j} + dx_4\mathbf{k}$$

and the conjugate quaternion of dx

$$d\bar{x} = dx_1 - dx_2\mathbf{i} - dx_3\mathbf{j} - dx_4\mathbf{k}.$$

Let $f(x)$ be a function of the quaternion variable x with quaternion values. Then we can write A as

$$A = \mathrm{Im}(f(x)dx), \tag{2.15}$$

where

$$f(x) = f_1(x) + f_2(x)\mathbf{i} + f_3(x)\mathbf{j} + f_4(x)\mathbf{k}.$$

Using the rules of multiplication (2.2) we have

$$A_1(x) = f_2(x)\mathbf{i} + f_3(x)\mathbf{j} + f_4(x)\mathbf{k},$$
$$A_2(x) = f_1(x)\mathbf{i} + f_4(x)\mathbf{j} - f_3(x)\mathbf{k},$$

$$A_3(x) = -f_4(x)\mathbf{i} + f_1(x)\mathbf{j} + f_2(x)\mathbf{k},$$
$$A_4(x) = f_3(x)\mathbf{i} - f_2(x)\mathbf{j} + f_1(x)\mathbf{k}.$$

Using (2.15) we can rewrite (2.9) as follows

$$F = \mathrm{Im}(df(x)\wedge dx + f(x)dx \wedge f(x)dx). \tag{2.16}$$

Note that calculation of the imaginary part of $f(x)dx$ and computing its curvature commute.

Let us take the following expression for $f(x)$:

$$f(x) = \frac{\bar{x}}{1+|x|^2}. \tag{2.17}$$

Then the connection 1-form A is defined by

$$A = \mathrm{Im}\left\{\frac{\bar{x}dx}{1+|x|^2}\right\}. \tag{2.18}$$

The explicit components A_μ can be written as

$$A_1(x) = \frac{-x_2\mathbf{i} - x_3\mathbf{j} - x_4\mathbf{k}}{1+|x|^2}, \qquad A_2(x) = \frac{x_1\mathbf{i} - x_4\mathbf{j} + x_3\mathbf{k}}{1+|x|^2},$$
$$A_3(x) = \frac{x_4\mathbf{i} + x_1\mathbf{j} - x_2\mathbf{k}}{1+|x|^2}, \qquad A_4(x) = \frac{-x_3\mathbf{i} + x_2\mathbf{j} + x_1\mathbf{k}}{1+|x|^2}. \tag{2.19}$$

Putting (2.17) in (2.16) we get the pure imaginary expression

$$F = \frac{d\bar{x}\wedge dx}{(1+|x|^2)^2}. \tag{2.20}$$

It is easy to show that the 2-form $d\bar{x}\wedge dx$ is anti-self-dual. Hence F is anti-self-dual too and the connection (2.18) describes an anti-instanton . See for details [1].

Similarly, if we take

$$A = \mathrm{Im}\left\{\frac{xd\bar{x}}{1+|x|^2}\right\}, \tag{2.21}$$

then we obtain the self-dual 2-form

$$F = \frac{dx\wedge d\overline{x}}{(1+|x|^2)^2}. \tag{2.22}$$

Thus the curvature is self-dual and (2.21) describes an instanton .

3 Double Complex

We will need the double complex construction described in [21]. In with section for the convenience of the reader we repeat the relevant material from [21] without proofs, thus making our presentation self-contained.

Let the tensor product $C(4)=C\otimes C\otimes C\otimes C$ of an 1-dimensional complex C be a combinatorial model of Euclidean space $\mathbb{R}^4$ (see for details also [6]). The 1-dimensional complex C is defined in the following way. Let C^0 denotes the real linear space of 0-dimensional chains generated by basis elements x_j (points), $j\in\mathbb{Z}$. It is convenient to introduce the shift operators τ,σ in the set of indices by

$$\tau j=j+1, \qquad \sigma j=j-1. \tag{3.1}$$

We denote the open interval $(x_j,\ x_{\tau j})$ by e_j. We'll regards the set $\{e_j\}$ as a set of basis elements of the real linear space C^1 of 1-dimensional chains. Then the 1-dimensional complex (combinatorial real line) is the direct sum of the introduced spaces $C=C^0\oplus C^1$. The boundary operator ∂ on the basis elements of C is given by

$$\partial x_j=0, \qquad \partial e_j=x_{\tau j}-x_j. \tag{3.2}$$

The definition is extended to arbitrary chains by linearity.

Multiplying the basis elements x_j,e_j in various ways we obtain basis elements of $C(4)$. Let $s_k^{(p)}$, where $k=(k_1,k_2,k_3,k_4)$ and $k_i\in\mathbb{Z}$, be an arbitrary basis element of $C(4)$. Then a p-dimensional chain is given by

$$c_p=\sum_k\sum_p c^k_{(p)}s_k^{(p)}, \quad c^k_{(p)}\in\mathbb{R}. \tag{3.3}$$

We suppose that the superscript (p) contains the whole requisite information about the quantity and places of 1-dimensional elements e_j in $s_k^{(p)}$. For example, the 1-dimensional basis elements e_k^i of $C(4)$ can be written as

$$\begin{aligned} e_k^1&=e_{k_1}\otimes x_{k_2}\otimes x_{k_3}\otimes x_{k_4}, \qquad & e_k^2&=x_{k_1}\otimes e_{k_2}\otimes x_{k_3}\otimes x_{k_4},\\ e_k^3&=x_{k_1}\otimes x_{k_2}\otimes e_{k_3}\otimes x_{k_4}, \qquad & e_k^4&=x_{k_1}\otimes x_{k_2}\otimes x_{k_3}\otimes e_{k_4} \end{aligned} \tag{3.4}$$

and for the 2-dimensional basis elements ε_k^{ij} we have

$$\begin{aligned} \varepsilon_k^{12}&=e_{k_1}\otimes e_{k_2}\otimes x_{k_3}\otimes e_{k_4}, \qquad & \varepsilon_k^{23}&=x_{k_1}\otimes e_{k_2}\otimes e_{k_3}\otimes x_{k_4},\\ \varepsilon_k^{13}&=e_{k_1}\otimes x_{k_2}\otimes e_{k_3}\otimes x_{k_4}, \qquad & \varepsilon_k^{24}&=x_{k_1}\otimes e_{k_2}\otimes x_{k_3}\otimes e_{k_4},\\ \varepsilon_k^{14}&=e_{k_1}\otimes x_{k_2}\otimes x_{k_3}\otimes e_{k_4}, \qquad & \varepsilon_k^{34}&=x_{k_1}\otimes x_{k_2}\otimes e_{k_3}\otimes e_{k_4}. \end{aligned} \tag{3.5}$$

Using (3.2) we define the boundary operator ∂ on chains of $C(4)$ in the following way: if $c_p,\ c_q$ are chains of the indicated dimension, belonging to the complexes being multiplied, then

$$\partial(c_p\otimes c_q)=\partial c_p\otimes c_q+(-1)^p c_p\otimes\partial c_q. \tag{3.6}$$

For example, for the basis element ε_k^{24} we have

$$\partial\varepsilon_k^{24}=\partial(x_{k_1}\otimes e_{k_2})\otimes x_{k_3}\otimes e_{k_4}-x_{k_1}\otimes e_{k_2}\otimes\partial(x_{k_3}\otimes e_{k_4})$$

$$
\begin{aligned}
&= \partial x_{k_1}\otimes e_{k_2}\otimes x_{k_3}\otimes e_{k_4} + x_{k_1}\otimes \partial e_{k_2}\otimes x_{k_3}\otimes e_{k_4}\\
&- x_{k_1}\otimes e_{k_2}\otimes \partial x_{k_3}\otimes e_{k_4} - x_{k_1}\otimes e_{k_2}\otimes x_{k_3}\otimes \partial e_{k_4}\\
&= x_{k_1}\otimes x_{\tau k_2}\otimes x_{k_3}\otimes e_{k_4} - x_{k_1}\otimes x_{k_2}\otimes x_{k_3}\otimes e_{k_4}\\
&- x_{k_1}\otimes x_{k_2}\otimes x_{k_3}\otimes x_{\tau k_4} + x_{k_1}\otimes x_{k_2}\otimes x_{k_3}\otimes x_{k_4}.
\end{aligned}
$$

For convenience we also introduce the shift operators τ_i and σ_i which act in the set of indices $k=(k_1,k_2,k_3,k_4)$, $k_i\in\mathbb{Z}$, as

$$\tau_i k = (k_1,\dots\tau k_i,\dots k_4), \qquad \sigma_i k = (k_1,\dots\sigma k_i,\dots k_4), \tag{3.7}$$

where τ and σ are given by (3.1).

Let us introduce the construction of a double complex. Together with the complex $C(4)$ we consider its double, namely the complex $\tilde{C}(4)$ of exactly the same structure. Define the one-to-one correspondence

$$*: C(4)\to \tilde{C}(4), \qquad *: \tilde{C}(4)\to C(4) \tag{3.8}$$

in the following way. Let $s_k^{(p)}$ be an arbitrary p-dimensional basis element of $C(4)$, i.e., the product $s_k^{(p)} = s_{k_1}\otimes s_{k_2}\otimes s_{k_3}\otimes s_{k_4}$ contains exactly p of 1-dimensional elements e_{k_i} and $4-p$ of 0-dimensional elements x_{k_i}, $p=0,1,2,3,4$, $k_i\in\mathbb{Z}$. Then

$$*: s_k^{(p)} \to \pm\tilde{s}_k^{(4-p)}, \qquad *: \tilde{s}_k^{(4-p)} \to \pm s_k^{(p)}, \tag{3.9}$$

where

$$\tilde{s}_k^{(4-p)} = *s_{k_1}\otimes *s_{k_2}\otimes *s_{k_3}\otimes *s_{k_4}$$

and $*s_{k_i}=\tilde{e}_{k_i}$ if $s_{k_i}=x_{k_i}$ and $*s_{k_i}=\tilde{x}_{k_i}$ if $s_{k_i}=e_{k_i}$. In the first of mapping (3.9) we take "+" if the permutation $((p),\ (4-p))$ of $(1,2,3,4)$ is even and "−" if the permutation $((p),\ (4-p))$ is odd. Recall that in symbol (p) the number of basis element is contained. For example, for the 2-dimensional basis element $\varepsilon_k^{13} = e_{k_1}\otimes x_{k_2}\otimes e_{k_3}\otimes x_{k_4}$ we have $*\varepsilon_k^{13} = -\tilde{\varepsilon}_k^{24}$ since the permutation $(1,3,2,4)$ is odd. The mapping $*: \tilde{s}_k^{(4-p)} \to \pm s_k^{(p)}$ is defined by analogy.

Proposition 3.1. *Let $c_r\in C(4)$ be an r-dimensional chain (3.3). Then we have*

$$**c_r = (-1)^{r(4-r)}c_r. \tag{3.10}$$

Proof. See [21]. □

Now we consider a dual object of the complex $C(4)$. Let $K(4)$ be a cochain complex with $gl(2,\mathbb{C})$-valued coefficients, where $gl(2,\mathbb{C})$ is the Lie algebra of the group $GL(2,\mathbb{C})$. Recall that $gl(2,\mathbb{C})$ consists of all complex 2×2 matrices $M(2,\mathbb{C})$ with bracket operation $[\cdot,\cdot]$.

We suppose that the complex $K(4)$, which is a conjugate of $C(4)$, has a similar structure: $K(4) = K \otimes K \otimes K \otimes K$, where K is a conjugate of the 1-dimensional complex C. Basis elements of K can be written as x^j, e^j. Then an arbitrary basis element of $K(4)$ is given by $s^k_{(p)} = s^{k_1} \otimes s^{k_2} \otimes s^{k_3} \otimes s^{k_4}$, where s^{k_j} is either x^{k_j} or e^{k_j}. For example, we denote the 1-, 2-dimensional basis elements of $K(4)$ by e^k_i, ε^k_{ij} respectively, cf. (3.4), (3.5). For a p-dimensional cochain $\varphi \in K(4)$ we have

$$\varphi = \sum_k \sum_p \varphi^{(p)}_k s^k_{(p)}, \tag{3.11}$$

where $\varphi^{(p)}_k \in gl(2,\mathbb{C})$. We will call cochains forms, emphasizing their relationship with the corresponding continual objects, differential forms.

We define the pairing operation $<\cdot\,,\,\cdot>$ for arbitrary basis elements $\varepsilon_k \in C(4)$, $s^k \in K(4)$ by the rule

$$<\varepsilon_k, as^k> = \begin{cases} 0, \ \varepsilon_k \neq s_k \\ a, \ \varepsilon_k = s_k, \ a \in gl(2,\mathbb{C}). \end{cases} \tag{3.12}$$

Here for simplicity the superscript (p) is omitted. The operation (3.12) is linearly extended to cochains.

The operation ∂ (3.6) induces the dual operation d^c on $K(4)$ in the following way:

$$<\partial\varepsilon_k, as^k> = <\varepsilon_k, ad^c s^k>. \tag{3.13}$$

For example, if $\varphi = \sum_k \varphi_k x^k$, where $x^k = x^{k_1} \otimes x^{k_2} \otimes x^{k_3} \otimes x^{k_4}$, is a 0-form, then

$$d^c\varphi = \sum_k \sum_{i=1}^4 (\Delta_i \varphi_k) e^k_i, \tag{3.14}$$

where $\Delta_i \varphi_k = \varphi_{\tau_i k} - \varphi_k$ and e^k_i is the 1-dimensional basis elements of $K(4)$. The coboundary operator d^c is an analog of the exterior differentiation operator.

Now we describe a cochain product on the forms of $K(4)$. See [6] for details. We denote this product by $\cup$. In terms of the homology theory this is the so-called Whitney product. First we introduce the $\cup$-product on the chains of the 1-dimensional complex K. For the basis elements of K the $\cup$-product is defined as follows

$$x^j \cup x^j = x^j, \quad e^j \cup x^{\tau j} = e^j, \quad x^j \cup e^j = e^j, \quad j \in \mathbb{Z},$$

supposing the product to be zero in all other case. To arbitrary forms the $\cup$-product be extended linearly. Let us introduce an r-dimensional complex $K(r)$, $r = 1,2,3$, in an obvious notation. Let $s^k_{(p)}$ be an arbitrary p-dimensional basis element of $K(r)$. It is convenient to write the basis element of $K(r+1)$ in the form $s^k_{(p)} \otimes s^j$, where $s^k_{(p)}$ is a basis element of $K(r)$

and s^j is either e^j or x^j, $j \in \mathbb{Z}$. Then, supposing that the $\cup$-product in $K(r)$ has been defined, we introduce it for basis elements of $K(r+1)$ by the rule

$$(s^k_{(p)} \otimes s^j) \cup (s^k_{(q)} \otimes s^\mu) = Q(j,q)(s^k_{(p)} \cup s^k_{(q)}) \otimes (s^j \cup s^\mu), \tag{3.15}$$

where the signum function $Q(j,q)$ is equal to -1 if the dimension of both elements s^j, $s^k_{(q)}$ is odd and to $+1$ otherwise. The extension of the $\cup$-product to arbitrary forms of $K(r+1)$ is linear. Note that the coefficients of forms multiply as matrices.

Proposition 3.2. *Let φ and ψ be arbitrary forms of $K(4)$. Then*

$$d^c(\varphi \cup \psi) = d^c\varphi \cup \psi + (-1)^p \varphi \cup d^c\psi, \tag{3.16}$$

where p is the dimension of a form φ.

The proof of Proposition 3.2 is totally analogous to one in [6, p. 147] for the case of discrete forms with real coefficients.

The complex of the cochains $\tilde{K}(4)$ over the double complex $\tilde{C}(4)$ with the operator d^c defined in it by (3.13) has the same structure as $K(4)$. The operation (3.8) induces the respective mapping

$$* : K(4) \to \tilde{K}(4), \qquad * : \tilde{K}(4) \to K(4)$$

by the rule:

$$< \tilde{c},\ *\varphi > = < *\tilde{c},\ \varphi >, \qquad < c,\ *\tilde{\psi} > = < *c,\ \tilde{\psi} >, \tag{3.17}$$

where $c \in C(4)$, $\tilde{c} \in \tilde{C}(4)$, $\varphi \in K(4)$, $\tilde{\psi} \in \tilde{K}(4)$. Hence for the basic elements of $K(4)$ or $\tilde{K}(4)$ we have relations (3.9). It is obviously that Proposition 3.1 is true for any r-dimensional cochain $c^r \in K(4)$. So we have

$$* * \varphi = (-1)^{r(4-r)}\varphi$$

for any discrete r-form φ on $K(4)$ and note that the same relation holds for the Hodge star operator. Thus this operator is a combinatorial analog of the Hodge star operator.

Let us introduce the following operation

$$\tilde{\iota} : K(4) \to \tilde{K}(4), \qquad \tilde{\iota} : \tilde{K}(4) \to K(4)$$

by setting

$$\tilde{\iota} s^k_{(p)} = \tilde{s}^k_{(p)}, \qquad \tilde{\iota}\tilde{s}^k_{(p)} = s^k_{(p)}, \tag{3.18}$$

where $s^k_{(p)}$ and $\tilde{s}^k_{(p)}$ are basis elements of $K(4)$ and $\tilde{K}(4)$. Hence for a p-form $\varphi \in K(4)$ we have $\tilde{\iota}\varphi = \tilde{\varphi}$. Recall that the coefficients of $\tilde{\varphi} \in \tilde{K}(4)$ and $\varphi \in K(4)$ are the same.

Proposition 3.3. *The following hold*

$$\tilde{\iota}^2 = Id, \quad \tilde{\iota}* = *\tilde{\iota}, \quad \tilde{\iota}d^c = d^c\tilde{\iota}, \tag{3.19}$$
$$\tilde{\iota}(\varphi \cup \psi) = \tilde{\iota}\varphi \cup \tilde{\iota}\psi,$$

where $\varphi,\ \psi \in K(4)$.

Proposition 3.4. *Let h be a discrete 0-form. Then for an arbitrary p-form $\varphi \in K(4)$ we have*

$$\tilde{\iota} * (h \cup \varphi) = h \cup \tilde{\iota} * \varphi. \tag{3.20}$$

Proof. See [21]. □

Note that the definition of inner product in the double complex and a discrete analog of the Yang-Mills actions (2.11) can be found in [21].

4 Quaternions and Discrete Forms

Let us consider a discrete 0-form with coefficients belonging to $M(2,\mathbb{C})$. We put

$$f = \sum_k f_k x^k, \tag{4.1}$$

where $x^k = x^{k_1} \otimes x^{k_2} \otimes x^{k_3} \otimes x^{k_4}$ is the 0-dimensional basis element of $K(4)$, $k = (k_1, k_2, k_3, k_4)$, $k_i \in \mathbb{Z}$. Suppose that the matrices $f_k \in M(2,\mathbb{C})$ look like (2.5), i. e.

$$f_k = \begin{pmatrix} f_k^1 + f_k^2 i & f_k^3 + f_k^4 i \\ -f_k^3 + f_k^4 i & f_k^1 - f_k^2 i \end{pmatrix}, \tag{4.2}$$

where $f_k^s \in \mathbb{R}$, $s = 1,2,3,4$. Then f_k in quaternionic form can be expressed as

$$f_k = f_k^1 + f_k^2 \mathbf{i} + f_k^3 \mathbf{j} + f_k^4 \mathbf{k}. \tag{4.3}$$

Hence the form (4.1) can be considered as a discrete form with quaternionic coefficients. We will call it simply the quaternionic form when no confusion can arise. In a proper way we define the quaternionic 0-form $\bar{f}$ with coefficients $\bar{f}_k$ regarded as the conjugate quaternions of f_k. Let f^{-1} be the quaternionic form, where f_k^{-1} is given by (2.4). Then we have

$$f \cup f^{-1} = \sum_k f_k f_k^{-1} x^k = \sum_k x^k. \tag{4.4}$$

Proposition 4.1. *Let f be a discrete 0-form and $f \neq 0$. Then we have*

$$d^c f \cup f^{-1} = -f \cup d^c f^{-1}. \tag{4.5}$$

Proof. By definition (3.14) and according to (4.4), we have $d^c(f \cup f^{-1}) = 0$. Using Proposition 3.2 we immediately obtain (4.5). □

Let us denote by e the following quaternionic 1-form

$$e = \sum_k e^k = \sum_k (e_1^k + e_2^k \mathbf{i} + e_3^k \mathbf{j} + e_4^k \mathbf{k}), \tag{4.6}$$

where e_i^k is the 1-dimensional basis elements of $K(4)$. Let $A \in K(4)$ be a discrete 1-form. We define the discrete $SU(2)$-connection A to be

$$A = \sum_k \sum_{i=1}^{4} A_k^i e_i^k, \tag{4.7}$$

where $A_k^i \in su(2)$ and $k = (k_1, k_2, k_3, k_4)$, $k_i \in \mathbb{Z}$. Using (4.3) and (4.6) we write (4.7) in quaternionic form as

$$A = \mathrm{Im}(f \cup e) = \mathrm{Im}\Big(\sum_k f_k e^k\Big). \tag{4.8}$$

Then the A_k^i are given by

$$\begin{aligned} A_k^1 &= f_k^2\mathbf{i} + f_k^3\mathbf{j} + f_k^4\mathbf{k}, \qquad & A_k^2 &= f_k^1\mathbf{i} + f_k^4\mathbf{j} - f_k^3\mathbf{k}, \\ A_k^3 &= -f_k^4\mathbf{i} + f_k^1\mathbf{j} + f_k^2\mathbf{k}, \qquad & A_k^4 &= f_k^3\mathbf{i} - f_k^2\mathbf{j} + f_k^1\mathbf{k}. \end{aligned} \tag{4.9}$$

Define the quaternionic 0-form x by

$$x = \sum_k \kappa x^k, \quad \kappa = k_1 + k_2\mathbf{i} + k_3\mathbf{j} + k_4\mathbf{k}, \tag{4.10}$$

where $k_i \in \mathbb{Z}$. It is easy to check that

$$d^c x = e. \tag{4.11}$$

Therefore we can rewrite (4.8) as

$$A = \mathrm{Im}(f \cup d^c x). \tag{4.12}$$

Let g be a quaternionic 0-form (4.1) with the components of unit norm, i.e., $|g_k| = 1$ for any k. It means that the corresponding discrete form is $SU(2)$-valued. We now define a gauge transformation for the discrete potential A which is analogous to (2.8). This is

$$A \to g^{-1} \cup A \cup g + g^{-1} \cup d^c g, \tag{4.13}$$

where A is given by (4.8) or (4.12). Note that the gauge transformed discrete form A is $su(2)$-valued too. It is not so obviously as in the continual case but follows immediately from the definition of $\cup$-multiplication and formula (3.16). More generally, if we assume that the gauge transformation g is an arbitrary quaternionic 0-form, then we take the imaginary part of $g^{-1}\cup A\cup g+g^{-1}\cup d^c g$ in (4.13). For a deeper discussion of gauge invariant discrete models of the Yang-Mills theory we refer the reader to [19, 21].

An arbitrary discrete 2-form $F\in K(4)$ can be written as follows

$$F=\sum_k\sum_{i<j}F_k^{ij}\varepsilon_{ij}^k, \tag{4.14}$$

where $F_k^{ij}\in gl(2,\mathbb{C})$, ε_{ij}^k is the 2-dimensional basis element of $K(4)$ and $1\leq i,j\leq 4$, $k=(k_1,k_2,k_3,k_4)$, $k_i\in\mathbb{Z}$. Let F is given by

$$F=d^cA+A\cup A. \tag{4.15}$$

Combining (4.7) and (4.15) and using (3.12), (3.13) and (3.15), we obtain

$$F_k^{ij}=\Delta_iA_k^j-\Delta_jA_k^i+A_k^iA_{\tau_ik}^j-A_k^jA_{\tau_jk}^i, \tag{4.16}$$

where $\Delta_iA_k^j=A_{\tau_ik}^j-A_k^j$ and τ_ik is given by (3.7).

Let us define a discrete analog of the exterior covariant differentiation operator (2.10) as follows

$$d_A^c\Omega=d^c\Omega+A\cup\Omega+(-1)^{p+1}\Omega\cup A, \tag{4.17}$$

where Ω is an arbitrary p-form of $K(4)$ looking like (3.11). Then a discrete analog of Equations (2.13) can be written as

$$d_A^cF=0,\quad d_A^c*\tilde{\iota}F=0, \tag{4.18}$$

where $\tilde{\iota}$ is given by (3.18). It is easy to check that the combinatorial Bianchi identity:

$$d^cF+A\cup F-F\cup A=0 \tag{4.19}$$

holds for the discrete curvature form (4.15) (cf. (2.13)).

Remark 4.2. *In the continual case the curvature form F (2.9) takes values in the algebra $su(2)$ for any $su(2)$-valued connection form A. Unfortunately, this is not true in the discrete case because, generally speaking, the components $A_k^iA_{\tau_ik}^j-A_k^jA_{\tau_jk}^i$ of the form $A\cup A$ (see (4.16)) do not belong to $su(2)$.*

To define an $su(2)$-valued discrete analog of the curvature 2-form we use the quaternionic form of A (4.8) and put in (4.15). Then the discrete curvature form F is given by

$$F=\mathrm{Im}\{d^cf\cup e+(f\cup e)\cup(f\cup e)\}. \tag{4.20}$$

It should be noted that in the discrete case calculation of the imaginary part of $f\cup e$ and computing its curvature do not commute.

Proposition 4.3. *If* $A = \mathrm{Im}(x^{-1} \cup d^c x)$*, where* x *is given by (4.10), then* $F = 0$*.*

Proof. Using (4.5) and putting $f = x^{-1}$ in (4.20) we get

$$
\begin{aligned}
F &= \mathrm{Im}(d^c(x^{-1} \cup d^c x) + (x^{-1} \cup d^c x) \cup (x^{-1} \cup d^c x)) \\
&= \mathrm{Im}(d^c x^{-1} \cup d^c x - d^c x^{-1} \cup x \cup x^{-1} \cup d^c x).
\end{aligned}
$$

According to (4.4) the form $x \cup x^{-1}$ has unit components. Hence

$$d^c x^{-1} \cup x \cup x^{-1} \cup d^c x = d^c x^{-1} \cup d^c x.$$

□

We now write down the components of (4.14) using quaternions. Putting (4.9) in (4.16) we find that

$$
\begin{aligned}
F_k^{12} = {} & (\Delta_1 f_k^1 - \Delta_2 f_k^2 - f_k^3 f_{\tau_1 k}^3 - f_k^4 f_{\tau_1 k}^4 - f_k^3 f_{\tau_2 k}^3 - f_k^4 f_{\tau_2 k}^4)\mathbf{i} \\
& + (\Delta_1 f_k^4 - \Delta_2 f_k^3 + f_k^2 f_{\tau_1 k}^3 + f_k^4 f_{\tau_1 k}^1 + f_k^1 f_{\tau_2 k}^4 + f_k^3 f_{\tau_2 k}^2)\mathbf{j} \\
& + (-\Delta_1 f_k^3 - \Delta_2 f_k^4 + f_k^2 f_{\tau_1 k}^4 - f_k^3 f_{\tau_1 k}^1 - f_k^1 f_{\tau_2 k}^3 + f_k^4 f_{\tau_2 k}^2)\mathbf{k} \\
& - f_k^2 f_{\tau_1 k}^1 - f_k^3 f_{\tau_1 k}^4 + f_k^4 f_{\tau_1 k}^3 + f_k^1 f_{\tau_2 k}^2 + f_k^4 f_{\tau_2 k}^3 - f_k^3 f_{\tau_2 k}^4,
\end{aligned}
$$

$$
\begin{aligned}
F_k^{13} = {} & (-\Delta_1 f_k^4 - \Delta_3 f_k^2 + f_k^3 f_{\tau_1 k}^2 - f_k^4 f_{\tau_1 k}^1 - f_k^1 f_{\tau_3 k}^4 + f_k^2 f_{\tau_3 k}^3)\mathbf{i} \\
& + (\Delta_1 f_k^1 - \Delta_3 f_k^3 - f_k^2 f_{\tau_1 k}^2 - f_k^4 f_{\tau_1 k}^4 - f_k^4 f_{\tau_3 k}^4 - f_k^2 f_{\tau_3 k}^2)\mathbf{j} \\
& + (\Delta_1 f_k^2 - \Delta_3 f_k^4 + f_k^2 f_{\tau_1 k}^1 + f_k^3 f_{\tau_1 k}^4 + f_k^4 f_{\tau_3 k}^3 + f_k^1 f_{\tau_3 k}^2)\mathbf{k} \\
& + f_k^2 f_{\tau_1 k}^4 - f_k^3 f_{\tau_1 k}^1 - f_k^4 f_{\tau_1 k}^2 - f_k^4 f_{\tau_3 k}^2 + f_k^1 f_{\tau_3 k}^3 + f_k^2 f_{\tau_3 k}^4,
\end{aligned}
$$

$$
\begin{aligned}
F_k^{14} = {} & (\Delta_1 f_k^3 - \Delta_4 f_k^2 + f_k^3 f_{\tau_1 k}^1 + f_k^4 f_{\tau_1 k}^2 + f_k^2 f_{\tau_4 k}^4 + f_k^1 f_{\tau_4 k}^3)\mathbf{i} \\
& + (-\Delta_1 f_k^2 - \Delta_4 f_k^3 - f_k^2 f_{\tau_1 k}^1 + f_k^4 f_{\tau_1 k}^3 + f_k^3 f_{\tau_4 k}^4 - f_k^1 f_{\tau_4 k}^2)\mathbf{j} \\
& + (\Delta_1 f_k^1 - \Delta_4 f_k^4 - f_k^2 f_{\tau_1 k}^2 - f_k^3 f_{\tau_1 k}^3 - f_k^3 f_{\tau_4 k}^3 - f_k^2 f_{\tau_4 k}^2)\mathbf{k} \\
& - f_k^2 f_{\tau_1 k}^3 + f_k^3 f_{\tau_1 k}^2 - f_k^4 f_{\tau_1 k}^1 + f_k^3 f_{\tau_4 k}^2 - f_k^2 f_{\tau_4 k}^3 + f_k^1 f_{\tau_4 k}^4,
\end{aligned}
$$

$$
\begin{aligned}
F_k^{23} = {} & (-\Delta_2 f_k^4 - \Delta_3 f_k^1 + f_k^4 f_{\tau_2 k}^2 + f_k^3 f_{\tau_2 k}^1 + f_k^1 f_{\tau_3 k}^3 + f_k^2 f_{\tau_3 k}^4)\mathbf{i} \\
& + (\Delta_2 f_k^1 - \Delta_3 f_k^4 - f_k^1 f_{\tau_2 k}^2 + f_k^3 f_{\tau_2 k}^4 + f_k^4 f_{\tau_3 k}^3 - f_k^2 f_{\tau_3 k}^1)\mathbf{j} \\
& + (\Delta_2 f_k^2 + \Delta_3 f_k^3 + f_k^1 f_{\tau_2 k}^1 + f_k^4 f_{\tau_2 k}^4 + f_k^4 f_{\tau_3 k}^4 + f_k^1 f_{\tau_3 k}^1)\mathbf{k} \\
& + f_k^1 f_{\tau_2 k}^4 - f_k^4 f_{\tau_2 k}^1 + f_k^3 f_{\tau_2 k}^2 - f_k^4 f_{\tau_3 k}^1 + f_k^1 f_{\tau_3 k}^4 - f_k^2 f_{\tau_3 k}^3,
\end{aligned}
$$

$$
F_k^{24} = (\Delta_2 f_k^3 - \Delta_4 f_k^1 + f_k^4 f_{\tau_2 k}^1 - f_k^3 f_{\tau_2 k}^2 - f_k^2 f_{\tau_4 k}^3 + f_k^1 f_{\tau_4 k}^4)\mathbf{i}
$$

$$
\begin{aligned}
&+(-\Delta_2 f_k^2 - \Delta_4 f_k^4 - f_k^1 f_{\tau_2 k}^1 - f_k^3 f_{\tau_2 k}^3 - f_k^3 f_{\tau_4 k}^3 - f_k^1 f_{\tau_4 k}^1)\mathbf{j}\\
&+(\Delta_2 f_k^1 + \Delta_4 f_k^3 - f_k^1 f_{\tau_2 k}^2 - f_k^4 f_{\tau_2 k}^3 - f_k^3 f_{\tau_4 k}^4 - f_k^2 f_{\tau_4 k}^1)\mathbf{k}\\
&-f_k^1 f_{\tau_2 k}^3 + f_k^4 f_{\tau_2 k}^2 + f_k^3 f_{\tau_2 k}^1 + f_k^3 f_{\tau_4 k}^1 - f_k^2 f_{\tau_4 k}^4 - f_k^1 f_{\tau_4 k}^3,
\end{aligned}
$$

$$
\begin{aligned}
F_k^{34} = &(\Delta_3 f_k^3 + \Delta_4 f_k^4 + f_k^1 f_{\tau_3 k}^1 + f_k^2 f_{\tau_3 k}^2 + f_k^2 f_{\tau_4 k}^2 + f_k^1 f_{\tau_4 k}^1)\mathbf{i}\\
&+(-\Delta_3 f_k^2 - \Delta_4 f_k^1 + f_k^4 f_{\tau_3 k}^1 + f_k^2 f_{\tau_3 k}^3 + f_k^3 f_{\tau_4 k}^2 + f_k^1 f_{\tau_4 k}^4)\mathbf{j}\\
&+(\Delta_3 f_k^1 - \Delta_4 f_k^2 + f_k^4 f_{\tau_3 k}^2 - f_k^1 f_{\tau_3 k}^3 - f_k^3 f_{\tau_4 k}^1 + f_k^2 f_{\tau_4 k}^4)\mathbf{k}\\
&+f_k^4 f_{\tau_3 k}^3 + f_k^1 f_{\tau_3 k}^2 - f_k^2 f_{\tau_3 k}^1 - f_k^3 f_{\tau_4 k}^4 - f_k^2 f_{\tau_4 k}^1 + f_k^1 f_{\tau_4 k}^2.
\end{aligned}
$$

To obtain the components of (4.20) we must take the imaginary part of these equations.

Proposition 4.4. *The discrete curvature 2-form F (4.15) is $su(2)$-valued if and only if*

$$
\begin{aligned}
-f_k^2 f_{\tau_1 k}^1 - f_k^3 f_{\tau_1 k}^4 + f_k^4 f_{\tau_1 k}^3 + f_k^1 f_{\tau_2 k}^2 + f_k^4 f_{\tau_2 k}^3 - f_k^3 f_{\tau_2 k}^4 &= 0,\\
f_k^2 f_{\tau_1 k}^4 - f_k^3 f_{\tau_1 k}^1 - f_k^4 f_{\tau_1 k}^2 - f_k^4 f_{\tau_3 k}^2 + f_k^1 f_{\tau_3 k}^3 + f_k^2 f_{\tau_3 k}^4 &= 0,\\
-f_k^2 f_{\tau_1 k}^3 + f_k^3 f_{\tau_1 k}^2 - f_k^4 f_{\tau_1 k}^1 + f_k^3 f_{\tau_4 k}^2 - f_k^2 f_{\tau_4 k}^3 + f_k^1 f_{\tau_4 k}^4 &= 0,\\
f_k^1 f_{\tau_2 k}^4 - f_k^4 f_{\tau_2 k}^1 + f_k^3 f_{\tau_2 k}^2 - f_k^4 f_{\tau_3 k}^1 + f_k^1 f_{\tau_3 k}^4 - f_k^2 f_{\tau_3 k}^3 &= 0,\\
-f_k^1 f_{\tau_2 k}^3 + f_k^4 f_{\tau_2 k}^2 + f_k^3 f_{\tau_2 k}^1 + f_k^3 f_{\tau_4 k}^1 - f_k^2 f_{\tau_4 k}^4 - f_k^1 f_{\tau_4 k}^3 &= 0,\\
f_k^4 f_{\tau_3 k}^3 + f_k^1 f_{\tau_3 k}^2 - f_k^2 f_{\tau_3 k}^1 - f_k^3 f_{\tau_4 k}^4 - f_k^2 f_{\tau_4 k}^1 + f_k^1 f_{\tau_4 k}^2 &= 0.
\end{aligned}
$$

Proof. From the above it follows immediately. □

Proposition 4.5. *Let e is given by (4.6). Then the 2-form $e \cup \bar{e}$ is self-dual, i.e.,*

$$e \cup \bar{e} = *\tilde{\iota}(e \cup \bar{e}), \tag{4.21}$$

and $\bar{e} \cup e$ is anti-self-dual, i.e.,

$$\bar{e} \cup e = -*\tilde{\iota}(\bar{e} \cup e). \tag{4.22}$$

Proof. Denote

$$e_i = \sum_k e_i^k, \qquad \varepsilon_{ij} = \sum_k \varepsilon_{ij}^k.$$

Recall that e_i^k and ε_{ij}^k are the 1-dimensional and 2-dimensional basic elements of $K(4)$ (see also (3.4) and (3.5)). From this by (3.15) we obtain $e_i \cup e_j = \varepsilon_{ij}$ and $e_j \cup e_i = -\varepsilon_{ij}$ for all $i < j$. Then we have

$$
\begin{aligned}
e \cup \bar{e} &= (e_1 + e_2\mathbf{i} + e_3\mathbf{j} + e_4\mathbf{k}) \cup (e_1 - e_2\mathbf{i} - e_3\mathbf{j} - e_4\mathbf{k})\\
&= -2\{(e_1 \cup e_2 + e_3 \cup e_4)\mathbf{i} + (e_1 \cup e_3 - e_2 \cup e_4)\mathbf{j} + (e_1 \cup e_4 + e_2 \cup e_3)\mathbf{k}\}\\
&= -2\{(\varepsilon_{12} + \varepsilon_{34})\mathbf{i} + (\varepsilon_{13} - \varepsilon_{24})\mathbf{j} + (\varepsilon_{14} + \varepsilon_{23})\mathbf{k}\}.
\end{aligned}
$$

Using (3.17) and (3.19) we get

$$*\tilde{\iota}(e\cup\bar{e}) = -2\tilde{\iota}\{(\tilde{\varepsilon}_{34}+\tilde{\varepsilon}_{12})\mathbf{i}+(-\tilde{\varepsilon}_{24}+\tilde{\varepsilon}_{13})\mathbf{j}+(\tilde{\varepsilon}_{23}+\tilde{\varepsilon}_{14})\mathbf{k}\} = e\cup\bar{e}.$$

In the same way we obtain (4.22). □

Corollary 4.6. *For any quaternionic 0-form f the form $f\cup e\cup\bar{e}$ is self-dual and $f\cup\bar{e}\cup e$ is anti-self-dual.*

Proof. This follows immediately from (3.20). □

Discrete self-dual and anti-self-dual equations (discrete analogs of Equations (2.13)) are defined by

$$F = \tilde{\iota} * F, \qquad F = -\tilde{\iota} * F, \tag{4.23}$$

where F is the discrete curvature form (4.4). Using (4.5), by the definitions of $\tilde{\iota}$ and $*$, the first equation (self-dual) of (4.23) can be rewritten as follows

$$F_k^{12} = F_k^{34}, \qquad F_k^{13} = -F_k^{24}, \qquad F_k^{14} = F_k^{23}. \tag{4.24}$$

By analogue with the continual case solutions of (4.23) (or (4.24)) are called instantons and anti-instantons respectively.

5 Discrete Instanton and Anti-Instanton

In further analogy with the continual case consider the discrete $SU(2)$-connection A. Let A be the quaternionic 1-form (4.8), where the components of f are given by

$$f_k = \frac{\bar{\kappa}}{1+|\kappa|^2}, \tag{5.1}$$

where $\kappa = k_1 + k_2\mathbf{i} + k_3\mathbf{j} + k_4\mathbf{k}$, $k_i \in \mathbb{Z}$. Putting the last in (4.9) we obtain

$$\begin{aligned} A_k^1 &= \frac{-k_2\mathbf{i}-k_3\mathbf{j}-k_4\mathbf{k}}{1+|\kappa|^2}, \qquad A_k^2 = \frac{k_1\mathbf{i}-k_4\mathbf{j}+k_3\mathbf{k}}{1+|\kappa|^2}, \\ A_k^3 &= \frac{k_4\mathbf{i}+k_1\mathbf{j}-k_2\mathbf{k}}{1+|\kappa|^2}, \qquad A_k^4 = \frac{-k_3\mathbf{i}+k_2\mathbf{j}+k_1\mathbf{k}}{1+|\kappa|^2}. \end{aligned} \tag{5.2}$$

It is convenient to denote

$$M_i = \frac{1}{(1+|\kappa|^2)(1+|\tau_i\kappa|^2)}, \qquad i = 1,2,3,4. \tag{5.3}$$

Recall that the shift operator τ_i is given by (3.7). Substituting (5.2) in (4.16) and using (5.3) we find that

$$F_k^{12} = \{M_1(1+k_2^2-k_1^2-k_1)+M_2(1+k_1^2-k_2^2-k_2)\}\mathbf{i}$$

$$
\begin{aligned}
&+\{M_1(k_4k_1+k_2k_3)-M_2(k_3k_2+k_4k_1)\}\mathbf{j}\\
&+\{M_1(k_2k_4-k_1k_3)+M_2(k_1k_3-k_2k_4)\}\mathbf{k}\\
&+M_1(k_1k_2+k_2)-M_2(k_1k_2+k_1),
\end{aligned}
$$

$$
\begin{aligned}
F_k^{13} = &\{M_1(k_2k_3-k_1k_4)+M_3(k_1k_4-k_2k_3)\}\mathbf{i}\\
&+\{M_1(1+k_3^2-k_1^2-k_1)+M_3(1+k_1^2-k_3^2-k_3)\}\mathbf{j}\\
&+\{M_1(k_1k_2+k_3k_4)-M_3(k_3k_4+k_1k_2)\}\mathbf{k}\\
&+M_1(k_1k_3+k_3)-M_3(k_1k_3+k_1),
\end{aligned}
$$

$$
\begin{aligned}
F_k^{14} = &\{M_1(k_1k_3+k_2k_4)-M_4(k_2k_4+k_1k_3)\}\mathbf{i}\\
&+\{M_1(k_3k_4-k_1k_2)+M_4(k_1k_2-k_3k_4)\}\mathbf{j}\\
&+\{M_1(1+k_4^2-k_1^2-k_1)+M_4(1+k_1^2-k_4^2-k_4)\}\mathbf{k}\\
&+M_1(k_1k_4+k_4)-M_4(k_1k_4+k_1),
\end{aligned}
$$

$$
\begin{aligned}
F_k^{23} = &\{-M_2(k_2k_4+k_1k_3)+M_3(k_1k_3+k_2k_4)\}\mathbf{i}\\
&+\{M_2(k_3k_4-k_1k_2)+M_3(k_1k_2-k_3k_4)\}\mathbf{j}\\
&-\{M_2(1+k_3^2-k_2^2-k_2)+M_3(1+k_2^2-k_3^2-k_3)\}\mathbf{k}\\
&+M_2(k_2k_3+k_3)-M_3(k_2k_3+k_2),
\end{aligned}
$$

$$
\begin{aligned}
F_k^{24} = &\{M_2(k_2k_3-k_4k_1)+M_4(k_1k_4-k_2k_3)\}\mathbf{i}\\
&+\{M_2(1+k_4^2-k_2^2-k_2)+M_4(1+k_2^2-k_4^2-k_4)\}\mathbf{j}\\
&-\{M_2(k_1k_2+k_3k_4)-M_4(k_3k_4+k_1k_2)\}\mathbf{k}\\
&+M_2(k_2k_4+k_4)-M_4(k_2k_4+k_2),
\end{aligned}
$$

$$
\begin{aligned}
F_k^{34} = &-\{M_3(1+k_4^2-k_3^2-k_3)+M_4(1+k_3^2-k_4^2-k_4)\}\mathbf{i}\\
&+\{M_3(-k_2k_3-k_1k_4)+M_4(k_1k_4+k_2k_3)\}\mathbf{j}\\
&+\{M_3(k_2k_4-k_1k_3)+M_4(k_1k_3-k_2k_4)\}\mathbf{k}\\
&+M_3(k_3k_4+k_4)-M_4(k_3k_4+k_3).
\end{aligned}
$$

Proposition 5.1. *The 2-form F with components F_k^{ij} above is $su(2)$-valued if and only if*

$$k_1 = k_2 = k_3 = k_4. \tag{5.4}$$

Proof. From Proposition 4.4 F is $su(2)$-valued if and only if

$$M_i(k_ik_j+k_j)-M_j(k_ik_j+k_i)=0$$

for any $k_i \in \mathbb{Z}$, $i,j = 1,2,3,4$ and $i<j$. It follows immediately (5.4). □

Thus, the $su(2)$-valued discrete curvature 2-form F can be written in the quaternionic form as follows

$$F = \sum_{k,\ k_i=\mu} M_\mu(2-2\mu)\{(\varepsilon^k_{12} - \varepsilon^k_{34})\mathbf{i} + (\varepsilon^k_{13} + \varepsilon^k_{24})\mathbf{j} + (\varepsilon^k_{14} - \varepsilon^k_{23})\mathbf{k}\}. \tag{5.5}$$

From (5.2) here we have $M_\mu = \frac{1}{2(1+4\mu^2)(1+\mu+2\mu^2)}$. Since $k_i = \mu$, in (5.5) we can write ε^μ_{ij} instead of ε^k_{ij}.

If we consider the 0-form

$$\omega = \sum_\mu M_\mu(1-\mu)x^\mu, \qquad \mu \in \mathbb{Z} \tag{5.6}$$

and use the following relation (see the proof of Proposition 4.5)

$$\bar{e} \cup e = 2\{(\varepsilon_{12} - \varepsilon_{34})\mathbf{i} + (\varepsilon_{13} + \varepsilon_{24})\mathbf{j} + (\varepsilon_{14} - \varepsilon_{23})\mathbf{k}\},$$

then F can be written as

$$F = \omega \cup \bar{e} \cup e. \tag{5.7}$$

In view of Corollary 4.6 F is anti-self-dual, i.e., $F = -\tilde{\iota} * F$. Thus under condition (5.4) A with components (5.1) describes an anti-instanton.

In the same manner we can see that the following quaternionic 1-form

$$A = \mathrm{Im}(f \cup \bar{e}), \tag{5.8}$$

where f has the components

$$f_k = \frac{\kappa}{1+|\kappa|^2}, \tag{5.9}$$

leads to an instanton solution of (4.24). Indeed, substituting (5.8) and (5.9) in (4.16) we now obtain

$$\begin{aligned}
F^{12}_k = {} & \{-M_1(1+k_2^2-k_1^2-k_1) - M_2(1+k_1^2-k_2^2-k_2)\}\mathbf{i} \\
& + \{M_1(k_4k_1 - k_2k_3) + M_2(k_3k_2 - k_4k_1)\}\mathbf{j} \\
& + \{M_1(-k_2k_4 - k_1k_3) + M_2(k_1k_3 + k_2k_4)\}\mathbf{k} \\
& + M_1(k_1k_2 + k_2) - M_2(k_1k_2 + k_1),
\end{aligned}$$

$$\begin{aligned}
F^{13}_k = {} & \{M_1(-k_2k_3 - k_1k_4) + M_3(k_1k_4 + k_2k_3)\}\mathbf{i} \\
& - \{M_1(1+k_3^2-k_1^2-k_1) + M_3(1+k_1^2-k_3^2-k_3)\}\mathbf{j} \\
& + \{M_1(k_1k_2 - k_3k_4) + M_3(k_3k_4 - k_1k_2)\}\mathbf{k}
\end{aligned}$$

$$
\begin{aligned}
&+M_1(k_1k_3+k_3)-M_3(k_1k_3+k_1),\\[2ex]
F_k^{14} &= \{M_1(k_1k_3-k_2k_4)+M_4(k_2k_4-k_1k_3)\}\mathbf{i}\\
&+\{M_1(-k_3k_4-k_1k_2)+M_4(k_1k_2+k_3k_4)\}\mathbf{j}\\
&-\{M_1(1+k_4^2-k_1^2-k_1)+M_4(1+k_1^2-k_4^2-k_4)\}\mathbf{k}\\
&+M_1(k_1k_4+k_4)-M_4(k_1k_4+k_1),\\[2ex]
F_k^{23} &= \{M_2(-k_2k_4+k_1k_3)+M_3(-k_1k_3+k_2k_4)\}\mathbf{i}\\
&+\{M_2(k_3k_4+k_1k_2)-M_3(k_1k_2+k_3k_4)\}\mathbf{j}\\
&-\{M_2(1+k_3^2-k_2^2-k_2)+M_3(1+k_2^2-k_3^2-k_3)\}\mathbf{k}\\
&+M_2(k_2k_3+k_3)-M_3(k_2k_3+k_2),\\[2ex]
F_k^{24} &= \{M_2(k_2k_3+k_4k_1)-M_4(k_1k_4+k_2k_3)\}\mathbf{i}\\
&+\{M_2(1+k_4^2-k_2^2-k_2)+M_4(1+k_2^2-k_4^2-k_4)\}\mathbf{j}\\
&+\{M_2(k_1k_2-k_3k_4)+M_4(k_3k_4-k_1k_2)\}\mathbf{k}\\
&+M_2(k_2k_4+k_4)-M_4(k_2k_4+k_2),\\[2ex]
F_k^{34} &= -\{M_3(1+k_4^2-k_3^2-k_3)+M_4(1+k_3^2-k_4^2-k_4)\}\mathbf{i}\\
&+\{M_3(-k_2k_3+k_1k_4)+M_4(-k_1k_4+k_2k_3)\}\mathbf{j}\\
&+\{M_3(k_2k_4+k_1k_3)-M_4(k_1k_3+k_2k_4)\}\mathbf{k}\\
&+M_3(k_3k_4+k_4)-M_4(k_3k_4+k_3).
\end{aligned}
$$

Again, under condition (5.4) we can write F as

$$F=\sum_{\mu} M_\mu(2\mu-2)\{(\varepsilon_{12}^\mu+\varepsilon_{34}^\mu)\mathbf{i}+(\varepsilon_{13}^\mu-\varepsilon_{24}^\mu)\mathbf{j}+(\varepsilon_{14}^\mu+\varepsilon_{23}^\mu)\mathbf{k}\},$$

where $\mu\in\mathbb{Z}$. Therefore

$$F=\omega\cup e\cup\bar{e}, \tag{5.10}$$

where ω is given by (5.6). Thus the discrete curvature form (5.10) is self-dual and we can say that (5.8) describes an instanton.

Now to complete the analogy with the continual case we describe more precisely how the anti-instanton given by (5.1) behaves as $|\kappa|\to\infty$. It is clear that f_k is asymptotically $\frac{\bar{\kappa}}{|\kappa|^2}=\kappa^{-1}$. Then

$$A\sim \mathrm{Im}(x^{-1}\cup d^c x)\quad\text{as}\quad |\kappa|\to\infty. \tag{5.11}$$

Here x is given by (4.10). By virtue of Proposition 4.3 the discrete curvature $F=0$ at infinity.

Proposition 5.2. *The anti-instanton (5.1) has the same form at ∞ as it has near 0.*

Proof. Introduce the quaternionic 0-form

$$y=\sum_k y_k x^k, \qquad \text{where} \qquad y_k=\frac{1}{\kappa}$$

and remind $\kappa=k_1+k_2\mathbf{i}+k_3\mathbf{j}+k_4\mathbf{k}$. Clearly, $y=x^{-1}$. We first compute $x\cup f\cup e\cup x^{-1}$, where f is given by (5.1). To do this, take (4.10), (4.11) and use the $\cup$-product definition. We have

$$\begin{aligned} x\cup f\cup e &= \Big(\sum_k \kappa x^k\Big)\cup\Big(\sum_k \frac{\bar\kappa}{1+|\kappa|^2}x^k\Big)\cup e \\ &= \Big(\sum_k \frac{|\kappa|^2}{1+|\kappa|^2}x^k\Big)\cup e = e-\Big(\sum_k \frac{1}{1+|\kappa|^2}x^k\Big)\cup e \\ &= d^cx-\Big(\sum_k \frac{1}{1+|\kappa|^2}x^k\Big)\cup d^cx. \end{aligned}$$

From this by (4.5) we get

$$x\cup f\cup e\cup x^{-1}=-x\cup d^cx^{-1}+\Big(\sum_k \frac{\kappa}{1+|\kappa|^2}x^k\Big)\cup d^cx^{-1}. \tag{5.12}$$

Now gauge transform the form $f\cup e$ by the gauge transformation $g=x^{-1}$. We must take the imaginary part of (4.13). This yields by (5.12)

$$\begin{aligned} \mathrm{Im}(g^{-1}\cup f\cup e\cup g+g^{-1}\cup d^cg) &= \mathrm{Im}\Big(\Big(\sum_k \frac{\kappa}{1+|\kappa|^2}x^k\Big)\cup d^cx^{-1}\Big) \\ &= \mathrm{Im}\Big(\Big(\sum_k \frac{\bar y_k}{1+|y_k|^2}x^k\Big)\cup d^cy\Big). \end{aligned}$$

Hence the gauge transformed anti-instanton A has precisely the form (5.11) near $y=0$. ☐

The same conclusion can be drawn for the instanton (5.8).

In the continual theory Proposition 5.2 shows that the anti-instanton (or instanton) extends to the 4-sphere S^4. This follows from the fact that S^4 can be obtained from $\mathbb{R}^4$ by adding the point at infinity, i.e., $S^4\simeq\mathbb{R}^4\cup\{\infty\}$. To obtain the same result for our discrete model we need to construct a suitable combinatorial analog of the 4-sphere. It would be interesting to connect the above constructions with discrete model of S^4 described in [21]. This connection must be investigated and we hope to treat its further in future work.

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Calculations in New Sequence Spaces and Application to Statistical Convergence

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ABSTRACT

In this paper we recall recent results that are direct consequences of the fact that $(w_\infty(\lambda), w_\infty(\lambda))$ is a Banach algebra. Then we define the set $W_\tau = D_\tau w_\infty$ and characterize the sets $W_\tau(A)$ where A is either of the operators Δ, Σ, $\Delta(\lambda)$, or $C(\lambda)$. Afterwards we consider the sets $[A_1, A_2]_{W_\tau}$ of all sequences X such that $A_1(\lambda)\left(\left|A_2(\mu)X\right|\right) \in W_\tau$ where A_1 and A_2 are of the form $C(\xi)$, $C^+(\xi)$, $\Delta(\xi)$, or $\Delta^+(\xi)$ and it is given necessary conditions to get $\left[A_1(\lambda), A_2(\mu)\right]_{W_\tau}$ in the form W_ξ. Finally we apply the previous results to statistical convergence. So we have conditions to have $x_k \to L(S(A))$ where A is either of the infinite matrices $D_{1/\tau}C(\lambda)C(\mu)$, $D_{1/\tau}\Delta(\lambda)\Delta(\mu)$, $D_{1/\tau}\Delta(\lambda)C(\mu)$. We also give conditions to have $x_k \to 0(S(A))$ where A is either of the operators $D_{1/\tau}C^+(\lambda)\Delta(\mu)$, $D_{1/\tau}C^+(\lambda)C(\mu)$, $D_{1/\tau}C^+(\lambda)C^+(\mu)$, or $D_{1/\tau}\Delta(\lambda)C^+(\mu)$.

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RESUMEN

Recordamos resultados recientes que son consecuencia directa del hecho de que $(w_\infty(\lambda), w_\infty(\lambda))$ es una algebra de Banach. Entonces nosotros definimos el conjunto $W_\tau = D_\tau w_\infty$ y caracterizamos los conjuntos $W_\tau(A)$ donde A es uno de los siguientes operadores Δ, Σ, $\Delta(\lambda)$, o $C(\lambda)$. Después consideramos los conjuntos $[A_1, A_2]_{W_\tau}$ de todas las sucesiones X tal que $A_1(\lambda)(|A_2(\mu)X|) \in W_\tau$ donde A_1 y A_2 son de la forma $C(\xi)$, $C^+(\xi)$, $\Delta(\xi)$, o $\Delta^+(\xi)$ y son dadas condiciones necesarias para obtener $[A_1(\lambda), A_2(\mu)]_{W_\tau}$ en la forma W_ξ. Finalmente, aplicamos los resultados previos para tener $x_k \to L(S(A))$ donde A es una de las matrices infinitas $D_{1/\tau}C(\lambda)C(\mu)$, $D_{1/\tau}\Delta(\lambda)\Delta(\mu)$, $D_{1/\tau}\Delta(\lambda)C(\mu)$. Nosotros también damos condiciones para tener $x_k \to 0(S(A))$ donde A es uno de los operadores $D_{1/\tau}C^+(\lambda)\Delta(\mu)$, $D_{1/\tau}C^+(\lambda)C(\mu)$, $D_{1/\tau}C^+(\lambda)C^+(\mu)$, o $D_{1/\tau}\Delta(\lambda)C^+(\mu)$.

Key words and phrases: *Banach algebra, statistical convergence, $A-$statistical convergence, infinite matrix.*

Math. Subj. Class.: *40C05, 40F05, 40J05, 46A15.*

1 Introduction

In this paper we consider spaces generalizing the well-known sets w^0 and w_∞ introduced and studied by Maddox [12, 13]. Recall that w^0 and w_∞ are the sets of *strongly summable and strongly bounded sequences*. In [15] Malkowsky and Rakočević gave characterizations of matrix maps between w^0, w, or w_∞ and w_∞^p and between w^0, w, or w_∞ and l_1. In [2] de Malafosse defined the spaces $w_\alpha(\lambda)$, $w_\alpha^{(c)}(\lambda)$ and $w_\alpha^0(\lambda)$ of all sequences that are *$\alpha-$strongly bounded, summable and summable to zero* respectively. For instance recall that $w_\alpha(\lambda)$ is the set of all sequences $(x_n)_n$ such that $1/\lambda_n \sum_{m=1}^n |x_m| = \alpha_n O(1)$ as n tends to infinity. It was shown that these spaces can be written in the form s_ξ, $s_\xi^{(c)}$ and s_ξ^0 under some condition on α and λ.

More recently in [5] it was shown that if λ is a *sequence exponentially bounded* then $(w_\infty(\lambda), w_\infty(\lambda))$ is a Banach algebra. This result led to consider bijective operators mapping between $w_\infty(\lambda)$. Here we will use these results to study sets of the form $W_\tau = D_\tau w_\infty$, $W_\tau(\Delta(\lambda))$, $W_\tau(C(\lambda))$ and $W_\tau(C^+(\lambda))$ generalizing the well-known *set of strongly bounded sequences* $c_\infty = w_\infty(\Delta(\mu))$ where $\mu_n = n$ for all n. These results lead to the study of *statistical convergence* which was introduced by Steinhaus in 1949, see [16], and studied by several authors such as Fast [7], Fridy, Orhan [8-11] and Connor. Here we will deal with the notion of *$A-$ statistical convergence* which generalizes the notion of *statistical convergence,* see [6], where A belongs to a special class of operators.

The paper is organized as follows. In Section 2 among other things we recall a recent result on the operators Δ_ρ and Δ_ρ^T considered as map from $w_\infty(\lambda)$ to itself. In Sections 3 and 4 our aim is to give necessary conditions to have $W_\tau(A)$ in the form W_ξ when A is either one of the matrices $\Delta(\lambda)$, $C(\lambda)$ or $C^+(\lambda)$. Then we consider spaces generalizing the well-known set of all strongly bounded sequences $[C,\Delta] = c_\infty$ defined and studied by Maddox. Then we will define the sets $[A_1,A_2]_{W_\tau}$ of all sequences X with $A_1(\lambda)\left(\left|A_2\left(\mu\right)X\right|\right) \in W_\tau$ where A_1 and A_2 are of the form $C(\xi)$, $C^+(\xi)$, $\Delta(\xi)$, or $\Delta^+(\xi)$ and we will give necessary conditions to get $\left[A_1(\lambda),A_2\left(\mu\right)\right]$ in the form W_τ. In Section 5 we apply these results to $A-$ *statistical convergence*, where A is equal to $D_{1/\tau}A_1A_2$ and A_1, A_2 are of the form $C(\xi)$, $\Delta(\xi)$, $\Delta\left(\mu\right)$, or $C^+(\xi)$.

2 Well Known Results

For a given infinite matrix $A = (a_{nm})_{n,m\geq 1}$ we define the operators A_n for any integer $n \geq 1$, by

$$A_n(X) = \sum_{m=1}^{\infty} a_{nm}x_m \tag{1}$$

where $X = (x_n)_{n\geq 1}$, the series intervening in the second member being convergent. So we are led to the study of the infinite linear system

$$A_n(X) = b_n \qquad n = 1,2,... \tag{2}$$

where $B = (b_n)_{n\geq 1}$ is a one-column matrix and X the unknown, see [2-5]. The equations (2) can be written in the form $AX = B$, where $AX = (A_n(X))_{n\geq 1}$. In this paper we shall also consider A as an operator from a sequence space into another sequence space.

We will write s for the set of all complex sequences and ℓ_∞ for the set of all *bounded sequences*.

Let E and F be any subsets of s. When A maps E into F we write that $A \in (E,F)$. So for every $X \in E$, $AX \in F$, ($AX \in F$ means that for each $n \geq 1$ the series defined by $y_n = \sum_{m=1}^{\infty} a_{nm}x_m$ is convergent and $(y_n)_{n\geq 1} \in F$).

Body Math For any subset E of s, we put

Body Math

$$AE = \{Y \in s \ : \ Y = AX \text{ for some } X \in E\}. \tag{3}$$

If F is a subset of s, we shall denote

$$F(A) = F_A = \{X \in s \ : \ Y = AX \in F\}. \tag{4}$$

In all what follows we will use the set

$$U^{+}=\left\{(u_n)_{n\geq 1}\in s\ :\quad u_n>0 \text{ for all } n\right\}$$

and the notation $e=(1,\dots,1,\dots)$. So for $\lambda=(\lambda_n)_{n\geq 1}\in U^{+}$ we will consider the sets of *strongly bounded and strongly summable sequences*, respectively, that is

$$\begin{aligned} w_{\infty}(\lambda) &= \left\{X=(x_n)_{n\geq 1}\in s\ :\ \sup_n \frac{1}{\lambda_n}\sum_{m=1}^{n}|x_m|<\infty\right\}, \\ w^{0}(\lambda) &= \left\{X=(x_n)_{n\geq 1}\in s\ :\ \lim_{n\to\infty}\frac{1}{\lambda_n}\sum_{m=1}^{n}|x_m|=0\right\} \end{aligned}$$

and

$$w(\lambda)=\left\{X=(x_n)_{n\geq 1}\in s\ :\ X-le\in w^{0}(\lambda) \text{ for some } l\in\mathbb{C}\right\}$$

were studied by Malkowsky, with the concept of *exponentially bounded sequences*, see [3]. Recall that Maddox [12, 13], defined and studied the sets $w_{\infty}(\lambda)=w_{\infty}$, $w_0(\lambda)=w^0$ and $w(\lambda)=w$ where $\lambda_n=n$ for all n.

A Banach space E of complex sequences with the norm $\|\,\|_E$ *is a BK space if each projection $P_n: X\mapsto P_nX=x_n$ is continuous. A BK space E is said to have AK if every sequence $X=(x_n)_{n\geq 1}\in E$ has a unique representation $X=\sum_{n=1}^{\infty}x_ne_n$* where e_n is the sequence with 1 in the n-th position and 0 otherwise.

Recall that a nondecreasing sequence $\lambda=(\lambda_n)_{n\geq 1}\in U^{+}$ is *exponentially bounded* if there is an integer $m\geq 2$ such that for all non-negative integers ν there is at least one term $\lambda_n\in I_m^{(\nu)}=\left[m^{\nu},m^{\nu+1}-1\right]$. It was shown (cf. [14, Lemma 1]) that a non-decreasing sequence $\lambda=(\lambda_n)_{n\geq 1}$ is *exponentially bounded* if and only if there are reals $s\leq t$ such that for some subsequence $\left(\lambda_{n_i}\right)_{i\geq 1}$

$$0<s\leq\frac{\lambda_{n_i}}{\lambda_{n_{i+1}}}\leq t<1 \text{ for all } i=1,2,\dots;$$

such a sequence is called an *associated subsequence*. Consider now the norm

$$\|X\|_{\lambda}=\sup_n\left(\frac{1}{\lambda_n}\sum_{m=1}^{n}|x_m|\right).$$

In [5] it was shown that if $\lambda=(\lambda_n)_{n\geq 1}\in U^{+}$ is *exponentially bounded* the class $(w_{\infty}(\lambda),w_{\infty}(\lambda))$ is a *Banach algebra* with the norm

$$\|A\|_{(w_{\infty}(\lambda),w_{\infty}(\lambda))}=\sup_{X\neq 0}\left(\frac{\|AX\|_{\lambda}}{\|X\|_{\lambda}}\right). \tag{5}$$

For $\rho = (\rho_n)_{n\geq 1}$ consider now the following matrices

$$\Delta_\rho^+ = \begin{pmatrix} 1 & -\rho_1 & & & \\ & . & . & & \\ & & 1 & -\rho_n & \\ & 0 & & . & . \\ & & & & \end{pmatrix} \text{ and } \Delta_\rho = \begin{pmatrix} 1 & & & & \\ -\rho_1 & 1 & & & 0 \\ & . & . & & \\ & & -\rho_{n-1} & 1 & . \\ & & & . & . \end{pmatrix}.$$

It can easily be shown that if $\rho = (\rho_n)_{n\geq 1}$ and $(\lambda_{n+1}/\lambda_n)_{n\geq 1} \in \ell_\infty$ then $\Delta_\rho^+ \in (w_\infty(\lambda), w_\infty(\lambda))$. We also see that $\Delta_\rho \in (w_\infty(\lambda), w_\infty(\lambda))$ for ρ, $(\lambda_{n-1}/\lambda_n)_{n\geq 2} \in \ell_\infty$. Recall the next result which is a direct consequence of [5, Theorem 5.1 and Theorem 5.12].

Lemma 2.1. *Let $\lambda \in U^+$ be a sequence exponentially bounded.*

(i) If

$$\overline{\lim_{n\to\infty}}\left(\frac{\lambda_{n+1}}{\lambda_n}\right) < \infty \text{ and } \overline{\lim_{n\to\infty}}|\rho_n| < \frac{1}{\overline{\lim_{n\to\infty}}\left(\frac{\lambda_{n+1}}{\lambda_n}\right)}, \tag{6}$$

for given $B \in w_\infty(\lambda)$ the equation $\Delta_\rho^+ X = B$ has a unique solution in $w_\infty(\lambda)$.

(ii) If

$$\overline{\lim_{n\to\infty}}|\rho_n| < \frac{1}{\overline{\lim_{n\to\infty}}\left(\frac{\lambda_{n-1}}{\lambda_n}\right)}, \tag{7}$$

then for any given $B \in w_\infty(\lambda)$ the equation $\Delta_\rho X = B$ has a unique solution in $w_\infty(\lambda)$.

When λ is a strictly increasing sequence tending to infinity we obtain similar results on the Banach algebra $(w^0(\lambda), w^0(\lambda))$ with the norm $\|A\|_{(w_\infty(\lambda), w_\infty(\lambda))}$.

3 On the Sets $W_\tau(A)$ Where A is Either $\Delta(\lambda)$, $C(\lambda)$ or $C^+(\lambda)$

In the following we will use the operators represented by $C(\lambda)$ and $\Delta(\lambda)$. Let U be the set of all sequences $(u_n)_{n\geq 1}$ with $u_n \neq 0$ for all n. We define $C(\lambda)$ for $\lambda = (\lambda_n)_{n\geq 1} \in U$, by

$$[C(\lambda)]_{nm} = \begin{cases} \dfrac{1}{\lambda_n} & \text{if } m \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We will write $C(\lambda)^T = C^+(\lambda)$, $C(e) = \Sigma$, $\Sigma^+ = \Sigma^T$, and for $\lambda_n = n$, the matrix $C_1 = C((n)_n)$ is called the Cesaro operator. If It can be proved that the matrix $\Delta(\lambda)$ with

$$[\Delta(\lambda)]_{nm} = \begin{cases} \lambda_n & \text{if } m = n, \\ -\lambda_{n-1} & \text{if } m = n-1 \text{ and } n \geq 2, \\ 0 & \text{otherwise,} \end{cases}$$

is the inverse of $C(\lambda)$, see [2, 3]. We will use the following sets

$$\begin{aligned} \Gamma &= \left\{ X \in U^+ : \overline{\lim_{n\to\infty}} \left(\frac{x_{n-1}}{x_n} \right) < 1 \right\}, \\ \Gamma^+ &= \left\{ X \in U^+ : \overline{\lim_{n\to\infty}} \left(\frac{x_{n+1}}{x_n} \right) < 1 \right\}. \end{aligned}$$

Note that $X \in \Gamma^+$ if and only if $1/X \in \Gamma$.

For given sequence $\tau = (\tau_n)_{n\geq 1} \in U^+$, we write D_τ for the diagonal matrix defined by $[D_\tau]_{nn} = \tau_n$ for all n. For any subset E of s, we write

$$D_\tau E = \left\{ X = (x_n)_{n\geq 1} \in s \ : \left(\frac{x_n}{\tau_n} \right)_n \in E \right\}.$$

We put $W_\tau = D_\tau w_\infty$ for $\tau \in U^+$, that is

$$W_\tau = \left\{ X : \|X\|_{W_\tau} = \sup_n \left(\frac{1}{n} \sum_{m=1}^{\infty} \frac{|x_m|}{\tau_m} \right) < \infty \right\}.$$

It can easily be seen that $W_\tau = w_\infty(D_{1/\tau})$ is a BK space with norm $\|\|_{W_\tau}$, (cf. [17, Theorem 4.3.6, p. 52]). In all that follows we will use the convention that the entries with subscripts strictly less than 1 are equal to zero. Then we are interested in the study of the following sets where $\lambda, \tau \in U^+$.

$$\begin{aligned} W_\tau(\Delta(\lambda)) &= \left\{ X : \sup_n \left(\frac{1}{n} \sum_{m=1}^{n} \frac{1}{\tau_m} |\lambda_m x_m - \lambda_{m-1} x_{m-1}| \right) < \infty \right\}, \\ W_\tau(C(\lambda)) &= \left\{ X : \sup_n \frac{1}{n} \sum_{m=1}^{n} \left(\frac{1}{\lambda_m \tau_m} \sum_{k=1}^{m} |x_k| \right) < \infty \right\}, \\ W_\tau\left(C^+(\lambda)\right) &= \left\{ X : \sup_n \frac{1}{n} \sum_{m=1}^{n} \left(\frac{1}{\tau_m} \sum_{k=m}^{\infty} \frac{|x_k|}{\lambda_k} \right) < \infty \right\}. \end{aligned}$$

Note that for $\lambda_n = n$ and $\tau = e$, $W_\tau(\Delta(\lambda))$ is the well known set of all strongly and bounded sequences c_∞. We obtain the following result that is a direct consequence of Lemma 2.1.

Proposition 3.1. *(i) If $\tau \in \Gamma$ then the operators Δ and Σ are bijective from W_τ into itself and*

$$W_\tau(\Delta) = W_\tau,\ W_\tau(\Sigma) = W_\tau.$$

(ii) a) If $\lambda\tau \in \Gamma$ then

$$W_\tau(C(\lambda)) = W_{\lambda\tau}.$$

b) If $\tau \in \Gamma$ then

$$W_\tau(\Delta(\lambda)) = W_{\tau/\lambda}.$$

(iii) Let $\tau \in \Gamma^+$. Then

a) the operators Δ^+ and Σ^+ are bijective from W_τ into itself and

$$W_\tau\left(\Sigma^+\right) = W_\tau.$$

b) the operator $C^+(\lambda)$ is bijective from $W_{\lambda\tau}$ into W_τ and

$$W_\tau\left(C^+(\lambda)\right) = W_{\lambda\tau}.$$

Proof. (i) By Lemma 2.1 where $\rho_n = \tau_{n-1}/\tau_n$ and $\lambda_n = n$ for all n, we easily see that if

$$\overline{\lim_{n\to\infty}} \frac{\tau_{n-1}}{\tau_n} < \frac{1}{\lim_{n\to\infty}\left(\frac{n-1}{n}\right)} = 1,$$

that is $\tau \in \Gamma$, then $D_{1/\tau}\Delta D_\tau$ is bijective from w_∞ to itself. This means that Δ is bijective from $D_\tau w_\infty$ to itself. Since Σ is also bijective from $D_\tau w_\infty$ to itself, this shows $W_\tau(\Delta) = W_\tau$ and $W_\tau(\Sigma) = W_\tau$.

(ii) We have $X \in W_\tau(C(\lambda))$ if and only if $\Sigma X \in D_{\lambda\tau} w_\infty = W_{\lambda\tau}$. This means that $X \in W_{\lambda\tau}(\Sigma)$ and by (i) the condition $\lambda\tau \in \Gamma$ implies $W_{\lambda\tau}(\Sigma) = W_{\lambda\tau}$. Then $W_\tau(C(\lambda)) = W_{\lambda\tau}$ and $C(\lambda)$ is bijective from $W_{\lambda\tau}$ to W_τ. Since $\Delta(\lambda) = C(\lambda)^{-1}$ we conclude $\Delta(\lambda)$ bijective from W_τ to $W_{\lambda\tau}$ and $W_{\lambda\tau}(\Delta(\lambda)) = W_\tau$. We deduce that for $\tau \in \Gamma$, $W_\tau(\Delta(\lambda)) = W_{\tau/\lambda}$.

(iii) a) By Lemma 2.1 with $\rho_n = \tau_{n+1}/\tau_n$ and $\lambda_n = n$ we have $\Delta^+_\rho = D_{1/\tau}\Delta^+ D_\tau$ and Δ^+ is bijective from $D_\tau w_\infty = W_\tau$ into itself for $\tau \in \Gamma^+$ and it is the same for Σ^+. Now the equation $\Sigma^+ X = Y$ for $Y \in W_\tau$ is equivalent to

$$\sum_{m=n}^{\infty} x_m = y_n \text{ for all } n. \tag{8}$$

We deduce (8) has a unique solution $X = (y_n - y_{n+1})_{n\geq 1} = \Delta^+ Y \in W_\tau$ and $W_\tau\left(\Sigma^+\right) = W_\tau$.

b) We have

$$W_\tau\left(C^+(\lambda)\right) = \left\{X : \Sigma^+ D_{1/\lambda} X \in W_\tau\right\} = D_\lambda W_\tau\left(\Sigma^+\right).$$

Now as we have seen above since $\tau \in \Gamma^+$ we get $W_\tau\left(\Sigma^+\right) = W_\tau$ and

$$W_\tau\left(C^+(\lambda)\right) = D_\lambda W_\tau\left(\Sigma^+\right) = D_\lambda W_\tau = W_{\lambda\tau}.$$

This gives the conclusion. $\square$

4 Calculations in New Sequence Spaces

4.1 The sets $[C,\Delta]_{W_\tau}$, $[C,C]_{W_\tau}$, $[C^+,\Delta]_{W_\tau}$, $[C^+,C]_{W_\tau}$ and $[C^+,C^+]_{W_\tau}$.

In [4], were defined and studied the sets

$$[A_1,A_2]=[A_1(\lambda),A_2(\mu)]=\{X\in s\ :\ A_1(\lambda)(|A_2(\mu)X|)\in D_\tau l_\infty\}$$

where $|X|=(|x_n|)_{n\geq 1}$, A_1 and A_2 of the form $C(\xi)$, $C^+(\xi)$, $\Delta(\xi)$, or $\Delta^+(\xi)$ for $\xi\in U^+$. It was given necessary conditions to get $[A_1(\lambda),A_2(\mu)]$ in the form s_γ.

Similarly in the following we will put

$$[A_1,A_2]_{W_\tau}=[A_1(\lambda),A_2(\mu)]_{W_\tau}=\{X\in s\ :\ A_1(\lambda)(|A_2(\mu)X|)\in W_\tau\}$$

for λ, μ, $\tau\in U^+$. We can explicitly write the previous sets $[A_1,A_2]_{W_\tau}$ as follows.

$$\begin{aligned}
[C,\Delta]_{W_\tau} &= \left\{X:\sup_n\left(\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\lambda_m\tau_m}\sum_{k=1}^{m}|\mu_kx_k-\mu_{k-1}x_{k-1}|\right)<\infty\right\},\\
[C,C]_{W_\tau} &= \left\{X:\sup_n\left(\frac{1}{n}\sum_{m=1}^{n}\left(\frac{1}{\lambda_m\tau_m}\sum_{k=1}^{m}\frac{1}{\mu_k}\left|\sum_{i=1}^{k}x_i\right|\right)\right)<\infty\right\},\\
[C^+,\Delta]_{W_\tau} &= \left\{X:\sup_n\left(\frac{1}{n}\sum_{m=1}^{n}\left(\frac{1}{\tau_m}\sum_{k=m}^{\infty}\frac{1}{\lambda_k}|\mu_kx_k-\mu_{k-1}x_{k-1}|\right)\right)<\infty\right\},\\
[C^+,C]_{W_\tau} &= \left\{X:\sup_n\left(\frac{1}{n}\sum_{m=1}^{n}\left(\frac{1}{\tau_m}\sum_{k=m}^{\infty}\frac{1}{\lambda_k}\frac{1}{\mu_k}\left|\sum_{i=1}^{k}x_i\right|\right)\right)<\infty\right\},\\
[C^+,C^+]_{W_\tau} &= \left\{X:\sup_n\left(\frac{1}{n}\sum_{m=1}^{n}\left(\frac{1}{\tau_m}\sum_{k=m}^{\infty}\frac{1}{\lambda_k}\left|\sum_{i=k}^{\infty}\frac{x_i}{\mu_i}\right|\right)\right)<\infty\right\}.
\end{aligned}$$

Note that if $\lambda_n=\mu_n$ for all n we get the well known set of sequences that are strongly bounded $[C,\Delta]_{W_e}=c_\infty(\lambda)$. We can state the following.

Theorem 4.1. *Let λ, $\mu,\tau\in U^+$.*

(i) If $\lambda\tau\in\Gamma$ then

$$[C,\Delta]_{W_\tau}=W_{\lambda\tau/\mu};$$

(ii) if $\lambda\tau$, $\lambda\mu\tau\in\Gamma$ then

$$[C,C]_{W_\tau}=W_{\lambda\mu\tau};$$

(iii) if $\tau\in\Gamma^+$ and $\lambda\tau\in\Gamma$ then

$$[C^+,\Delta]_{W_\tau}=W_{\lambda\tau/\mu};$$

(iv) if $\tau \in \Gamma^+$ and $\lambda\mu\tau \in \Gamma$ then

$$\left[C^+, C\right]_{W_\tau} = W_{\lambda\mu\tau};$$

(v) if τ, $\lambda\tau \in \Gamma^+$ then

$$\left[C^+, C^+\right]_{W_\tau} = W_{\lambda\mu\tau}.$$

Proof. In the following we will use the fact that for any $\xi \in U^+$ we have $|X| \in W_\xi$ if and only if $X \in W_\xi$.

(i) We have $C(\lambda)\left(\left|\Delta(\mu)X\right|\right) \in W_\tau$ if and only if $\left|\Delta(\mu)X\right| \in W_\tau(C(\lambda))$ and by Proposition 3.1, since $\lambda\tau \in \Gamma$ we get $W_\tau(C(\lambda)) = W_{\lambda\tau}$. Then by Proposition 3.1 (ii) we have $W_{\lambda\tau}\left(\Delta(\mu)\right) = W_{\lambda\tau/\mu}$ and we conclude $\Delta(\mu)X \in W_{\lambda\tau}$ if and only if $X \in W_{\lambda\tau}\left(\Delta(\mu)\right) = W_{\lambda\tau/\mu}$, that is $[C,\Delta]_{W_\tau} = W_{\lambda\tau/\mu}$.

(ii) Here we have $C(\lambda)\left(\left|C(\mu)X\right|\right) \in W_\tau$ if and only if $\left|C(\mu)X\right| \in W_\tau(C(\lambda))$; and since $\lambda\tau \in \Gamma$ by Proposition 3.1 we have $W_\tau(C(\lambda)) = W_{\lambda\tau}$. So $X \in [C,C]_{W_\tau}$ if and only if $C(\mu)X \in W_{\lambda\tau}$, that is $X \in W_{\lambda\tau}\left(C(\mu)\right)$. Then by Proposition 3.1 (ii) a) $\lambda\mu\tau \in \Gamma$ implies $W_{\lambda\tau}\left(C(\mu)\right) = W_{\lambda\mu\tau}$ and we have shown (ii).

(iii) For any given $X \in \left[C^+,\Delta\right]_{W_\tau}$ we have $\Delta(\mu)X \in W_\tau\left(C^+(\lambda)\right)$ and for $\tau \in \Gamma^+$ we have $W_\tau\left(C^+(\lambda)\right) = W_{\lambda\tau}$. Now the condition $\lambda\tau \in \Gamma$ implies $X \in \left[C^+,\Delta\right]_{W_\tau}$ if and only if $X \in W_{\lambda\tau}\left(\Delta(\mu)\right) = W_{\lambda\tau/\mu}$ and we have shown (iii).

(iv) Let $X \in \left[C^+,C\right]_{W_\tau}$. We have $\tau \in \Gamma^+$ implies $W_\tau\left(C^+(\lambda)\right) = W_{\lambda\tau}$ and so $X \in \left[C^+,C\right]_{W_\tau}$ if and only if $C(\mu)X \in W_{\lambda\tau}$. Now since $\lambda\mu\tau \in \Gamma$ we have $W_{\lambda\tau}\left(C(\mu)\right) = W_{\lambda\mu\tau}$ and we conclude $\left[C^+,C\right]_{W_\tau} = W_{\lambda\mu\tau}$.

(v) As above $X \in \left[C^+,C^+\right]_{W_\tau}$ if and only if $C^+(\mu)X \in W_\tau\left(C^+(\lambda)\right)$ and the condition $\tau \in \Gamma^+$ implies $W_\tau\left(C^+(\lambda)\right) = W_{\lambda\tau}$. Since $\lambda\tau \in \Gamma^+$ we conclude $W_{\lambda\tau}\left(C^+(\mu)\right) = W_{\lambda\mu\tau}$ that is $\left[C^+,C^+\right]_{W_\tau} = W_{\lambda\mu\tau}$. □

Now we are led to study sets of the form $[\Delta, A_2]_{W_\tau}$ for $A_2 \in \left\{\Delta,\Delta,C^+\right\}$.

4.2 The sets $[\Delta,\Delta]_{W_\tau}$, $[\Delta,C]_{W_\tau}$ and $\left[\Delta,C^+\right]_{W_\tau}$

Using the convention $\mu_0 = 0$, and the notation $\Delta(\mu)x_m = \mu_m x_m - \mu_{m-1}x_{m-1}$ for $m \geq 1$ we explicitly have

$$\begin{aligned}
[\Delta,\Delta]_{W_\tau} &= \left\{X : \sup_n\left(\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\tau_m}\left|\lambda_m\left|\Delta(\mu)x_m\right| - \lambda_{m-1}\left|\Delta(\mu)x_{m-1}\right|\right|\right) < \infty\right\},\\
[\Delta,C]_{W_\tau} &= \left\{X : \sup_n\left(\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\tau_m}\left|\lambda_m\left|\frac{1}{\mu_m}\sum_{k=1}^{m}x_k\right| - \lambda_{m-1}\left|\frac{1}{\mu_{m-1}}\sum_{k=1}^{m-1}x_k\right|\right|\right) < \infty\right\},
\end{aligned}$$

$$\left[\Delta, C^{+}\right]_{W_\tau} = \left\{X : \sup_n \left(\frac{1}{n}\sum_{m=1}^{n}\frac{1}{\tau_m}\left|\lambda_m\left|\sum_{k=m}^{\infty}\frac{x_k}{\mu_k}\right| - \lambda_{m-1}\left|\sum_{k=m-1}^{\infty}\frac{x_k}{\mu_k}\right|\right|\right) < \infty\right\}.$$

As a direct consequence of Proposition 3.1 we also obtain the following results.

Theorem 4.2. *Let λ, μ, $\tau \in U^{+}$. Then*

(i) If τ, $\tau/\lambda \in \Gamma$ then

$$[\Delta, \Delta]_{W_\tau} = W_{\tau/\lambda\mu}.$$

(ii) If τ, $\tau\mu/\lambda \in \Gamma$ then

$$[\Delta, C]_{W_\tau} = W_{\tau\mu/\lambda}.$$

(iii) If τ, $\tau/\lambda \in \Gamma^{+}$ then

$$\left[\Delta, C^{+}\right]_{W_\tau} = W_{\tau\mu/\lambda}.$$

Proof. (i) Let $X \in [\Delta, \Delta]_{W_\tau}$. Since $\tau \in \Gamma$ we have $W_\tau(\Delta(\lambda)) = W_{\tau/\lambda}$ and $\Delta(\lambda)\left|\Delta(\mu)X\right| \in W_\tau$ means $\Delta(\mu)X \in W_{\tau/\lambda}$. We conclude $W_{\tau/\lambda}(\Delta(\mu)) = W_{\tau/\lambda\mu}$ for $\tau/\lambda \in \Gamma$.

(ii) Reasoning as above since $\tau \in \Gamma$ we have $X \in [\Delta, C]_{W_\tau}$ if and only if $C(\mu)X \in W_{\tau/\lambda}$. We conclude since the condition $\tau\mu/\lambda \in \Gamma$ implies $W_{\tau/\lambda}(C(\mu)) = W_{\tau\mu/\lambda}$.

(iii) Here under the conditions τ, $\tau/\lambda \in \Gamma^{+}$, we have $X \in \left[\Delta, C^{+}\right]_{W_\tau}$ if and only if $X \in W_{\tau/\lambda}\left(C^{+}(\mu)\right) = W_{\tau\mu/\lambda}$. □

The previous results can be applied to the case when w_∞ is replaced by w^0.

4.3 The sets $[A_1, A_2]_{W_\tau^0}$

Using the Banach algebra $\left(w^0(\lambda), w^0(\lambda)\right)$ we get similar results to those given above replacing $w_\infty(\lambda)$ by $w^0(\lambda)$ and W_τ by $W_\tau^0 = D_\tau w^0$. Note that $X \in W_\tau^0$ if and only if

$$\frac{1}{n}\sum_{m=1}^{n}\frac{|x_m|}{\tau_m} \to 0 \ (n \to \infty).$$

By [17, Theorem 4.3.6, p. 52] the set W_τ^0 is a BK space with AK normed by $\|\|_{W_\tau}$. So we can state the following.

Proposition 4.3. *Let λ, $\mu \in U^{+}$.*

(i) If $\lambda\tau \in \Gamma$ then $[C, \Delta]_{W_\tau^0} = W_{\lambda\tau/\mu}^0$;

(ii) if $\lambda\tau$, $\lambda\mu\tau \in \Gamma$ then $[C, C]_{W_\tau^0} = W_{\lambda\mu\tau}^0$;

(iii) if $\tau \in \Gamma^{+}$ and $\lambda\tau \in \Gamma$ then $\left[C^{+}, \Delta\right]_{W_\tau^0} = W_{\lambda\tau/\mu}^0$;

(iv) if $\tau \in \Gamma^{+}$ *and* $\lambda\mu\tau \in \Gamma$ *then* $\left[C^{+}, C\right]_{W_{\tau}^{0}} = W_{\lambda\mu\tau}^{0}$;

(v) if τ, $\lambda\tau \in \Gamma^{+}$ *then* $\left[C^{+}, C^{+}\right]_{W_{\tau}^{0}} = W_{\lambda\mu\tau}^{0}$;

(vi) if τ, $\tau/\lambda \in \Gamma$ *then* $[\Delta, \Delta]_{W_{\tau}^{0}} = W_{\tau/\lambda\mu}^{0}$;

(vii) if τ, $\tau\mu/\lambda \in \Gamma$ *then* $[\Delta, C]_{W_{\tau}^{0}} = W_{\tau\mu/\lambda}^{0}$;

(viii) if τ, $\tau/\lambda \in \Gamma^{+}$ *then* $\left[\Delta, C^{+}\right]_{W_{\tau}^{0}} = W_{\tau\mu/\lambda}^{0}$.

We immediatly get the next remark.

Remark 4.4. *It can easily be seen that in Proposition 4.3 each of the sets* $[A_1, A_2]_{W_{\tau}^{0}}$ *is equal to* $W_{\tau}^{0}(A_1A_2)$*. This result is a direct consequence of the previous proofs and of the fact that* W_{τ}^{0} *is of absolute type, that is* $|X| \in W_{\tau}^{0}$ *if and only if* $X \in W_{\tau}^{0}$*.*

These results can be applied to statistical convergence.

5 Application to $A-$Statistical Convergence

In this section we will give conditions to have $x_k \to L(S(A))$ where A is either of the infinite matrices $D_{1/\tau}C(\lambda)C(\mu)$, $D_{1/\tau}\Delta(\lambda)\Delta(\mu)$, or $D_{1/\tau}\Delta(\lambda)C(\mu)$. Then we give conditions to have $x_k \to 0(S(A))$ where A is either of the operators $D_{1/\tau}C^{+}(\lambda)\Delta(\mu)$, $D_{1/\tau}C^{+}(\lambda)C(\mu)$, $D_{1/\tau}C^{+}(\lambda)$ $C^{+}(\mu)$ and $D_{1/\tau}\Delta(\lambda)C^{+}(\mu)$.

The sequence $X = (x_n)_{n\geq 1}$ is said to be *statiscally convergent to the number* L if

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0 \text{ for all } \varepsilon > 0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we will write $x_k \to L(S)$ or $st - \lim X = L$.

Let $A \in (E, F)$ for given $L \in \mathbb{C}$ and for every $\varepsilon > 0$ we will use the notation

$$I_{\varepsilon}(A) = \{k \leq n : |[AX]_k - L| \geq \varepsilon\},$$

(where we assume that every series $[AX]_k = A_k(X) = \sum_{m=1}^{\infty} a_{km}x_m$ for $k \geq 1$ is convergent). We will say that $X = (x_n)_{n\geq 1}$ *is* $A-$ *statistically convergent to* L *if for every* $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} |I_{\varepsilon}(A)| = 0.$$

Then we will write $x_k \to L(S(A))$ and for $A = I$, $x_k \to L(S(I))$ means that $st - \lim X = L$, (cf. [6]).

Now we require a lemma where we will put $T^{-1}e = \widetilde{l} = (l_n)_{n\geq 1}$ for given triangle T, that is $T = (t_{nm})_{n,m\geq 1}$ with $t_{nn} \neq 0$ and $t_{nm} = 0$ if $m > n$ for all n, m.

We can state the following.

Lemma 5.1. *If $X - L\widetilde{l} \in w^0(T)$ then x_k is $T-$ statistically convergent to L.*

Proof. The condition $X - L\widetilde{l} \in w^0(T)$ means that $T\left(X - L\widetilde{l}\right) \in w^0$. Since

$$TX - Le = T\left(X - LT^{-1}e\right) = T\left(X - L\widetilde{l}\right)$$

for any $\varepsilon > 0$ we have

$$\begin{aligned} y_n &= \frac{1}{n}\sum_{k=1}^{n} \left|[TX]_k - L\right| = \frac{1}{n}\sum_{k=1}^{n}\left|\left[T\left(X - L\widetilde{l}\right)\right]_k\right| \\ &\geq \frac{1}{n}\sum_{k\in I_\varepsilon(T)}\left|\left[T\left(X - L\widetilde{l}\right)\right]_k\right| \\ &\geq \frac{1}{n}\sum_{k\in I_\varepsilon(T)}\varepsilon \\ &\geq \frac{\varepsilon}{n}\left|\left\{k \leq n : \left|[TX]_k - L\right| \geq \varepsilon\right\}\right|. \end{aligned}$$

We conclude that $X - L\widetilde{l} \in w^0(T)$ implies $y_n \to 0$ $(n \to \infty)$ and $x_k \to L(S(T))$. ☐

We are led to state the next results.

Theorem 5.2. *(i) Let $\lambda\tau$, $\lambda\tau\mu \in \Gamma$. If*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\left|x_k - L\left[\lambda_k\mu_k\tau_k + \left(\mu_{k-1} + \mu_k\right)\lambda_{k-1}\tau_{k-1} - \lambda_{k-2}\mu_{k-2}\tau_{k-2}\right]\right|}{\lambda_k\mu_k\tau_k} = 0 \tag{9}$$

then $x_k \to L\left(S\left(D_{1/\tau}C(\lambda)C\left(\mu\right)\right)\right)$, that is for every $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \leq n : \left|\frac{1}{\lambda_k\tau_k}\sum_{i=1}^{k}\frac{1}{\mu_i}\left(\sum_{j=1}^{i}x_j\right) - L\right| \geq \varepsilon\right\}\right| = 0.$$

(ii) Let τ, $\tau/\lambda \in \Gamma$. If

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\lambda_k\mu_k}{\tau_k}\left|x_k - L\left(\frac{1}{\mu_k}\sum_{i=1}^{k}\frac{1}{\lambda_i}\sum_{j=1}^{i}\tau_j\right)\right| = 0$$

then $x_k \to L\left(S\left(D_{1/\tau}\Delta(\lambda)\Delta\left(\mu\right)\right)\right)$, that is for every $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \leq n : \left|\frac{1}{\tau_k}\left[\lambda_k\Delta\left(\mu\right)x_k - \lambda_{k-1}\Delta\left(\mu\right)x_{k-1}\right] - L\right| \geq \varepsilon\right\}\right| = 0.$$

(iii) Let τ, $\tau\mu/\lambda \in \Gamma$. If

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\lambda_k}{\mu_k\tau_k}\left|x_k - L\left[\left(\frac{\mu_k}{\lambda_k} - \frac{\mu_{k-1}}{\lambda_{k-1}}\right)\sum_{i=1}^{k-1}\tau_i + \frac{\mu_k}{\lambda_k}\tau_k\right]\right| = 0$$

then $x_k \to L\left(S\left(D_{1/\tau}\Delta(\lambda)C\left(\mu\right)\right)\right)$*, that is for every* $\varepsilon > 0$

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : \left|\frac{1}{\tau_k}\left[\left(\frac{\lambda_k}{\mu_k}-\frac{\lambda_{k-1}}{\mu_{k-1}}\right)\sum_{i=1}^{k-1}x_i+\frac{\lambda_k}{\mu_k}x_k\right]-L\right|\ge\varepsilon\right\}\right|=0.$$

Proof. (i) First by Proposition 4.3 (ii) and Remark 4.4, we easily see that for $\lambda\tau$, $\lambda\tau\mu\in\Gamma$ we have $W_\tau^0\left(C(\lambda)C\left(\mu\right)\right)=W_{\lambda\mu\tau}^0$. Then putting $T=D_{1/\tau}C(\lambda)C\left(\mu\right)$ we get

$$w^0(T)=W_\tau^0\left(C(\lambda)C\left(\mu\right)\right)=W_{\lambda\mu\tau}^0. \tag{10}$$

Then $\widetilde{l}=T^{-1}e=\Delta\left(\mu\right)\Delta(\lambda)D_\tau e$ for each n with

$$l_n=\left[\Delta\left(\mu\right)\Delta(\lambda)D_\tau e\right]_n=\lambda_n\mu_n\tau_n+\left(\mu_{n-1}+\mu_n\right)\lambda_{n-1}\tau_{n-1}-\lambda_{n-2}\mu_{n-2}\tau_{n-2} \tag{11}$$

Using (10) and (11) we see that condition (9) is equivalent $X-L\widetilde{l}\in w^0(T)$. We conclude by Lemma 5.1 that $x_k\to L(S(T))$. This completes the proof of (i).

(ii) By Proposition 4.3 (vi) and Remark 4.4, since τ, $\tau/\lambda\in\Gamma$ we have $W_\tau^0\left(\Delta(\lambda)\Delta\left(\mu\right)\right)=W_{\tau/\lambda\mu}^0$. Then putting $T'=D_{1/\tau}\Delta(\lambda)\Delta\left(\mu\right)$ we get

$$w^0\left(T'\right)=W_\tau^0\left(\Delta(\lambda)\Delta\left(\mu\right)\right)=W_{\tau/\lambda\mu}^0. \tag{12}$$

Since $\widetilde{l'}=T'^{-1}e=C\left(\mu\right)C(\lambda)D_\tau e$ we have

$$l'_n=\left[C\left(\mu\right)C(\lambda)D_\tau e\right]_n=\frac{1}{\mu_n}\sum_{i=1}^n\frac{1}{\lambda_i}\left(\sum_{j=1}^i\tau_j\right)\text{ for all }n.$$

By Lemma 5.1 we conclude $x_k\to L\left(S\left(D_{1/\tau}\Delta(\lambda)\Delta\left(\mu\right)\right)\right)$ for all X with

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^n\left|x_k-Ll'_k\right|\frac{\lambda_k\mu_k}{\tau_k}=0$$

This shows (ii).

(iii) Again by Proposition 4.3 (vii) and Remark 4.4, since τ, $\tau\mu/\lambda\in\Gamma$ we have $W_\tau^0\left(\Delta(\lambda)C\left(\mu\right)\right)=W_{\tau\mu/\lambda}^0$. Then putting $T''=D_{1/\tau}\Delta(\lambda)C\left(\mu\right)$ we get

$$w^0\left(T''\right)=W_\tau^0\left(\Delta(\lambda)C\left(\mu\right)\right)=W_{\tau\mu/\lambda}^0. \tag{13}$$

Writing $\widetilde{l''}=T''^{-1}e=\Delta\left(\mu\right)C(\lambda)D_\tau e$ we successively get

$$D_\tau e=(\tau_n)_{n\ge1},C(\lambda)D_\tau e=\left(\left(\sum_{i=1}^n\tau_i\right)/\lambda_n\right)_{n\ge1}$$

and

$$\Delta\left(\mu\right)C(\lambda)D_\tau e=\left(\frac{\mu_n}{\lambda_n}\sum_{i=1}^n\tau_i-\frac{\mu_{n-1}}{\lambda_{n-1}}\sum_{i=1}^{n-1}\tau_i\right)_{n\ge1}.$$

So for each n we have

$$l''_n = [\Delta(\mu) C(\lambda) D_\tau e]_n = \left(\frac{\mu_n}{\lambda_n} - \frac{\mu_{n-1}}{\lambda_{n-1}}\right) \sum_{i=1}^{n-1} \tau_i + \frac{\mu_n}{\lambda_n} x_k.$$

We conclude that for every X with

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k - Ll''_k| \frac{\lambda_k}{\mu_k \tau_k} = 0$$

then $x_k \to L(S(T''))$. Finally we easily get

$$\begin{aligned} [T''X]_n &= \frac{1}{\tau_n}\left(\frac{\lambda_n}{\mu_n}\sum_{i=1}^{n} x_i - \frac{\lambda_{n-1}}{\mu_{n-1}}\sum_{i=1}^{n-1} x_i\right) \\ &= \frac{1}{\tau_n}\left[\left(\frac{\lambda_n}{\mu_n} - \frac{\lambda_{n-1}}{\mu_{n-1}}\right)\sum_{i=1}^{n-1} x_i + \frac{\lambda_n}{\mu_n} x_n\right]. \end{aligned}$$

This shows (iii). □

We are led to illustrate the previous results with some examples where we must have in mind that the condition $x_k/\tau_k \to 0$ $(k \to \infty)$ implies $X \in W_\tau^0$.

Example 5.3. *The condition*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left| \frac{x_k}{2^k} - \frac{7}{4} L \right| = 0$$

for given $L \in \mathbb{C}$ implies $x_k \to L(S(D_{(n/2^n)_n} C_1 \Sigma))$, that is, for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{1}{2^k} \sum_{i=1}^{k} \sum_{j=1}^{i} x_j - L \right| \ge \varepsilon \right\} \right| = 0. \tag{14}$$

Indeed it is enough to apply Theorem 5.2 (i) with $\lambda_k = k$, $\tau_k = 2^k/k$ and $\mu_k = 1$ for all k. Note that if $x_k/2^k \to 7L/4$ $(k \to \infty)$ then $x_k \to L(S(D_{(n/2^n)_n} C_1 \Sigma))$.

We can also state the next application.

Example 5.4. *If $\lim_{n\to\infty}(1/n)\sum_{k=1}^{n} |x_k|/k2^k = 0$ then $x_k \to L(S(D_{(2^{-n})_n} \Delta C_1))$, that is for each $\varepsilon > 0$*

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{1}{2^k} \left(\frac{1}{k} - \frac{1}{k-1} \right) \sum_{i=1}^{k-1} x_i + \frac{1}{k} x_k \right| \ge \varepsilon \right\} \right| = 0.$$

This result is a direct consequence of Theorem 5.2 (iii) with $\lambda_k = 1$, $\tau_k = 2^k$ and $\mu_k = k$ for all k. Again note that we have $x_k \to L(S(D_{(2^{-n})_n} \Delta C_1))$ if $x_k/k2^k \to 0$ $(k \to \infty)$.

In the following we will use the previous Proposition 4.3 and the expressions of $W_\tau^0\left(C^+(\lambda)\Delta(\mu)\right) = \left[C^+,\Delta\right]_{W_\tau^0}$, $W_\tau^0\left(C^+(\lambda)C(\mu)\right) = \left[C^+,C\right]_{W_\tau^0}$, $W_\tau^0\left(C^+(\lambda)C^+(\mu)\right) = \left[C^+,C^+\right]_{W_\tau^0}$ and $W_\tau^0\left(\Delta(\lambda)C^+(\mu)\right) = \left[\Delta,C^+\right]_{W_\tau^0}$. We now require a lemma which is a direct consequence of Lemma 5.1.

Lemma 5.5. *Let A be an infinite matrix. If $X \in w^0(A)$ then*

$$x_k \to 0(S(A)).$$

we deduce the next results.

Theorem 5.6. *(i) Let $\tau \in \Gamma^+$ and $\lambda\tau \in \Gamma$. If*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{|x_k|}{\lambda_k \tau_k} \mu_k = 0 \tag{15}$$

then $x_k \to 0\left(S\left(D_{1/\tau} C^+(\lambda)\Delta(\mu)\right)\right)$, that is for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{1}{\tau_k} \sum_{i=k}^{\infty} \frac{\mu_i x_i - \mu_{i-1} x_{i-1}}{\lambda_i} \right| \ge \varepsilon \right\} \right| = 0. \tag{16}$$

(ii) Let $\tau \in \Gamma^+$ and $\lambda\mu\tau \in \Gamma$. If

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{|x_k|}{\lambda_k \mu_k \tau_k} = 0 \tag{17}$$

then $x_k \to 0\left(S\left(D_{1/\tau} C^+(\lambda) C(\mu)\right)\right)$, that is for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{1}{\tau_k} \sum_{i=k}^{\infty} \frac{1}{\lambda_i} \left(\frac{1}{\mu_i} \sum_{j=1}^{i} x_j \right) \right| \ge \varepsilon \right\} \right| = 0. \tag{18}$$

(iii) Let τ, $\lambda\tau \in \Gamma^+$. If

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{|x_k|}{\lambda_k \mu_k \tau_k} = 0 \tag{19}$$

then $x_k \to 0\left(S\left(D_{1/\tau} C^+(\lambda) C^+(\mu)\right)\right)$, that is for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{1}{\tau_k} \sum_{i=k}^{\infty} \frac{1}{\lambda_i} \left(\sum_{j=i}^{\infty} \frac{x_j}{\mu_j} \right) \right| \ge \varepsilon \right\} \right| = 0. \tag{20}$$

(iv) Let τ, $\tau/\lambda \in \Gamma^+$. If

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\lambda_k |x_k|}{\mu_k \tau_k} = 0$$

then $x_k \to 0\left(S\left(D_{1/\tau} \Delta(\lambda) C^+(\mu)\right)\right)$, that is for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \frac{1}{\tau_k} \left| (\lambda_k - \lambda_{k-1}) \sum_{i=k-1}^{\infty} \frac{x_i}{\mu_i} + \frac{\lambda_k}{\mu_k} x_k \right| \ge \varepsilon \right\} \right| = 0. \tag{21}$$

Proof. (i) Condition (15) implies $X \in W^0_{\lambda\tau/\mu}$ and by Proposition 4.3 and Remark 4.4 since $\tau \in \Gamma^+$ and $\lambda\tau \in \Gamma$ we have $W^0_{\lambda\tau/\mu} = W^0_\tau\left(C^+(\lambda)\Delta(\mu)\right)$ and $X \in W^0_\tau\left(C^+(\lambda)\Delta(\mu)\right)$. Now it can be easily seen that

$$\left[D_{1/\tau}C^+(\lambda)\Delta(\mu)\right]_n = \frac{1}{\tau_n}\sum_{i=n}^{\infty}\frac{\mu_i x_i - \mu_{i-1}x_{i-1}}{\lambda_i},$$

so by Lemma 5.5 with $A = D_{1/\tau}C^+(\lambda)\Delta(\mu)$ we conclude $x_k \to 0\left(S\left(D_{1/\tau}C^+(\lambda)\Delta(\mu)\right)\right)$. This shows (i).

(ii) Here condition (17) means $X \in W^0_{\lambda\mu\tau}$ and by Proposition 4.3 and Remark 4.4 since $\tau \in \Gamma^+$ and $\lambda\mu\tau \in \Gamma$ we have $W^0_{\lambda\mu\tau} = W^0_\tau\left(C^+(\lambda)C(\mu)\right)$ and $X \in W^0_\tau\left(C^+(\lambda)C(\mu)\right)$. Now since

$$\left[D_{1/\tau}C^+(\lambda)C(\mu)\right]_n = \frac{1}{\tau_n}\sum_{i=n}^{\infty}\frac{1}{\lambda_i}\left(\frac{1}{\mu_i}\sum_{j=1}^{i}x_j\right),$$

by Lemma 5.5 where $A' = D_{1/\tau}C^+(\lambda)C(\mu)$, we conclude $x_k \to 0\left(S\left(D_{1/\tau}C^+(\lambda)C(\mu)\right)\right)$. So we have shown (ii).

(iii) can be obtained reasoning as above with $A'' = D_{1/\tau}C^+(\lambda)C^+(\mu)$ and so $x_k \to 0$ $\left(S\left(D_{1/\tau}C^+(\lambda)C^+(\mu)\right)\right)$.

(iv) can also be obtained similarly. It is enough to put $A''' = D_{1/\tau}\Delta(\lambda)C^+(\mu)$. An elementary calculation gives

$$\left[A'''X\right]_k = \frac{1}{\tau_k}\left[(\lambda_k - \lambda_{k-1})\sum_{i=k-1}^{\infty}\frac{x_i}{\mu_i} + \frac{\lambda_k}{\mu_k}x_k\right]$$

and we conclude that $x_k \to 0\left(S\left(D_{1/\tau}\Delta(\lambda)C^+(\mu)\right)\right)$, that is (21). □

We can state the next example

Example 5.7. *for each $\varepsilon > 0$ and for every $X \in W^0_{3/2}$ we have $x_k \to 0\left(S\left(D_{(2^n)_n}\Sigma^+C((3^n)_n)\right)\right)$, that is*

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \le n : \left|2^k\sum_{i=1}^{\infty}\frac{1}{3^i}\left(\sum_{j=1}^{i}x_j\right)\right| \ge \varepsilon\right\}\right| = 0. \tag{22}$$

It is enough to apply Theorem 5.6 (ii) with $\tau_k = 2^{-k}$, $\mu_k = 3^k$ and $\lambda_k = 1$ for all k. So if $(2/3)^k x_k \to 0$ $(k \to \infty)$ then (22) holds.

We also have the next example.

Example 5.8. *From Theorem 5.6 (iii) with $\lambda_k = \mu_k = k$ and $\tau_k = 2^{-k}$ the condition*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}2^k\frac{|x_k|}{k^2} = 0$$

implies $x_k \to 0\left(S\left(D_{(2^n)_n} C_1 C_1^+\right)\right)$ *that is, for each* $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| 2^k \sum_{i=k}^{\infty} \frac{1}{i} \left(\sum_{j=i}^{\infty} \frac{x_j}{j} \right) \right| \ge \varepsilon \right\} \right| = 0. \tag{23}$$

As in the previous cases (23) holds if $2^k x_k / k^2 \to 0$ $(k \to \infty)$.

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Some Generalizations of Mulit-Valued Version of Schauder's Fixed Point Theorem with Applications

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ABSTRACT

In this article, a generalization of a Kakutani-Fan fixed point theorem for multi-valued mappings in Banach spaces is proved under weaker upper semi-continuity condition and it is further applied to derive a generalized version of Krasnoselskii's fixed point theorem and some nonlinear alternatives of Leray-Schauder type for multi-valued closed mappings in Banach spaces.

RESUMEN

En este artículo probamos una generalización para el teorema del punto fijo de Kakutani-Fan para aplicaciones multi-valuadas en espacios de Banach, bajo condición de semi-continuidad superior debil. Este resultado es aplicado para obtener una versión generalizada del teorema del punto fijo Krasnoselskii y algunas alternativas de tipo Leray-Schauder para aplicaciones multi-valuadas cerradas en espacios de Banach.

Key words and phrases: *Multi-valued mappings, fixed point theorem, nonlinear alternative.*

Math. Subj. Class.: *47H10.*

1 Introduction

Throughout this paper, unless otherwise mentioned, let E be a Banach space and let $\mathscr{P}(E)$ denote the class of all subsets of E. Denote

$$\mathscr{P}_p(E) = \{A \subset E \mid A \text{ is non-empty and has a property } p\}.$$

Here, p may be the property p = closed (in short cl), or p = compact (in short cp), or p = convex (in short cv), or p = bounded (in short bd) etc. Thus, $\mathscr{P}_{bd}(E)$,$\mathscr{P}_{cl}(E)$,$\mathscr{P}_{cv}(E)$, $\mathscr{P}_{cp}(E)$, $\mathscr{P}_{cl,bd}(E)$, $\mathscr{P}_{cp,cv}(E)$ denote the classes of all bounded, closed, convex, compact, closed-bounded and compact-convex subsets of E respectively. Similarly, $\mathscr{P}_{cl,cv,bd}(E)$ and $\mathscr{P}_{rcp}(E)$ denote respectively the classes of closed, convex and bounded and relatively compact subsets of E.

A correspondence $Q : E \to \mathscr{P}_p(E)$ is called a multi-valued operator or multi-valued mapping on E into E. A point $u \in E$ is called a fixed point of Q if $u \in Qu$. For the sake of convenience, we denote $Q(A) = \bigcup_{x\in A} Tx$ for all subsets A of E.

Let E_1 and E_2 be two Banach spaces and let $Q : E_1 \to \mathscr{P}_p(E_2)$ be a multi-valued operator. Then for any non-empty subset A of E_2, define

$$Q^+(A) = \{x \in E_1 \mid Tx \subset A\},$$

$$Q^-(A) = \{x \in E_1 \mid Tx \cap A \neq \emptyset\},$$

and

$$Q^{-1}(A) = \{x \in E_1 \mid \cup_x Tx = A\}.$$

Definition 1.1. *A multi-valued operator $Q : E_1 \to \mathscr{P}_p(E_2)$ is called upper semi-continuous (resp. lower semi-continuous and continuous) if $Q^+(U)$ (resp. $Q^-(U)$ and $Q^{-1}(U)$) is open set in E_1 for every open subset U of E_2.*

In what follows,we confine ourselves only to the fixed point theory related to upper semi-continuous multi-valued mappings in Banach spaces. The first fixed point theorem in this direction is due to Kakutani-Fan [11] which is as follows.

Theorem 1.1. *Let K be a compact subset of a Banach space E and let $Q : E \to \mathscr{P}_{cp,cv}(E)$ be an upper semi-continuous multi-valued operator. Then Q has a fixed point.*

Note that following are the main three ingredients for the above Theorem 1.1.

(i) The domain space E,

(ii) The domain set K, and

(iii) The nature of the multi-valued operator Q.

Theorem 1.1 has been extended in the literature by generalizing or modifying the above three hypotheses with the same conclusion. In the following discussion, we do not change the hypothesis on the domain space, and thus keep us in the practical applicability of the so obtained fixed point theorem to other areas of mathematics. However, the generalizations of the above Theorem 1.1 with change of domain space may be found in the works of Browder-Fan [11] and Himmelberg [9] etc.

A first generalization of Theorem 1.1 is due to Bohnenblust-Karlin as given in Petruşel [12].

Theorem 1.2 (Bohnenblust-Karlin)**.** *Let X be a closed convex and bounded subset of a Banach algebra E and let $Q : X \to \mathscr{P}_{cp,cv}(X)$ be a upper semi-continuous multi-valued operator with a relatively compact range. Then Q has a fixed point.*

A multi-valued map $Q : X \to \mathscr{P}_{cp}(X)$ is called **compact** if $\overline{Q(X)}$ is a compact subset of X. Q is called **totally bounded** if for any bounded subset A of X, $Q(A) = \bigcup_{x\in A} Qx$ is a totally bounded subset of X. It is clear that every compact multi-valued operator is totally bounded, but the converse may not be true. However, these two notions are equivalent on a bounded subset of X. Finally, Q is called **completely continuous** if it is upper semi-continuous and totally bounded on X.

The upper semi-continuity is further weakened to closed graph operators as follows. If $Q : E_1 \to E_2$ is a multi-valued operator, then the graph $Gr(Q)$ of the operator Q is defined by

$$Gr(Q) = \{(x, y) \in E_1 \times E_2 \mid y \in Tx\}.$$

The graph $Gr(Q)$ of the operator Q is said to be closed if $\{(x_n, y_n)\}$ be a sequence in $Gr(Q)$ such that $(x_n, y_n) \to (x, y)$, then we have that $(x, y) \in Gr(Q)$.

Definition 1.2. *A multi-valued operator $Q : E_1 \to \mathscr{P}_{cl}(E_2)$ is called closed if it has a closed graph in $E_1 \times E_2$.*

The following result concerning the upper semi-continuity of multi-valued mappings in Banach spaces is very much useful in the study of multi-valued analysis. The details appears in Deimling [5].

Lemma 1.1. *A multi-valued operator $Q : E_1 \to \mathscr{P}_{cl}(E_2)$ is upper semi-continuous if and only if it is closed and has compact range.*

Theorem 1.3 (O'Regan [13]). *Let X be a closed convex and bounded subset of a Banach algebra E and let $Q : X \to \mathscr{P}_{cp,cv}(X)$ be a compact and closed multi-valued operator. Then Q has a fixed point.*

The compactness of Q in Theorem 1.3 is further weakened to condensing operators with the help of measure of noncompactness in the Banach space E. The Kuratowskii measure α and the ball or Hausdorff measure β of noncompactness of a bounded set in the Banach space E are the functions $\alpha, \beta : \mathscr{P}_{bd}(E) \to \mathbb{R}^+$ defined by

$$\alpha(A) = \inf\Big\{r > 0 : A \subset \cup_{i=1}^{n} A_i,\ \mathrm{diam}(S_i) \le r\ \forall\, i\Big\}, \tag{1.1}$$

and

$$\beta(A) = \inf\Big\{r > 0 \mid A \subset \bigcup_{i=1}^{n} \mathscr{B}_r(x_i) \text{ for some } x_i \in X\Big\} \tag{1.2}$$

for all $A \in \mathscr{P}_{bd}(E)$, where diam $(A_i) = \sup\{\|x - x\| : x, y \in A_i\}$ and $\mathscr{B}_r(x_i)$ are the open balls centered at x_i of radius r.

Definition 1.3. *A multi-valued operator $Q : E \to \mathscr{P}_{cl,bd}(E)$ is called β-condensing if for all bounded sets A in E, $Q(A)$ is bounded and $\beta(Q(A)) < \beta(A)$ for $\beta(A) > 0$.*

Theorem 1.4. *Let X be a closed convex and bounded subset of a Banach space E and let $Q : X \to \mathscr{P}_{cl,cv}(X)$ be a upper semi-continuous and β-condensing multi-valued operator. Then Q has a fixed point.*

In this article, we generalize Theorem 1.1 by weakening the upper semi-continuity as well as compactness of the multi-valued operator Q in a Banach space E and discuss some of its applications.

2 Fixed Point Theory

A function $d_H : \mathscr{P}_p(E) \times \mathscr{P}_p(E) \to \mathbb{R}^+$ defined by

$$d_H(A,B) = \max\Big\{\sup_{a \in A} D(a,B)\,,\ \sup_{b \in B} D(b,A)\Big\} \tag{2.1}$$

satisfies all the conditions of a metric on $\mathscr{P}_p(E)$ and is called a Hausdorff-Pompeiu metric on E, where $D(a,B) = \inf\{\|a - b\| : b \in B\}$. It is known that the hyperspace $\big(\mathscr{P}_{cl}(E), d_H\big)$ is a complete metric space.

The axiomatic way of defining the measures of noncompactness has been adopted in several papers in the literature. See Akhmerov *et al.* [2], Banas and Goebel [3], and the references given therein. In this paper, we define the measure of noncompactness in a Banach space on the lines of Dhage [6] which is slightly different manner from that given in the above monographs.

Definition 2.1. *A sequence* $\{A_n\}$ *of non-empty sets in* $\mathscr{P}_p(E)$ *is said to converge to a set* A*, called the limiting set, if* $d_H(A_n, A) \to 0$ *as* $n \to \infty$.

Definition 2.2. *A mapping* $\mu : \mathscr{P}_p(E) \to \mathbb{R}^+$ *is continuous if for any sequence* $\{A_n\}$ *in* $\mathscr{P}_p(E)$*, we have that*

$$d_H(A_n, A) \to 0 \quad \textit{implies} \quad |\mu(A_n) - \mu(A)| \to 0 \quad \textit{as} \quad n \to \infty.$$

Definition 2.3. A mapping $\mu : \mathscr{P}_p(E) \to \mathbb{R}^+$ is called nondecreasing if $A, B \in \mathscr{P}_p(E)$ are any two sets with $A \subseteq B$, then $\mu(A) \le \mu(B)$, where $\subseteq$ is a order relation by inclusion in $\mathscr{P}_p(E)$.

Now we are equipped with the necessary details to define the measures of noncompactness of a bounded subset of the Banach space E.

Definition 2.4. *A function* $\mu : \mathscr{P}_{cl,bd}(E) \to \mathbb{R}^+$ *is called a measure of noncompactness if it satisfies*

(μ_1) $\emptyset \neq \mu^{-1}(0) \subset \mathscr{P}_{rcp}(E)$,

(μ_2) $\mu(\overline{A}) = \mu(A)$, *where* $\overline{A}$ *denotes the closure of* A,

(μ_3) $\mu(Conv\, A) = \mu(A)$, *where* $Conv\, A$ *denotes the convex hull of* A,

(μ_4) μ *is nondecreasing, and*

(μ_5) *if* $\{A_n\}$ *is a decreasing sequence of sets in* $\mathscr{P}_{cl,bd}(E)$ *satisfying* $\lim\limits_{n\to\infty} \mu(A_n) = 0$*, then the limiting set* $A_\infty = \lim\limits_{n\to\infty} \overline{A}_n$ *is non-empty.*

Note that the functions α and β defined by (1.1) and (1.2) satisfy the conditions (μ_1) through (μ_5). Hence α and β are the measures of noncompactness on E. Moreover, they are locally Lipschitz and hence are locally continuous on $\mathscr{P}_{cl,bd}(E)$. Some nice properties of α and β have been discussed in Akhmerov *et al.* [2] and Banas and Goebel [3].

We remark that if (μ_4) holds, then $A_\infty \in \mathscr{P}_{rcp}(E)$. To see this, let $\lim_{n\to\infty} \mu(A_n) = 0$. As $A_\infty \subseteq A_n$ for each $n = 0, 1, 2, \ldots$; by the monotonicity of μ, we obtain

$$\mu(A_\infty) \le \lim_{n\to\infty} \overline{A}_n = \lim_{n\to\infty} \mu(A_n) = 0.$$

Hence, by assumption (μ_1), we get A_∞ is nonempty and $A_\infty \in \mathscr{P}_{rcp}(E)$.

A measure μ is called *complete* or *full* if the kernel of μ consists of all possible relatively compact subsets of E. Next, a measure μ is called *sublinear* if it satisfies

(μ_6) $\mu(\lambda A) = |\lambda|\mu(A)$ for $\lambda \in \mathbb{R}$, and

(μ_7) $\mu(A+B) \leq \mu(A)+\mu(B)$ for $A,B \in \mathscr{P}_{cl,bd}(E)$.

There do exist the sublinear measures of noncompactness in Banach spaces E. Indeed, the measures α and β of noncompactness defined by (1.1) and (1.2) are sublinear on E.

Now we prove a fixed point theorem for the mappings in Banach spaces involving the measures of noncompactness. Before going to the main results, we give a useful definition.

Definition 2.5. *A multi-valued mapping $Q : E \to \mathscr{P}_{cl,bd}(E)$ is called $\mathscr{D}$-***set-Lipschitz** *if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(Q(A)) \leq \psi(\mu(A))$ for all $A \in \mathscr{P}_{cl,bd}(E)$ with $Q(A) \in \mathscr{P}_{cl,bd}(E)$, where $\psi(0) = 0$. Sometimes we call the function ψ to be a $\mathscr{D}$-***function** *of Q on E. In the special case, when $\psi(r) = kr, k > 0$, Q is called a k-***set-Lipschitz** *mapping and if $k < 1$, then Q is called a k-***set-contraction** *on E. Further, if $\psi(r) < r$ for $r > 0$, then Q is called a* **nonlinear $\mathscr{D}$-set-contraction** *on E.*

We need the following lemma in the sequel.

Lemma 2.1 (Dhage [8])**.** *If ψ is a $\mathscr{D}$-function with $\psi(r) < r$ for $r > 0$, then $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t \in [0,\infty)$.*

Theorem 2.1. *Let X be a non-empty, closed, convex and bounded subset of a Banach space E and let $Q : X \to \mathscr{P}_{cl,cv}(X)$ be a closed and nonlinear $\mathscr{D}$-set-contraction. Then Q has a fixed point.*

Proof. Define a sequence $\{X_n\}$ of sets in $\mathscr{P}_{cl,bd}(E)$ by

$$X_0 = X,\ X_{n+1} = \overline{\mathrm{Conv}\, Q(X_n)}, n = 0,1,...$$

Clearly,

$$X_0 \supset X_1 \supset \cdots \supset X_n \supset X_{n+1} \cdots .$$

and so, $\{X_n\}$ is a decreasing sequence of subsets of E. Since

$$\mu(X_{n+1}) = \mu\left(\overline{\mathrm{Conv}\, Q(X_n)}\right) = \mu(Q(X_n)) \leq \psi(\mu(X_n))$$

for all $n = 0,1,2,\ldots$, we have

$$\mu(X_{n+1}) \leq \psi^n(\mu(X_0)).$$

Therefore

$$\limsup_{n\to\infty} \mu(X_{n+1}) \leq \limsup_{n\to\infty} \psi^n(\mu(X_0)) = 0.$$

From the monotonicity of μ it follows that $\lim\limits_{n\to\infty} X_n = X_\infty$ is a compact subset of E. As $X_{n+1} \subset X_n$ and $Q : X_n \to X_n$ for all $n = 0, 2, \ldots$, we have

$$X_\infty = \lim_{n\to\infty} X_n = \bigcap_{n=1}^{\infty} X_n \neq \emptyset$$

is a convex subset of E and $Q : X_\infty \to \mathscr{P}_{cp,cv}(X_\infty)$ which is upper-semi-continuous in view of Lemma 1.1. Now the desired conclusion follows by an application of Theorem 1.1 to the operator Q on X_∞. This completes the proof. □

Remark 2.1. The fixed point set Fix(Q) of the multi-valued operator Q in above Theorem 2.1 is compact. In fact if $\mu(\text{Fix}(Q)) > 0$, then from nonlinear $\mathscr{D}$-set-contraction it follows that $\mu(\text{Fix}(Q)) = \mu(Q(\text{Fix}(Q))) \leq \psi(\mu(\text{Fix}(Q)))$ which is a contradiction since $\psi(r) < r$ for $r > 0$.

As a consequence of Theorem 2.1 we obtain a fixed point theorem of Darbo [3] type for linear set-contractions,

Corollary 2.1. *Let X be a closed, convex and bounded subset of a Banach space E and let $Q : X \to \mathscr{P}_{cl,cv}(X)$ be a closed and k-set-contraction. Then Q has a fixed point.*

Before stating the generalization of Theorem 2.1of Sadovskii [14] type, we give a useful definition.

Definition 2.6. *A multi-valued mapping $Q : E \to \mathscr{P}(E)$ is called μ-condensing if for any bounded subset A of E, $Q(A)$ is bounded and $\mu(Q(A)) < \mu(A)$ for $\mu(A) > 0$.*

Theorem 2.2. *Let X be a nonempty, closed, convex and bounded subset of a Banach space E and let $Q : X \to \mathscr{P}_{cl,cv}(X)$ be a closed and μ-condensing mapping. Then Q has a fixed point.*

Thus, we have a one way implication that *Sadovskii's type theorem* ⇒ *Theorem 2.1* ⇒ *Darbo's type theorem*. However, it is rather difficult to find the operators satisfying the conditions on Banach spaces given in Sadovskii's type fixed point theorem.

3 Applications

3.1 Hybrid fixed point theory

First, we derive a Krasnoselskii type fixed point theorem for the sum of two multi-valued mappings in Banach spaces. Before stating this result, we need the following definition.

Definition 3.1. *A multi-valued mapping $Q : E \to \mathscr{P}_{cl,cv}(E)$ is said to be nonlinear $\mathscr{D}$-contraction if there is a $\mathscr{D}$-function ψ such that*

$$d_H(Qx, Qy) \leq \psi(d(x, y))$$

for all $x,y \in E$, where $\psi(r) < r$.

Theorem 3.1. *Let X be a closed, convex and bounded subset of a Banach space E and let μ be a sublinear measure of noncompactness in it. Let $S,T : X \to \mathscr{P}_{cl,cv}(E)$ be two operators such that*

(a) S is closed and nonlinear $\mathscr{D}$-set-contraction,

(b) T is compact and closed, and

(c) $Sx + Tx \subset X$ for all $x \in X$.

Then the operator inclusion $x \in Sx + Tx$ has a solution and the set of all solutions is compact in E.

Proof. Define a mapping $Q : X \to \mathscr{P}_{cl,cv}(X)$ by

$$Qx = Sx + Tx. \tag{3.1}$$

We show that Q satisfies all the conditions of Theorem 2.1. Obviously, by hypothesis (c), Q defines a mapping $Q : X \to \mathscr{P}_{cl,cv}(X)$. Since S and T are closed, the sum $Q = S + T$ is also closed on X. As hypothesis (a) holds, there is a $\mathscr{D}$-function ψ such that $\psi(r) < r$ for $r > 0$. Further, let A be a non-empty subset of X.Then A bounded and

$$Q(A) \subseteq X \quad \text{and} \quad Q(A) \subseteq S(A) + T(A),$$

and hence $Q(A)$ is bounded. By sublinearity of μ, we obtain

$$\mu(Q(A)) \le \mu(S(A)) + \mu(T(A)) \le \psi(\mu(A))$$

where, $\psi(r) < r$ for $r > 0$. This shows that Q is a nonlinear $\mathscr{D}$-set-contraction on X into itself. Now an application of Theorem 2.1 yields that Q has a fixed point. Consequently, the operator equation $x \in Sx + Tx$ has a solution. This completes the proof. □

The following lemma is obvious and the proof may be found in the monographs of Deimling [5] and Hu and Papageorgiou [10].

Lemma 3.1. *If $Q : E \to \mathscr{P}_{cp,cv}(E)$ is nonlinear contraction. Then for any bounded subset A of E with $Q(A)$ bounded, we have $\beta(Q(A)) \le \psi(\beta(A))$, where β is a ball measure of noncompactness in E defined by (1.2).*

Theorem 3.2. *Let X be a closed, convex and bounded subset of a Banach space E and let $S,T : X \to \mathscr{P}_{cp,cv}(E)$ be two multi-valued operators such that* *(a) S is a nonlinear $\mathscr{D}$-contraction,*

(b) T is compact and closed, and

(c) $Sx + Tx \subset X$ for all $x \in X$.

Then the operator inclusion $x \in Sx + Tx$ has a solution and the set of all solutions is compact in E.

Proof. Since S is nonlinear $\mathscr{D}$-contraction, it is closed on X and there is a $\mathscr{D}$-function ψ of S on X with the properties that $\psi(r) < r$ for $r > 0$. Again from Lemma 3.1, it follows that it is a also nonlinear $\mathscr{D}$-set-contraction with respect to the Hausdorff measure of noncompactness β and with a $\mathscr{D}$-function ψ on X. Now the desired conclusion follows by a direct application of Theorem 2.1. □

3.2 Nonlinear alternative

The following nonlinear alternative for multi-valued mappings in Banach spaces is well-known in the literature.

Theorem 3.3 (O'Regan [13]). *Let U be a open bounded subset of a Banach space E with $0 \in U$ and let $Q : \overline{U} \to \mathscr{P}_{cl,cv}(E)$ be a compact and closed multi-valued operator. Then either*

(i) the operator inclusion $x \in Qx$ has a solution in $\overline{U}$, or

(ii) there is an element $u \in \partial U$ such that $\lambda u \in Qu$ for some $\lambda > 1$, where ∂U is the boundary of U in E.

A generalization of above Theorem 3.4 is

Theorem 3.4. *Let U be a open bounded subset of a Banach space E with $0 \in U$ and let $Q : \overline{U} \to \mathscr{P}_{cl,cv}(E)$ be a μ-condensing and closed multi-valued operator. Then either*

(i) the operator inclusion $x \in Qx$ has a solution in $\overline{U}$ and the set of all solutions is compact in E, or

(ii) there is an element $u \in \partial U$ such that $\lambda u \in Qu$ for some $\lambda > 1$, where ∂U is the boundary of U in E.

Proof. The proof is similar to that given for Theorem 3.3 in O'Regan [13](see also Agarwal *et al.* [1]) and now the conclusion follows by an application of Theorem 3.2. □

As a consequence of Theorem 3.4, we obtain

Corollary 3.1. *Let $\mathscr{B}_r(0)$ be a open ball in a Banach space E centered at origin $0 \in E$ of radius r and let $Q : \overline{\mathscr{B}_r}(0) \to \mathscr{P}_{cl,cv}(E)$ be a μ-condensing and closed multi-valued operator. Then either*

(i) the operator inclusion $x \in Qx$ has a solution in $\overline{\mathscr{B}_r}(0)$ and the set of all solutions is compact in E, or

(ii) there is an element $u \in E$ such that $\|u\| = r$ satisfying $\lambda u \in Qu$ for some $\lambda > 1$.

Corollary 3.2. *Let E be a Banach space and let $Q : E \to \mathscr{P}_{cl,cv}(E)$ be a μ-condensing and closed multi-valued operator. Then, either*

(i) the operator inclusion $x \in Qx$ has a solution and the set of all solutions is compact in E, or

(ii) the set $\mathscr{E} = \{u \in E \mid \lambda u \in Qu\}$ is in unbounded for some $\lambda > 1$.

The above Corollary 3.1 includes the following fixed point result due to Martelli [10] which has been used by several authors in the literature for proving the existence theorems for differential and integral inclusions.

Corollary 3.3. *Let E be a Banach space and let $Q : E \to \mathscr{P}_{cl,cv}(E)$ be a upper semi-continuous and α-condensing(or β-condensing) multi-valued operator. Then, either*

(i) the operator inclusion $x \in Qx$ has a solution in X, or

(ii) the set $\mathscr{E} = \{u \in E \mid \lambda u \in Qu\}$ is in unbounded for some $\lambda > 1$.

Similarly, we can apply Theorem 3.4 to obtain the following nonlinear alternatives for sum of the two multi-valued operators in Banach spaces.

Theorem 3.5. *Let U be a open bounded subset of a Banach space E with $0 \in U$ and let $S, T : \overline{U} \to \mathscr{P}_{cl,cv}(E)$ be two multi-valued operators such that*

(a) S is closed and nonlinear $\mathscr{D}$-set-contraction, and

(b) T is compact and closed.

Then, either

(i) the operator inclusion $x \in Sx + Tx$ has a solution in $\overline{U}$ and the set of all solutions is compact in E, or

(ii) there is an element $u \in \partial U$ such that $\lambda u \in Su + Tu$ for some $\lambda > 1$, where ∂U is the boundary of U in E.

Theorem 3.6. *Let U be a open bounded subset of a Banach space E with $0 \in U$ and let $S,T : \overline{U} \to \mathscr{P}_{cp,cv}(E)$ be two multi-valued operators such that*

(a) S is nonlinear $\mathscr{D}$-contraction, and

(b) T is compact and closed.

Then, either

(i) the operator inclusion $x \in Sx + Tx$ has a solution in $\overline{U}$ and the set of all solutions is compact in E, or

(ii) there is an element $u \in \partial U$ such that $\lambda u \in Su + Tu$ for some $\lambda > 1$, where ∂U is the boundary of U in E.

Corollary 3.4. *Let $\mathscr{B}_r(0)$ be a open ball in a Banach space E centered at origin $0 \in E$ of radius r and let $S,T : \overline{\mathscr{B}_r}(0) \to \mathscr{P}_{cp,cv}(E)$ be two multi-valued operators such that*

(a) S is nonlinear $\mathscr{D}$-contraction, and

(b) T is compact and closed.

Then, either

(i) the operator inclusion $x \in Sx + Tx$ has a solution in $\overline{\mathscr{B}_r}(0)$ and the set of all solutions is compact in E, or

(ii) there is an element $u \in E$ such that $\|u\| = r$ satisfying $\lambda u \in Su + Tu$ for some $\lambda > 1$.

Corollary 3.5. *Let E be a Banach space E and let $S,T : E \to \mathscr{P}_{cp,cv}(E)$ be two multi-valued operators such that*

(a) S is nonlinear $\mathscr{D}$-contraction, and

(b) T is compact and closed.

Then, either

(i) the operator inclusion $x \in Sx + Tx$ has a solution and the set of all solutions is compact in E, or

(ii) the set $\mathscr{E} = \{u \in E \mid \lambda u \in Su + Tu\}$ *is in unbounded for some* $\lambda > 1$.

Remark 3.1. Note that our Theorem 3.6 and Corollary 3.4 improve the hybrid fixed point theorems for multi-valued mappings proved in Dhage [6, 7] under weaker upper semi-continuity conditions.

4 The Conclusion

Finally, while concluding, we remark that the multi-valued fixed point theorems of this paper have some nice applications to differential and integral inclusions for proving the existence as well as some characterizations of solutions such as global and local asymptotic attractivity of solutions on bounded and unbounded intervals of real line. The investigations of these and other similar problems form the scope for further research work in the theory of differential and integral inclusions under weaker upper semi-continuity conditions. Some of the results in this direction will be reported elsewhere.

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Existence of Periodic Solutions for a Class of Second-Order Neutral Differential Equations with Multiple Deviating Arguments[1]

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ABSTRACT

Using Kranoselskii fixed point theorem and Mawhin's continuation theorem we establish the existence of periodic solutions for a second order neutral differential equation with multiple deviating arguments.

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RESUMEN

Usando el teorema del punto fijo de Kranoselskii y el teorema de continuación de Mawhin establecemos la existencia de soluciones periódicas de una ecuación diferencial neutral de segundo orden con argumento de desviación multiple.

Key words and phrases: *Periodic solution, Multiple deviating arguments, Neutral differential equation, Kranoselskii fixed point theorem, Mawhin's continuation theorem.*

Math. Subj. Class.: *34K15; 34C25.*

1 Introduction

In this paper, we discuss the second-order neutral differential equation with multiple deviating arguments of the form

$$x^{''}(t)+cx^{''}(t-\tau)+a(t)x(t)+g(t,x(t-\tau_1(t)),x(t-\tau_2(t))\cdots,x(t-\tau_n(t)))=p(t), \tag{1.1}$$

where $|c|<1$, τ is a constant, $\tau_i(t)(i=1,2,\cdots,n)$, $a(t)$ and $p(t)$ are real continuous functions defined on $\mathbf{R}$ with positive period T and $g(t,x_1,x_2,\cdots,x_n)\in C(\mathbf{R}\times\mathbf{R}\times\mathbf{R}\times\cdots\times\mathbf{R},\mathbf{R})$ and is $T-$periodic in t.

Periodic solutions for differential equations were studied in [2-4, 6-10, 12, 15] and we note that most of the results in the literatue concern delay problems. There are only a few papers[1, 5, 11, 13, 14] which discuss neutral problems.

For the sake of completeness, we first state Kranoselskii fixed point theorem and Mawhin's continuation theorem [3].

Theorem A (Kranoselskii). Suppose that Ω is a Banach space and X is a bounded, convex and closed subset of Ω. Let $U,S:X\to\Omega$ satisfy the following conditions:

(1) $Ux+Sy\in X$ for any $x,y\in X$;

(2) U is a contraction mapping;

(3) S is completely continuous.

Then $U+S$ has a fixed point in X.

Let X and Y be two Banach space and $L:DomL\subset X\longrightarrow Y$ is a linear mapping and $N:X\longrightarrow Y$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dimKerL=codimImL<+\infty$, and ImL is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P:X\longrightarrow X$ and $Q:Y\longrightarrow Y$ such that

$ImP = KerL$ and $ImL = KerQ = Im(I-Q)$. It follows that $L|_{DomL\cap KerP} : (I-P)X \longrightarrow ImL$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called $L-$compact on Ω if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N(\overline{\Omega})$ is compact. Since ImQ is isomorphic to $KerL$, there exists an isomorphism $J : ImQ \longrightarrow KerL$.

Theorem B (Mawhin's continuation theorem[3]). Let L be a Fredholm mapping of index zero, and let N be $L-$compact on $\overline{\Omega}$. Suppose

(1) for each $\lambda \in (0,1)$ and $x \in \partial\Omega, Lx \neq \lambda Nx$ and

(2) for each $x \in \partial\Omega \cap Ker(L), QNx \neq 0$ and $deg(QN, \Omega \cap Ker(L), 0) \neq 0$.

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap D(L)$.

2 Main Results

Now we make the following assumption on $a(t)$:

(H_1) $\quad (\frac{\pi}{T})^2 > M = \max_{t\in[0,T]} a(t) \geq a(t) \geq m = \min_{t\in[0,T]} a(t) > 0.$

Our main results are the following theorems.

Theorem 2.1 Suppose (H_1) holds and also assume there exists a constant $K_1 > 0$ such that (H_2)

$$\|g\|_0 \leq m - 3|c|M - \|p\|_0,$$

where $\|g\|_0 = \max_{\{t\in[0,T],|x_1|\leq K_1,\cdots,|x_n|\leq K_1\}} |g(t, x_1, x_2, \cdots, x_n)|$ and $\|p\|_0 = \max_{t\in[0,T]} |p(t)|$.
Then Eq.(1.1) possesses a nontrivial $T-$periodic solution.

Theorem 2.2 Suppose (H_1) holds and also assume
(H_3)

$$|g(t, x_1, x_2, \cdots, x_n)| \leq \gamma \textstyle\sum_{i=1}^{n} |x_i|.$$

Then Eq.(1.1) has at least one $T-$periodic solution as $0 < \gamma < \frac{1}{n}[(1-|c|)m - |c|M]$.

In order to prove the main theorems we need some preliminaries. Set

$$X := \{x | x \in C^2(\mathbf{R}, \mathbf{R}), x(t+T) = x(t), \forall t \in \mathbf{R}\}$$

and $x^{(0)}(t) = x(t)$ and define the norm on X as follows

$$||x|| = \max_{t\in[0,T]} |x(t)| + \max_{t\in[0,T]} |x^{'}(t)| + \max_{t\in[0,T]} |x^{''}(t)|.$$

Remark 2.3 If $x \in X$, then it follows that $x^{(i)}(0) = x^{(i)}(T)(i = 0,1,2)$.

In order to prove our main results, we need the following Lemma [10].

Lemma 2.4 ([10]). Suppose that M is a positive number and satisfies $0 < M < (\frac{\pi}{T})^2$. Then for any function φ defined in $[0,T]$, the following equation

$$\begin{cases} x''(t) + Mx(t) = \varphi(t), \\ x(0) = x(T), x'(0) = x'(T) \end{cases}$$

has a unique solution

$$x(t) = \int_0^T G(t,s)\varphi(s)ds,$$

where

$$G(t,s)\begin{cases} w(t-s), \quad (k-1)T \le s \le t \le kT \\ w(T+t-s), \quad (k-1)T \le t \le s \le kT(k \in \mathbf{N}), \end{cases}$$

$$w(t) = \frac{\cos\alpha(t-\frac{T}{2})}{2\alpha\sin\frac{\alpha T}{2}}$$

and $\alpha = \sqrt{M}$. Here

$$\max_{t\in[0,T]}\int_0^T |G(t,s)|ds = \frac{1}{M}.$$

Proof of Theorem 2.1: For $\forall x \in X$, define the operators $U : X \longrightarrow X$ and $S : X \longrightarrow X$ respectively by

$$(Ux)(t) = -cx(t-\tau) \tag{2.1}$$

and

$$\begin{aligned} (Sx)(t) \quad &= cx(t-\tau) + \int_0^T G(t,s)[-cx''(s-\tau)(M-a(s))x(s) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds. \end{aligned} \tag{2.2}$$

It is clear that a fixed point of $U+S$ is a $T-$periodic solution of Eq.(1.1).

We are going to demonstrate that U and S satisfy the conditions of Theorem A.

Let $x, y \in X$ and $|x| \le K_1, |y| \le K_1$(here K_1 is as in the statement of Theorem 2.1). Now we prove that $|Ux+Sy| \le K_1$ holds.

First, we have the following equality:

$$\int_0^T G(t,s)x''(s-\tau)ds = M\int_0^T G(t,s)x(s-\tau)ds. \tag{2.3}$$

In fact, we have from Lemma 2.4

$$
\begin{aligned}
\int_0^T G(t,s)x''(s-\tau)ds &= \int_0^t \frac{\cos\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}d[x'(s-\tau)] + \int_t^T \frac{\cos\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}d[x'(s-\tau)] \\
&= \frac{\cos\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}x'(s-\tau)|_0^t - \alpha\int_0^t \frac{\sin\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}d[x(s-\tau)] \\
&+ \frac{\cos\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}x'(s-\tau)|_t^T - \alpha\int_t^T \frac{\sin\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}d[x(s-\tau)] \\
&= -\alpha[\frac{\sin\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}x(s-\tau)|_0^t + \frac{\sin\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}x(s-\tau)|_t^T] \\
&+\alpha^2[\int_0^t \frac{\cos\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}x(s-\tau)ds + \int_t^T \frac{\cos\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}x(s-\tau)ds] \\
&= M\int_0^T G(t,s)x(s-\tau)ds,
\end{aligned}
\tag{2.4}
$$

so (2.3) holds.

From (H_1), (H_2) and (2.1)-(2.3), we have

$$
\begin{aligned}
|(Uy)(t)+(Sx)(t)| &\le |(Uy)(t)|+|(Sx)(t)| \\
&\le 2|c|K_1 + |\int_0^T G(t,s)(M-a(s))x(s) - cx''(s-\tau)+p(s) \\
&-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds| + |c|K_1 \\
&\le 2|c|K_1 + \frac{M-m}{M}K_1 + \frac{\|g\|_0}{M} + |c|M||\int_0^T G(t,s)x(s-\tau)ds| \\
&\le 3|c|K_1 + \frac{M-m}{M}K_1 + \frac{\|g\|_0+\|p\|_0}{M} \\
&\le K_1, \quad x,y\in X,
\end{aligned}
\tag{2.5}
$$

where $\|g\|_0$ and $\|p\|_0$ are given in (H_2).

Set

$$K_2 = \frac{\rho_0[(M-m)K_1+|c|K_3+\|g\|_0+\|p\|_0]}{1-2|c|}, \tag{2.6}$$

where $\rho_0 = \frac{T}{2\sin\frac{T\alpha}{2}}$,

$$K_3 = \frac{MK_1+\|g\|_0+\|p\|_0}{1-|c|} \tag{2.7}$$

and

$$G = \{x\in X : |x(t)|\le K_1, |x'(t)|\le K_2, |x''(t)|\le K_3\}.$$

It is clear that G is a bounded, convex and closed subset of X.

(1) For $\forall x,y\in G$, we will show that

$$|\frac{d}{dt}[(Uy)(t)+(Sx)(t)]| \le K_2 \tag{2.8}$$

and

$$|\frac{d^2[(Uy)(t)+(Sx)(t)]}{dt^2}| \leq K_3. \tag{2.9}$$

From (2.1) we have

$$\frac{d}{dt}[(Ux)(t)] = -cx^{'}(t-\tau) \tag{2.10}$$

and

$$\frac{d^2[(Ux)(t)]}{dt^2} = -cx^{''}(t-\tau). \tag{2.11}$$

Also from Lemma 2.4 and (2.2) we have

$$\begin{aligned} \frac{d}{dt}[(Sx)(t)] &= \int_0^T G_t(t,s)[(M-a(s))x(s) - cx^{''}(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx^{''}(t-\tau), \end{aligned} \tag{2.12}$$

where

$$G_t(t,s)\begin{cases} \widetilde{w}(t-s), & (k-1)T \leq s \leq t \leq kT \\ \widetilde{w}(T+t-s), & (k-1)T \leq t \leq s \leq kT(k \in \mathbf{N}) \end{cases}$$

and

$$\widetilde{w}(t) = \frac{\sin\alpha(t-\frac{T}{2})}{2\sin\frac{\alpha T}{2}},$$

since

$$\begin{aligned} \frac{d}{dt}[(Sx)(t)] &= \{\int_0^T G_t(t,s)[(M-a(s))x(s) - cx^{''}(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx^{''}(t-\tau)\}^{'} \\ &= \{\int_0^t \frac{\cos\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}[(M-a(s))x(s) - cx^{''}(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx^{''}(t-\tau)\}^{'} \\ &+\{\int_t^s \frac{\cos\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}[(M-a(s))x(s) - cx^{''}(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx^{''}(t-\tau)\}^{'} \\ &= \alpha\{\int_0^t \frac{\cos\alpha(t-s-\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}[(M-a(s))x(s) - cx^{''}(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx^{''}(t-\tau)\} \\ &+\alpha\{\int_t^s \frac{\cos\alpha(t-s+\frac{T}{2})}{2\alpha\sin\frac{T\alpha}{2}}[(M-a(s))x(s) - cx^{''}(s-\tau) + p(s) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s))\cdots,x(s-\tau_n(s)))]ds + cx^{''}(t-\tau)\}. \end{aligned}$$

Note

$$\int_0^T |G_t(t,s|ds \le \frac{T}{2\sin\frac{\alpha T}{2}} = \rho_0$$

and

$$\frac{d^2[(Sx)(t)]}{dt^2} = p(t) - a(t)x(t) - g(t, x(t-\tau_1(t)), x(t-\tau_2(t)) \cdots, x(t-\tau_n(t))). \tag{2.13}$$

From (2.6),(2.7) and (2.10)-(2.13), we have

$$\begin{aligned} |\frac{d}{dt}[(Uy)(t) + (Sx)(t)]| &\le |\frac{d}{dt}[(Uy)(t)]| + |\frac{d}{dt}[(Sx)(t)]| \\ &\le 2|c|K_2 + \rho_0[(M-m)K_1 + |c|K_3 + \|g\|_0 + \|p\|_0] \\ &\le K_2 \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} |\frac{d^2[(Uy)(t)+(Sx)(t)]}{dt^2}| &= |(M-a(t))x(t) - cy''(t-\tau) + p(t) \\ &\quad - g(t, x(t-\tau_1(t)), x(t-\tau_2(t)) \cdots, x(t-\tau_n(t)))| \\ &\le (M-m)K_1 + |c|K_3 + \|g\|_0 + \|p\|_0 \\ &\le K_3. \end{aligned} \tag{2.15}$$

From (2.5), (2.14) and (2.15), we have $Ux + Sy \in G$ for $\forall x, y \in G$.

(2) U is a contraction mapping.

Let $x, y \in G$ and we from (2.1) that

$$\begin{aligned} \|Ux - Uy\| &= \max_{t\in[0,T]}|cx(t-\tau) - cy(t-\tau)| + \max_{t\in[0,T]}|cx'(t-\tau) - cy'(t-\tau)| \\ &\quad + \max_{t\in[0,T]}|cx''(t-\tau) - cy''(t-\tau)| \\ &= |c|[\max_{t\in[0,T]}|x(t-\tau) - y(t-\tau)| + \max_{t\in[0,T]}|x'(t-\tau) - y'(t-\tau)| \\ &\quad + \max_{t\in[0,T]}|x''(t-\tau) - y''(t-\tau)|] \\ &= |c|\|x-y\|. \end{aligned}$$

Since $|c| < 1$, U is a contraction mapping.

(3) S is completely continuous.

We can obtain the continuity of S from the continuity of $a(t), p(t)$ and $g(t, x(t-\tau_1(t)), x(t-\tau_2(t)) \cdots, x(t-\tau_n(t)))$ for $t \in [0,T], x \in G$. In fact, suppose that $x_k \in G$ and $\|x_k - s\| \to 0$ as

$k \to +\infty$. Since G is closed convex subset of X, we have $x \in G$. Then

$$\begin{aligned} |Sx_k - Sx| \quad &= c[x_k(t-\tau) - x(t-\tau)] + c[x_k(t-\tau) - x(t-\tau)] \\ &+ \int_0^T G(t,s)\{(M-a(s))(x_k(s) - x(s)) - c[x_k''(s-\tau) - x''(s-\tau)] \\ &-[g(s, x_k(s-\tau_1(s)), x_k(s-\tau_2(s)) \cdots, x_k(s-\tau_n(s))) \\ &-g(s, x(s-\tau_1(s)), x(s-\tau_2(s)) \cdots, x(s-\tau_n(s)))]\}ds. \end{aligned} \tag{2.16}$$

Using the Lebesgue dominated convergence theorem, we have from (2.12), (2.13) and (2.16) that

$$\lim_{k\to+\infty} \|Sx_k - Sx\| = 0.$$

Then S is continuous.

Next, we prove that Sx is relatively compact. It suffices to show that the family of functions $\{Sx : x \in G\}$ is uniformly bounded and equicontinuous on $[0,T]$. From (2.2), (2.12) and(2.13), it is easy to see that $\{Sx : x \in G\}$ is uniformly bounded and equicontinuity. Since S is continuous and is relatively compact, S is completely continuous. By Theorem A (Kranoselskii fixed point theorem), we have a fixed point x of $U+S$. That means that x is a $T-$periodic solution of Eq.(1.1).

In order to prove Theorem 2.2, we need some preliminaries. Set

$$Z := \{x | x \in C^1(\mathbf{R}, \mathbf{R}), x(t+T) = x(t), \forall t \in \mathbf{R}\}$$

and $x^{(0)}(t) = x(t)$ and define the norm on Z as follows

$$||x|| = \max\{\max_{t\in[0,T]}|x(t)|, \max_{t\in[0,T]}|x'(t)|\},$$

and set

$$Y := \{y | y \in C(\mathbf{R}, \mathbf{R}), y(t+T) = y(t), \forall t \in \mathbf{R}\}.$$

We define the norm on Y as follow $||y||_0 = \max_{t\in[0,T]}|y(t)|$. Thus both $(Z, ||\cdot||)$ and $(Y, ||\cdot||_0)$ are Banach spaces.

Remark 2.5 If $x \in Z$, then it follows that $x^{(i)}(0) = x^{(i)}(T) (i = 0,1)$.

Define the operators $L : Z \longrightarrow Y$ and $N : Z \longrightarrow Y$ respectively by

$$Lx(t) = x''(t), \quad t \in \mathbf{R}, \tag{2.17}$$

and

$$\begin{aligned} Nx(t) &= -cx''(t-\tau) - a(t)x(t) + p(t) \\ &\quad -g(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \cdots, x(t-\tau_n(t))), \quad t \in \mathbf{R}. \end{aligned} \tag{2.18}$$

Clearly,

$$KerL = \{x \in Z : x(t) = c \in \mathbf{R}\} \tag{2.19}$$

and

$$ImL = \{y \in Y : \int_0^T y(t)dt = 0\} \tag{2.20}$$

is closed in Y. Thus L is a Fredholm mapping of index zero.

Let us define $P : Z \to Z$ and $Q : Y \to Y/Im(L)$ respectively by

$$Px(t) = \frac{1}{T}\int_0^T x(t)dt = x(0), \quad t \in \mathbf{R}, \tag{2.21}$$

for $x = x(t) \in X$ and

$$Qy(t) = \frac{1}{T}\int_0^T y(t)dt, \quad t \in \mathbf{R} \tag{2.22}$$

for $y = y(t) \in Y$. It is easy to see that $ImP = KerL$ and $ImL = KerQ = Im(I-Q)$. It follows that $L|_{DomL \cap KerP} : (I-P)Z \longrightarrow ImL$ has an inverse which will be denoted by K_P.

Let Ω be an open and bounded subset of Z, we can easily see that $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N(\overline{\Omega})$ is compact. Thus the mapping N is $L-$compact on $\overline{\Omega}$. That is, we have the following result.

Lemma 2.6. Let L, N, P and Q be defined by (2.17), (2.18), (2.21) and (2.22) respectively. Then L is a Fredholm mapping of index zero and N is $L-$compact on $\overline{\Omega}$, where Ω is any open and bounded subset of Z.

In order to prove Theorem 2.2, we need the following Lemma [12].

Lemma 2.7 ([12 and Remark 2.5]). Let $x(t) \in C^{(n)}(\mathbf{R}, \mathbf{R}) \cap C_T$. Then

$$||x^{(i)}||_0 \le \frac{1}{2}\int_0^T |x^{(i+1)}(s)|ds, \quad i = 1, 2, \cdots, n-1,$$

where $n \ge 2$ and $C_T := \{x | x \in C(R, R), x(t+T) = x(t), \forall t \in \mathbf{R}\}$.

Now, we consider the following auxiliary equation

$$\begin{aligned} x''(t) &+ c\lambda x''(t-\tau) + a(t)\lambda x(t) = \lambda p(t) \\ &-\lambda g(t, x(t-\tau_1(t)), x(t-\tau_2(t)), \cdots, x(t-\tau_n(t))), \end{aligned} \tag{2.23}$$

where $0 < \lambda < 1$.

Lemma 2.8. Suppose that conditions of Theorem 2.2 are satisfied. If $x(t)$ is a $T-$periodic solution of Eq.(2.23), then there are positive constants $D_i (i = 0,1)$, which are independent of λ, such that

$$||x^{(i)}||_0 \leq D_i, \quad t \in [0,T], \quad i = 0,1. \tag{2.24}$$

Proof: Suppose that $x(t)$ is a $T-$periodic solution of (2.23). We have from (H_3) and (2.23) that

$$|x^{''}(t)| \leq \max_{t\in[0,T]}|c||x^{''}(t)| + M||x||_0 + ||p||_0 + \gamma n||x||_0. \tag{2.25}$$

From (2.25), we have

$$\max_{t\in[0,T]}|x^{''}(t)| \leq \frac{1}{1-|c|}[(M+\gamma n)||x||_0 + ||p||_0]. \tag{2.26}$$

On the other hand, from Lemma 2.4 and (2.23), we get

$$\begin{aligned} x(t) \quad &= \int_0^T \widetilde{G}(t,s)\lambda[(M-a(s))x(s) + p(s) - cx^{''}(s-\tau) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s)),\cdots,x(s-\tau_n(s))]ds, \end{aligned} \tag{2.27}$$

where

$$\widetilde{G}(t,s)\begin{cases} \widetilde{w}(t-s), \quad (k-1)T \leq s \leq t \leq kT \\ \widetilde{w}(T+t-s), \quad (k-1)T \leq t \leq s \leq kT (k \in \mathbf{N}), \end{cases} \tag{2.28}$$

$$\widetilde{w}(t) = \frac{\cos\alpha_1(t-\frac{T}{2})}{2\alpha_1 \sin\frac{\alpha_1 T}{2}}, \tag{2.29}$$

$\alpha_1 = \sqrt{\lambda M}$ and

$$\max_{t\in[0,T]}\int_0^T |\widetilde{G}(t,s)|ds = \frac{1}{\lambda M}. \tag{2.30}$$

From (H_3), (2.27) and (2.30), we have

$$\begin{aligned} \|x\|_0 \quad &= \max_{t\in[0,T]}|\int_0^T \widetilde{G}(t,s)\lambda[(M-a(s))x(s) + p(s) - cx^{''}(s-\tau) \\ &-g(s,x(s-\tau_1(s)),x(s-\tau_2(s)),\cdots,x(s-\tau_n(s))]ds| \\ &\leq \frac{1}{M}[(M-m)\|x\|_0 + \|p\|_0 + |c|\max_{t\in[0,T]}|x^{''}(t)| + \gamma n\|x\|_0]. \end{aligned} \tag{2.31}$$

From (2.31), we have

$$\|x\|_0 \leq \frac{|c|\max_{t\in[0,T]}|x^{''}(t)| + \|p\|_0}{m-\gamma n}. \tag{2.32}$$

Thus combining (2.26) and (2.32), we see that

$$\max_{t\in[0,T]}|x^{''}(t)| \leq \frac{M+m}{m(1-|c|)-M|c|-\gamma n} = \xi \tag{2.33}$$

and

$$\|x\|_0 \leq \frac{|c|\xi + \|p\|_0}{m-\gamma n} = D_0. \tag{2.34}$$

Finally from Lemma 2.4, (2.33) and (2.34), we get

$$||x^{'}||_0 \leq D_1. \tag{2.35}$$

The proof of Lemma 2.8 is complete.

Proof of Theorem 2.2: Suppose that $x(t)$ is a T-periodic solution of Eq.(2.23). By Lemma 2.8, there exist positive constants $D_i(i = 0,1)$ which are independent of λ such that (2.24) is true. Consider any positive constant $\overline{D} > \max_{0\leq i\leq 1}\{D_i\} + \|p\|_0$.

Set

$$\Omega := \{x \in Z : ||x|| < \overline{D}\}.$$

We know that L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$(see [3]).

Recall

$$Ker(L) = \{x \in Z : x(t) = c \in \mathbf{R}\}$$

and the norm on Z is

$$||x|| = \max\{\max_{t\in[0,T]}|x(t)|, \max_{t\in[0,T]}|x^{'}(t)|\}.$$

Then we have

$$x = \overline{D} \quad \text{or} \quad x = -\overline{D} \quad \text{for} \quad x \in \partial\Omega \cap Ker(L). \tag{2.36}$$

From (H_3) and (2.36), we have(if $\overline{D}$ is chosen large enough)

$$a(t)\overline{D} + g(t,\overline{D},\overline{D},\cdots,\overline{D}) - \|p\|_0 > 0 \quad \text{for} \quad t \in [0,T] \tag{2.37}$$

and

$$x^{'}(t) = 0 \quad \text{and} \quad x^{''}(t) = 0, \quad \text{for} \quad t \in [0,T]. \tag{2.38}$$

Finally from (2.18), (2.22), (2.37) and (2.38), we have

$$\begin{aligned}(QNx) \quad &= \tfrac{1}{T}\int_0^T[-cx^{''}(t-\tau) - a(t)x(t) + p(t)]dt \\ &-g(t,x(t-\tau_1(t)),x(t-\tau_2(t)),\cdots,x(t-\tau_n(t)))]dt \\ &\neq 0, \quad \forall x \in \partial\Omega \cap Ker(L).\end{aligned}$$

Then, for any $x \in KerL \cap \partial\Omega$ and $\eta \in [0,1]$, we have

$$\begin{aligned}xH(x,\eta) \quad &= -\eta x^2 - \tfrac{x}{T}(1-\eta)\int_0^T[cx^{''}(t-\tau) + a(t)x(t) - p(t) \\ &+g(t,x(t-\tau_1(t)),x(t-\tau_2(t)),\cdots,x(t-\tau_n(t)))dt]dt \\ &\neq 0.\end{aligned}$$

Thus

$$\begin{aligned} deg\{QN, \quad & \Omega\cap Ker(L),0\} \\ & = deg\{-\frac{1}{T}\int_0^T[cx''(t-\tau)+a(t)x(t)-p(t) \\ & +g(t,x(t-\tau_1(t)),x(t-\tau_2(t)),\cdots,x(t-\tau_n(t)))]dt,\Omega\cap Ker(L),0\} \\ & = deg\{-x,\Omega\cap Ker(L),0\} \\ & \neq 0. \end{aligned}$$

From Lemma 2.8 for any $x\in\partial\Omega\cap Dom(L)$ and $\lambda\in(0,1)$ we have $Lx\neq\lambda Nx$. By Theorem B (Mawhin's continuation theorem), the equation $Lx = Nx$ has at least a solution in $Dom(L)\cap\overline{\Omega}$, so there exists a T-periodic solution of Eq.(1.1). The proof is complete.

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Special Section in
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by Rémi Vaillancourt

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Preface

The “Second international conference on pseudo-differential operators and related topics” was held in Växjö, Sweden, on 23–27 June 2008, to honour Professsor Luigi Rodino of Turin University, Italy, on the occasion of his 60th birthday. Related topics included differential equations, Fourier integral operators, micro-local analysis, time-frequency analysis, harmonic analysis, generalized functions and several other fields. The conference gathered around 75 participants.

Cubo offered to publish papers in a special issue on pseudodifferential operators. Professors Gianluca Garello (University of Torino, Torino, Italy), Joachim Toft (Linæus University, Växjö, Sweden) and Man Wah Wong (York University, Toronto, Canada) kindly accepted to receive the submitted papers, have them refereed and accepted the five papers which appear in the present issue of Cubo.

I thank the editors of Cubo for asking me to take the responsibility of this special issue of the journal.

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Generalized Spectrograms and τ-Wigner Transforms

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ABSTRACT

We consider in this paper Wigner type representations Wig_τ depending on a parameter $\tau \in [0,1]$ as defined in [2]. We prove that the Cohen class can be characterized in terms of the convolution of such Wig_τ with a tempered distribution. We introduce furthermore a class of "quadratic representations" Sp^τ based on the τ-Wigner, as an extension of the two window Spectrogram (see [2]). We give basic properties of Sp^τ as subclasses of the general Cohen class.

RESUMEN

Nosotros consideramos en este artículo representaciones de tipo Wigner Wig_τ dependiendo de um parámetro $\tau \in [0,1]$ como definido en [2]. Probamos que la clase Cohen puede ser caracterizada en terminos de la convolución de tales Wig_τ con una distribución temperada. Introducimos también la clase de "representaciones cuadraticas" Sp^τ basado en el τ-Wigner, como una extensión de dos ventanas espectrograma (ver [2]). Nosotros damos propiedades básicas de Sp^τ como subclases de la clase Cohen.

Key words and phrases: *Time-Frequency representation, τ-Wigner distribution, generalized Spectrogram.*

Math. Subj. Class.: *42B10, 47A07, 33C05.*

1 Introduction

One of the basic problems in time-frequency analysis is the representation of the energy of a signal simultaneously with respect to time and frequency. Considering for generality signals as square-integrable functions on $\mathbb{R}^d$, the classical mathematical tool used for this aim are sesquilinear maps $Q : L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d})$. For a given signal f, the function $Q(f,f)(x,\omega)$, or for short $Q(f)(x,\omega)$, plays a role corresponding to that of density of mass in classical mechanics or that of probability distribution in statistics. In contrast however to these situations, in the case of the energy of a signal the time-frequency distribution to be used is not unique. Many proposals have been presented in the literature, each having advantages and drawbacks, see [5], [6], [7], [8], [9] for detailed presentations of these topics.

This is due essentially to the presence of the Heisenberg uncertainty principle which makes some of the natural requirements of a joint time-frequency distribution incompatible (see [11]).

Two of the most used time-frequency representations are the *Wigner distribution*:

$$Wig(f,f)(x,\omega) = Wig(f)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i t\omega} f(x+t/2)\overline{f(x-t/2)}\,dt \tag{1.1}$$

and the *Spectrogram*

$$Sp_g(f)(x,\omega) = |V_g(f)(x,\omega)|^2 \tag{1.2}$$

where $V_g(f)$ is the *Gabor transform* (also known as *short-time Fourier transform*) and is defined by

$$V_g(f)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i t\omega} f(t)\overline{g(x-t)}\,dt \tag{1.3}$$

in dependence on the "window" $g(x)$, which in the most generality can be supposed to be a tempered distribution.

This paper is based on these two representations of which we present modifications depending on parameters. We shall analyze the properties of these new representations with respect to classical requirements such as reality of values, marginal distribution conditions, and their relations with the *Cohen class*. This is a very general class of time-frequency representations, introduced by L. Cohen, see [6], and widely studied since the 1970's. It can be defined as the set of representations of the form

$$C(f) = \sigma * Wig(f) \tag{1.4}$$

where, in our context, σ will be supposed to be a tempered distribution in $\mathscr{S}'(\mathbb{R}^{2d})$ and will be called *Cohen kernel*. The wide possibility of choice of the Cohen kernel permits to cover most time-frequency representations.

We recall next that some considerations concerning shifts of the ghost frequencies led in [2] to the introduction of the representations

$$Wig_\tau(f,g)(x,\omega) = \int_{\mathbb{R}^d} e^{-2\pi i t\omega} f(x+\tau t)\overline{g(x-(1-\tau)t)}\,dt \tag{1.5}$$

which are a parameterized version of the Wigner representation in dependence on $\tau \in [0,1]$. It was also showed in [2] that these representations constitute the natural "quadratic form" counterparts to the τ-pseudo-differential operators which are extensions of the Weyl calculus on $\mathbb{R}^d$; classical references on this subject are Shubin [14] and Wong [15], see also [1] for generalizations concerning global hypo-ellipticity.

In the present paper we analyze at first the role of (1.5) in the definition of the Cohen class, showing that we can replace $Wig(f)$ in (1.4) by $Wig_\tau(f)$, for an arbitrary fixed $\tau \in [0,1]$, getting equivalent definitions of the Cohen class. In the second part of the paper, we propose a new form based on the two window spectrogram and the τ-Wigner representation. The *two window spectrogram* was studied in [3]-[4] (called there *generalized spectrogram*) and is defined by

$$Sp_{\phi,\psi}(f,g)(x,w) = V_\phi f(x,w)\overline{V_\psi g}(x,w). \tag{1.6}$$

Using τ-Wigner distribution, we generalize here definition (1.6) by replacing the classical Wigner distribution with τ-Wigner distributions. We obtain new representations that we shall call *parameterized two window spectrograms* and we study some of their basic properties such as positivity, support properties and boundedness in the L^p context. We show that our definition is motivated by the fact that the parameterized two window spectrograms show in some basic cases reduced interference phenomena with respect to (1.6) without a loss in the quality of the time-frequency localization. Finally we prove that among the variety of time-frequency representations they constitute a peculiarity as they do not belong to the Cohen class [1].

2 τ-Wigner Representations and the Cohen Class

In the definition (1.4) of the Cohen class the Wigner representation plays a special role and one natural question is if it can be replaced by another representation. In general this can be achieved under some additional conditions. More precisely suppose

$$C_0(f) = \sigma_0 * Wig(f)$$

is a fixed representation in the Cohen class; then, as long as $\hat{C}_0(f)/\hat{\sigma}_0$ belongs to $\mathscr{S}'(\mathbb{R}^{2d})$ for every signal $f \in \mathscr{S}(\mathbb{R}^d)$, we have

$$Wig(f) = \mathscr{F}^{-1}(\hat{C}_0(f)/\hat{\sigma}_0).$$

But even under this somewhat restrictive condition it does not necessarily happen that $C_0 \to \mathscr{F}^{-1}(\hat{C}_0(f)/\hat{\sigma}_0)$ is a convolution. Actually only if this were the case we could write

$$\mathscr{F}^{-1}(\hat{C}_0(f)/\hat{\sigma}_0) = \sigma' * C_0(f)$$

for a suitable fixed $\sigma' \in \mathscr{S}'(\mathbb{R}^{2d})$, and then for any generic representation in the Cohen class $C = \sigma * Wig$, (with $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$), we would obtain

$$C(f) = \sigma * Wig(f) = (\sigma * \sigma') * C_0(f).$$

In this case, under the further condition that $\sigma * \sigma' \in \mathscr{S}'(\mathbb{R}^{2d})$, we would have that every element in the Cohen class could be expressed in terms of C_0 instead of Wig.

In view of these observations it is interesting, even if not surprising, that any Wig_τ representation can replace the Wigner representation in the expression of the Cohen class.

In order to prove this assertion we need the explicit expression of Wig_τ as a member of the Cohen class. We recall then from [2] the following result.

Proposition 1. *The representation $Wig_\tau(f)$ belongs to the Cohen class for every $\tau \in [0,1]$, in particular*

$$Wig_\tau(f)(x,\omega) = \big(\sigma_\tau * Wig(f)\big)(x,\omega), \tag{2.1}$$

for every $f \in \mathscr{S}(\mathbb{R}^d)$, where

$$\sigma_\tau = \begin{cases} \frac{2^d}{|2\tau-1|^d}\, e^{2\pi i \frac{2}{2\tau-1} x\omega} & \text{for } \tau \neq \frac{1}{2} \\ \delta & \text{for } \tau = \frac{1}{2} \end{cases} \tag{2.2}$$

and δ is the Dirac distribution.

[1] According to (1.4) we only consider signal independent kernels σ

We have now the following Proposition:

Proposition 2. *Let $\tau \in [0,1]$ be fixed, then Wig_τ can be used to express the entire Cohen class, i.e. every representation C in the Cohen class can be written in the form*

$$C(f) = \sigma' * Wig_\tau(f)$$

for a suitable $\sigma' \in \mathscr{S}'(\mathbb{R}^{2d})$.

Proof. Let

$$C(f) = \sigma * Wig(f) \tag{2.3}$$

with $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$, be the expression of $C(f)$ in the Cohen class. From the previous proposition we have

$$Wig_\tau(f) = \sigma_\tau * Wig(f)$$

and a straightforward computation yields:

$$\sigma_\tau * \sigma_{1-\tau} = \delta.$$

We have therefore

$$\sigma_{1-\tau} * Wig_\tau(f) = Wig(f)$$

and substituting in (2.3) we get formally:

$$C(f) = (\sigma * \sigma_{1-\tau}) * Wig_\tau(f)$$

This expression has actually a meaning if we show that $\sigma * \sigma_{1-\tau}$ is a well defined tempered distribution. As $\sigma * \sigma_{1-\tau} = \mathscr{F}^{-1}(\hat{\sigma}\hat{\sigma}_{1-\tau})$ and $\sigma \in \mathscr{S}'(\mathbb{R}^{2d})$, this is equivalent to prove that $\hat{\sigma}_{1-\tau}$ is a multiplier of $\mathscr{S}'(\mathbb{R}^{2d})$. Since $\int e^{2\pi i y\rho}\, dy\, d\rho = 1$ we have

$$\mathscr{F}\sigma_{1-\tau}(\xi, t) = e^{-\pi i(1-2\tau)t\xi} \tag{2.4}$$

which is a C^∞ function with derivatives with polynomial growth and therefore our assertion is proved. The thesis is then satisfied with $\sigma' = \sigma * \sigma_{1-\tau}$. □

We turn now our attention to the spectrograms with the aim of describing how the general context above applies to this specific case.

As already pointed out in the Introduction, the classical spectrogram, defined by

$$Sp_g(f)(x,w) = |V_g f(x,w)|^2, \tag{2.5}$$

is a way to represent the energy of a signal f simultaneously with respect to time and frequency; $V_g f$ is the short-time Fourier transform, or Gabor transform, with window g, see for reference [13], [16], [10]. In [3], the two window spectrogram has been introduced and studied: it depends on two windows and it is defined by the skew-linear form

$$Sp_{\phi,\psi}(f,g)(x,w) = V_\phi(f)\overline{V_\psi(g)}(x,w); \tag{2.6}$$

when $\phi = \psi, f = g$, formula (2.6) becomes the classical spectrogram.

The following relationship between Wigner distribution and two window spectrogram holds (see [3]):

$$Sp_{\phi,\psi}(f,g)(x,w) = Wig(\tilde{\psi},\tilde{\phi}) * Wig(f,g)(x,w), \tag{2.7}$$

where $\tilde{\phi}(s) := \phi(-s)$ and $\tilde{\psi}(s) := \psi(-s)$. Relation (2.7), valid in suitable functional settings, for example when $f,g,\phi,\psi \in \mathscr{S}(\mathbb{R}^d)$, gives us the expression of the two window spectrogram as an element of the Cohen class, where σ in (1.4) is given now by $Wig(\tilde{\psi},\tilde{\phi})$. As proved in Proposition 2, we can re-write $Sp_{\phi,\psi}(f,g)$ through the τ-Wigner transform. In the special case of the two window spectrogram this can be made more explicit as showed by the following result.

Proposition 3. *For every $f,g,\phi,\psi \in \mathscr{S}(\mathbb{R}^d)$ and for every $\tau \in [0,1]$, we have*

$$Sp_{\phi,\psi}(f,g) = Wig_{1-\tau}(\tilde{\psi},\tilde{\phi}) * Wig_\tau(f,g)(x,w).$$

Proof. Since

$$Wig_{1-\tau}(\tilde{\psi},\tilde{\phi}) = \overline{Wig_\tau(\tilde{\phi},\tilde{\psi})}, \tag{2.8}$$

we have to prove that

$$Sp_{\phi,\psi}(f,g) = \overline{Wig_\tau(\tilde{\phi},\tilde{\psi})} * Wig_\tau(f,g)(x,w). \tag{2.9}$$

Let us observe that, by a simple change of variables, we can write

$$Wig_\tau(f,g)(x-y,w-\eta) = \mathscr{F}_{t\to\eta}\Big(e^{2\pi i\omega t} f(x-y-\tau t)\overline{g(x-y+(1-\tau)t)}\Big).$$

Since

$$Wig_\tau(\tilde{\phi},\tilde{\psi})(y,\eta) = \mathscr{F}_{t\to\eta}\Big(\tilde{\phi}(y+\tau t)\overline{\tilde{\psi}(y-(1-\tau)t)}\Big),$$

by the standard properties of the Fourier transform we get

$$\begin{aligned}&\overline{Wig_\tau(\tilde{\phi},\tilde{\psi})} * Wig_\tau(f,g)(x,w)\\ &\quad = \Big(\tilde{\phi}(y+\tau t)\overline{\tilde{\psi}(y-(1-\tau)t)}, e^{2\pi i\omega t} f(x-y-\tau t)\overline{g(x-y+(1-\tau)t)}\Big)_{L^2(\mathbb{R}^{2d}_{y,t})}.\end{aligned}$$

Finally, by the change of variables

$$\begin{cases} y+\tau t = Y \\ y-(1-\tau)t = T \end{cases}$$

in the L^2-product, we have

$$\overline{Wig_\tau(\tilde{\phi},\tilde{\psi})} * Wig_\tau(f,g)(x,w) = \left(\tilde{\phi}(Y)\overline{\tilde{\psi}(T)}, e^{2\pi i\omega(Y-T)} f(x-Y)\overline{g(x-T)}\right)_{L^2(\mathbb{R}^{2d}_{Y,T})}.$$

This shows that $\overline{Wig_\tau(\tilde{\phi},\tilde{\psi})} * Wig_\tau(f,g)(x,w)$ is independent of $\tau \in [0,1]$, and so for every $\tau \in [0,1]$,

$$\overline{Wig_\tau(\tilde{\phi},\tilde{\psi})} * Wig_\tau(f,g)(x,w) = \overline{Wig(\tilde{\phi},\tilde{\psi})} * Wig(f,g)(x,w).$$

From (2.8), (2.7) and this last identity, we get (2.9). □

3 The Parameterized Two Window Spectrogram: Definition and Motivations

So far we have been concerned with relationships between $\tau-$Wigner and spectrograms representations within the frame of the Cohen class. In this section we want to consider relationships between these two types of representations under another point of view which will bring us to the definition of a further representation. We start with some preliminary remarks. It is well-known that the Wigner transform can be expressed in function of the spectrogram by the following equality

$$Wig(f,g)(x,w) = 2^d e^{4\pi i x w} V_{\tilde{g}} f(2x,2w), \tag{3.1}$$

and viceversa we have

$$V_g f(x,w) = 2^{-d} e^{-\pi i x w} Wig(f,\tilde{g})(\frac{x}{2},\frac{w}{2}). \tag{3.2}$$

From (2.6) it is then clear that we can then rewrite the two window spectrogram as

$$Sp_{\phi,\psi}(f,g)(x,w) = 4^{-d} Wig(f,\tilde{\phi})(\frac{x}{2},\frac{w}{2})\overline{Wig(g,\tilde{\psi})(\frac{x}{2},\frac{w}{2})}. \tag{3.3}$$

In view of this equality it is natural to introduce the following generalization of the spectrogram:

Definiton 4. Let $\tau_1,\tau_2 \in [0,1]$ be two parameters, the *parameterized two window spectrogram*, denoted $Sp^{(\tau_1,\tau_2)}_{\phi,\psi}(f,g)$, is defined by

$$Sp^{(\tau_1,\tau_2)}_{\phi,\psi}(f,g)(x,w) = 4^{-d} Wig_{\tau_1}(f,\tilde{\phi})(\frac{x}{2},\frac{w}{2})\overline{Wig_{\tau_2}(g,\tilde{\psi})(\frac{x}{2},\frac{w}{2})}, \tag{3.4}$$

where ϕ,ψ are window functions and f,g are signals in suitable functional or distributional spaces.

Remark 5. *When* $\tau_1 = \tau_2 = 1/2$*, the parameterized two window spectrogram becomes the two window spectrogram*

$$Sp^{(\tau_1,\tau_2)}_{\phi,\psi}(f,g)(x,w) = Sp_{\phi,\psi}(f,g)(x,w).$$

The introduction of this new family of parameterized representations is not due to pure search of mathematical generality. Actually, as we describe next, the form $Sp^{(\tau_1,\tau_2)}_{\phi,\psi}(f,g)$ shows an interesting behavior for what concerns localization properties and reduction of interference disturbances in particular in the cases where frequencies occur in time intervals very close to one another. To this aim let us consider a signal f containing the frequency $\omega = 2$ in the time interval $[-4,0]$ and the frequency $\omega = 3$ in the time interval $[0,4]$; we fix the window functions $\phi = \chi_{[-10,10]}$ and $\psi = \chi_{[-\frac{1}{10},\frac{1}{10}]}$, where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$ and we compare the pictures of the parameterized two window spectrograms $Sp^{(\tau_1,\tau_2)}_{\phi,\psi}(f,g)$ for different values of τ_1 and τ_2. The two window spectrogram $Sp_{\phi,\psi}(f,f)$, corresponding to case $\tau_1 = \tau_2 = \frac{1}{2}$, is visualized in Figure 1:

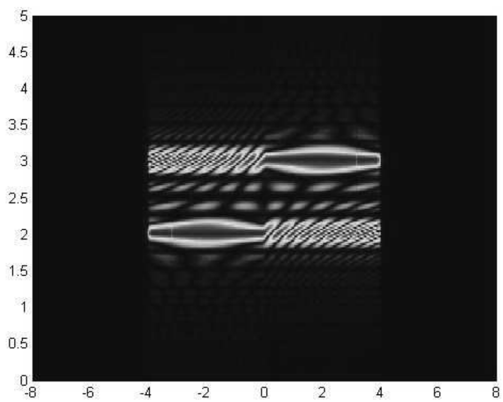


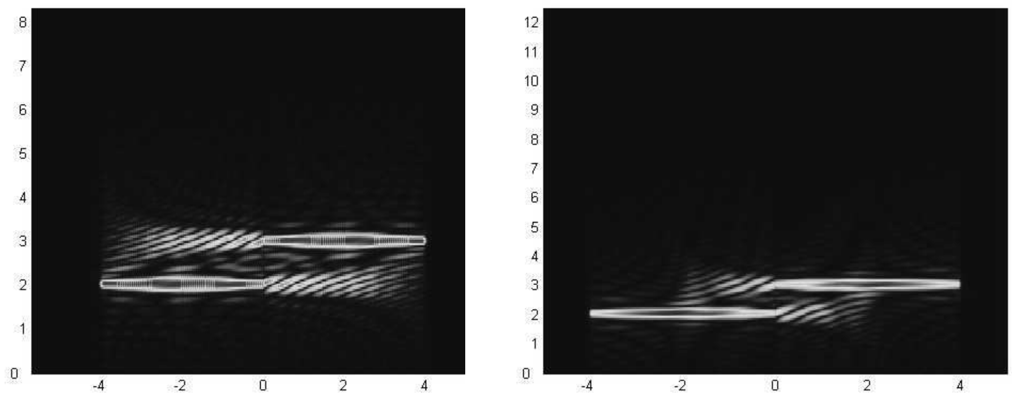
Figure 1: $Sp^{(\frac{1}{2},\frac{1}{2})}_{\phi,\psi}(f,f) = Sp_{\phi,\psi}(f,f)$

As we can see, although the localization is good both in time and in frequency, the picture presents disturbing interference patterns. The explanation of this fact is the following. The Gabor transform $V_\phi f$ with a large window ϕ gives better information regarding frequencies, and the Gabor transform $V_\psi f$ with a narrow window ψ gives better information concerning time. When we consider the two window spectrogram

$$Sp_{\phi,\psi}(f,g) = V_\phi f \overline{V_\psi g}$$

we take a product of one Gabor transform well localized in time and another one well localized in frequency, and so the reciprocal cut-off effect yields good localization both in time and frequency, see [4] for a detailed discussion on this subject. It could seem therefore that we have overcome the Heisenberg uncertainty principle but of course it is not so. Actually what is obtained in good localization, is "paid" terms of interference. More precisely, the fact that each Gabor transform is well localized in one variable and, consequently, badly localized in the other, implies that the supports of the two Gabor transforms also intersects in places where no frequency is present. This is what is observed in Figure 1 and clearly represents a considerable drawback in the use of the classical two window spectrogram.

Let us consider now the parameterized two window spectrogram, with the same windows and signal as above. In Picture 2 we have a representation of $Sp_{\phi,\psi}^{(0.3,0.3)}(f,f)$ and $Sp_{\phi,\psi}^{(0.2,0.2)}(f,f)$ (for simplicity we take here $\tau_1 = \tau_2$).



$Sp_{\phi,\psi}^{(0.3,0.3)}(f,f)$ $\qquad$ $Sp_{\phi,\psi}^{(0.2,0.2)}(f,f)$

Figure 2: Parameterized two window spectrogram for different values of τ_1, τ_2.

As we observe from the pictures, although the windows ϕ and ψ are kept fixed, the interference between the two frequencies is considerably reduced when the parameter τ in $Sp_{\phi,\psi}^{(\tau,\tau)}(f,f)$ becomes small, keeping on the other hand the good level of localization. Incidentally we also remark that the improvement of frequency localization is only apparent as it is essentially the consequence of an effect of vertical contraction and horizontal dilation compensated in the picture by a relabeling of the axis.

4 Properties of the Parameterized Two Window Spectrogram

In this section we analyze some properties of the representation $Sp_{\phi,\psi}^{(\tau_1,\tau_2)}(f,g)$ with $\tau_1, \tau_2 \in [0,1]$. More precisely we consider positivity, L^p-boundedness and support property, we conclude then our investigations by showing that the parameterized two window spectrogram does not belong to the Cohen class.

For what positivity is concerned we limit ourself to the following basic fact, we have

$$Sp_{\phi}^{(\tau)}(f)(x,w) := Sp_{\phi,\phi}^{(\tau,\tau)}(f,f)(x,w) = 4^{-d}|Wig_{\tau}(f,\tilde{\phi})(x,w)|^2 \geq 0.$$

and therefore the following property holds:

Proposition 6. *For $\tau_1 = \tau_2, f = g$ and $\phi = \psi$ the parameterized two window spectrogram is a positive time-frequency representation.*

We consider next the parameterized two window spectrogram in the context of the L^p spaces. For this purpose we shall need the following Proposition, which is proved in [2].

Proposition 7. *Let us fix q and p satisfying $q \geq 2$ and $q' \leq p \leq q, (\frac{1}{q} + \frac{1}{q'} = 1)$. Then:*

i) *For each $\tau \in (0,1)$, $Wig_{\tau} : L^{p'}(\mathbb{R}) \times L^p(\mathbb{R}) \to L^q(\mathbb{R}^{2d})$ is continuous, in particular:*

$$\|Wig_{\tau}(g,f)\|_{L^q} \leq \frac{1}{|1-\tau|^{d(\frac{1}{p}-\frac{1}{q})}} \frac{1}{|\tau|^{d(1-\frac{1}{p}-\frac{1}{q})}} \|g\|_{L^{p'}} \|f\|_{L^p}. \tag{4.1}$$

ii) *For $\tau = 0$, $Wig_0(g,f)(x,w) = R(g,f)(x,w)$ and $Wig_0 : L^q(\mathbb{R}) \times L^{q'}(\mathbb{R}) \to L^q(\mathbb{R}^{2d})$ is continuous, in particular*

$$\|Wig_0(g,f)\|_{L^q} \leq \|g\|_{L^{q'}} \|f\|_{L^q}. \tag{4.2}$$

iii) *For $\tau = 1$, $Wig_1(g,f)(x,w) = \overline{R(g,f)}(x,w)$ and $Wig_1 : L^{q'}(\mathbb{R}) \times L^q(\mathbb{R}) \to L^q(\mathbb{R}^{2d})$ is continuous, in particular*

$$\|Wig_1(g,f)\|_{L^q} \leq \|g\|_{L^q} \|f\|_{L^{q'}}. \tag{4.3}$$

Furthermore for p, q in the remaining cases the τ-Wigner transform is not bounded as sesquilinear map: $L^{p'}(\mathbb{R}) \times L^p(\mathbb{R}) \to L^q(\mathbb{R}^{2d})$.

The L^p behavior of the parameterized two window spectrogram is specified by the following proposition.

Theorem 8. *Let $q \geq 1$, $q_j \geq 2, p_j \geq 1, (j = 1,2)$ satisfy the following conditions: $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}; q_j' \leq p_j \leq q_j, (j = 1,2)$, where $\frac{1}{q_j} + \frac{1}{q_j'} = 1$. Then*

i) *The parameterized two window spectrogram $Sp^{(\tau_1,\tau_2)} : L^{p_1'} \times L^{p_1} \times L^{p_2'} \times L^{p_2} \to L^q$ is continuous $(0 < \tau_1, \tau_2 < 1)$, in particular*

$$\|Sp_{\phi,\psi}^{(\tau_1,\tau_2)}(f,g)\|_{L^q} \leq C \|f\|_{L^{p_1'}} \|\phi\|_{L^{p_1}} \|g\|_{L^{p_2'}} \|\psi\|_{L^{p_2}}, \tag{4.4}$$

where $C = C_1 C_2$ with $C_j = \frac{1}{|1-\tau|^{d(\frac{1}{p_j}-\frac{1}{q_j})}} \frac{1}{|\tau|^{d(1-\frac{1}{p_j}-\frac{1}{q_j})}}, j = 1,2.$

ii) *When* $\tau_1 = 1, \tau_2 = 0$ *then* $Sp^{(1,0)} : L^{q_1} \times L^{q_1'} \times L^{q_2'} \times L^{q_2} \to L^q$ *is continuous, in particular*

$$\|Sp^{(1,0)}_{\phi,\psi}(f,g)\|_{L^q} \le \|f\|_{L^{q_1}} \|\phi\|_{L^{q_1'}} \|g\|_{L^{q_2'}} \|\psi\|_{L^{q_2}}. \tag{4.5}$$

iii) *When* $\tau_1 = 0, \tau_2 = 1$ *then* $Sp^{(0,1)} : L^{q_1'} \times L^{q_1} \times L^{q_2} \times L^{q_2'} \to L^q$ *is continuous, in particular*

$$\|Sp^{(0,1)}_{\phi,\psi}(f,g)\|_{L^q} \le \|f\|_{L^{q_1'}} \|\phi\|_{L^{q_1}} \|g\|_{L^{q_2}} \|\psi\|_{L^{q_2'}}. \tag{4.6}$$

iv) *When* $\tau_1 = \tau_2 = 1$ *then* $Sp^{(1,1)} : L^{q_1} \times L^{q_1'} \times L^{q_2} \times L^{q_2'} \to L^q$ *is continuous, in particular*

$$\|Sp^{(1,1)}_{\phi,\psi}(f,g)\|_{L^q} \le \|f\|_{L^{q_1}} \|\phi\|_{L^{q_1'}} \|g\|_{L^{q_2}} \|\psi\|_{L^{q_2'}}. \tag{4.7}$$

v) *When* $\tau_1 = \tau_2 = 0$ *then* $Sp^{(0,0)} : L^{q_1'} \times L^{q_1} \times L^{q_2'} \times L^{q_2} \to L^q$ *is continuous, in particular*

$$\|Sp^{(0,0)}_{\phi,\psi}(f,g)\|_{L^q} \le \|f\|_{L^{q_1'}} \|\phi\|_{L^{q_1}} \|g\|_{L^{q_2'}} \|\psi\|_{L^{q_2}}. \tag{4.8}$$

Proof. It is an easy consequence of Proposition 7 and the generalized Hölder's inequality

$$\|fg\|_{L^q} \le \|f\|_{L^{q_1}} \|f\|_{L^{q_2}} \text{ for } \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}, \quad q_1 \ge q,$$

□

We recall now some notations. We indicate with $H(\text{supp} f)$ the convex hull of $\text{supp} f$ and with Π_x, Π_w the orthogonal projections on the first and the second factor in $\mathbb{R}^d_x \times \mathbb{R}^d_w$ respectively. Properties on the support of time-frequency representations is a widely studied subject because too large projections Π_x and Π_w of the support of a representation in comparison with the supports of the signal itself and its Fourier transform respectively would indicate a "spreading" of the energy that is seen as disturbance in the applications, see for instance [12]. We have the following basic results.

Lemma 9. *Let* $Wig_\tau(f,g)$ *be the* τ*-Wigner representation defined by* (1.5)*; then*

$$\Pi_x(\text{supp} Wig_\tau(f,g)) \subset H(\text{supp} f + \text{supp} g). \tag{4.9}$$

and

$$\Pi_w(\text{supp} Wig_\tau(f,g)) \subset H(\text{supp} \hat{f} + \text{supp} \hat{g}). \tag{4.10}$$

Proof. Suppose that $Wig_\tau(f,g)(x,\omega) \neq 0$, then there exists $t \in \mathbb{R}^d$ such that $f(y_1) \neq 0$ and $g(y_2) \neq 0$ with $y_1 = x + \tau t$ and $y_2 = x - (1-\tau)t$. On the other hand $x = \lambda y_1 + \mu y_2$ with $\lambda = 1 - \tau$ and $\mu = \tau$, i.e. x can be written as convex linear combination of y_1 and y_2. We have therefore that x belongs to the segment $[y_1, y_2]$ and (4.9) follows then immediately. To obtain (4.10) we just need to recall that

$$Wig_\tau(f,g)(x,w) = Wig_\tau(\hat{f},\hat{g})(w,-x).$$

and repeat the argument above with x replaced by ω. □

From (4.9), (4.10), and the equality $\mathrm{supp}(fg) = \mathrm{supp} f \cap \mathrm{supp} g$, we obtain the "support" property of the parameterized two window spectrogram.

Proposition 10. *The support of the parameterized two window spectrogram satisfies the following properties:*

$$\Pi_x(\mathrm{supp} Sp_{\phi,\psi}^{(\tau_1,\tau_2)}(f,g)) \subset H(\mathrm{supp} f + \mathrm{supp}\tilde{\phi}) \cap H(\mathrm{supp} g + \mathrm{supp}\tilde{\psi}) \tag{4.11}$$

and

$$\Pi_w(\mathrm{supp} Sp_{\phi,\psi}^{(\tau_1,\tau_2)}(f,g)) \subset H(\mathrm{supp}\hat{f} + \mathrm{supp}\hat{\tilde{\phi}}) \cap H(\mathrm{supp}\hat{g} + \mathrm{supp}\hat{\tilde{\psi}}). \tag{4.12}$$

Remark 11. *The meaning of the Proposition 10 becomes even more evident if we consider the case where $f = g$ is a signal and we suppose that one window is well localized in time and the other one in frequency. Assume for example that* $\mathrm{supp}\ \phi \subset B^\delta$ *and* $\mathrm{supp}\ \hat{\psi} \subset B^\delta$*, with B^δ ball of radius $\delta > 0$, then Proposition 10 implies that*

$$\mathrm{supp}\ Sp_{\phi,\psi}^{(\tau_1,\tau_2)}(f,f) \subset H(\mathrm{supp} f + B^\delta) \times H(\mathrm{supp}\ \hat{f} + B^\delta),$$

i.e. we have good localization both in time and in frequency, having reduced the spread of the energy to a ball of radius δ with respect to each variable.

Finally we prove that the parameterized two window spectrogram, in general, does not belong to the Cohen class. Let us consider for simplicity the case $\tau_1 = \tau_2 := \tau$ in Definition 4, with $\tau \neq \frac{1}{2}$ (actually for $\tau = \frac{1}{2}$, the representation $Sp_{\phi,\psi}^{(\frac{1}{2},\frac{1}{2})}(f,g)$ belongs to the Cohen class, since, as proved in [3], it coincides with $Sp_{\phi,\psi}(f,g)$). We denote for shortness $Sp_{\phi,\psi}^{\tau}(f,g) := Sp_{\phi,\psi}^{(\tau,\tau)}(f,g)$; the following proposition holds.

Proposition 12. *For $\tau \neq \frac{1}{2}$ there does not exist a tempered distribution $\sigma = \sigma_{\tau,\phi,\psi} \in \mathscr{S}'(\mathbb{R}^{2d})$ such that*

$$Sp_{\phi,\psi}^{\tau} = \sigma * Wig, \tag{4.13}$$

i.e. $Sp_{\phi,\psi}^{\tau}(f,g) = \sigma * Wig(f,g)$ for every $f,g \in \mathscr{S}(\mathbb{R}^d)$.

Proof. By Definition 4 and simple changes of variables we have:

$$
\begin{aligned}
Sp^{\tau}_{\phi,\psi}(f,g) &= 4^{-d}\int e^{-2\pi i t\frac{\omega}{2}} f\Big(\frac{x}{2}+\tau t\Big)\overline{\tilde{\phi}\Big(\frac{x}{2}-(1-\tau)t\Big)}\,dt \\
&\qquad \int e^{2\pi i t\frac{\omega}{2}}\overline{g\Big(\frac{x}{2}+\tau t\Big)}\tilde{\psi}\Big(\frac{x}{2}-(1-\tau)t\Big)\,dt \\
&= \int e^{-2\pi i s\omega} f(2\tau s)\overline{\phi\Big(2(1-\tau)s-\frac{x}{2\tau}\Big)}\,ds \\
&\qquad \int e^{-2\pi i s\omega}\overline{g(-2\tau s)}\psi\Big(-2(1-\tau)s-\frac{x}{2\tau}\Big)\,ds.
\end{aligned}
$$

By standard properties of the Fourier transform we can write the inverse Fourier transform of $Sp^{\tau}_{\phi,\psi}(f,g)(x,\omega)$ in the following way:

$$
\begin{aligned}
&\mathscr{F}^{-1}_{\substack{x\to t\\ \omega\to\xi}}\Big(Sp^{\tau}_{\phi,\psi}(f,g)(x,\omega)\Big) = \\
&= \mathscr{F}^{-1}_{x\to t}\Big[f(2\tau\xi)\overline{\phi\Big(2(1-\tau)\xi-\frac{x}{2\tau}\Big)}\Big] * \mathscr{F}^{-1}_{x\to t}\Big[\overline{g(-2\tau\xi)}\psi\Big(-2(1-\tau)\xi-\frac{x}{2\tau}\Big)\Big] \\
&= (2\tau)^{2d}\Big[e^{2\pi i(4\tau(1-\tau))t\xi} f(2\tau\xi)\widehat{\overline{\phi}}(2\tau t)\Big] * \Big[e^{-2\pi i(4\tau(1-\tau))t\xi}\overline{g(-2\tau\xi)}\,\hat{\psi}(2\tau t)\Big],
\end{aligned}
$$

where the convolution is performed in both the variables (t,ξ). Finally, writing explicitly the convolution, we obtain

$$
\begin{aligned}
\mathscr{F}^{-1}_{\substack{x\to t\\ \omega\to\xi}}\Big(Sp^{\tau}_{\phi,\psi}(f,g)(x,\omega)\Big) &= (2\tau)^{2d} e^{2\pi i(4\tau(1-\tau))t\xi} \\
&\qquad \int e^{-2\pi i(4\tau(1-\tau))tx} f(2\tau(\xi-x))\overline{g(-2\tau x)}\,dx \\
&\qquad \int e^{-2\pi i(4\tau(1-\tau))\xi s}\widehat{\overline{\phi}}(2\tau(t-s))\hat{\psi}(2\tau s)\,ds.
\end{aligned}
\tag{4.14}
$$

We observe that, by the definition of the Wigner transform,

$$
\begin{aligned}
\mathscr{F}^{-1}_{\substack{x\to t\\ \omega\to\xi}}\big(Wig(f,g)\big) &= \mathscr{F}^{-1}_{\substack{x\to t\\ \omega\to\xi}}\Big[\mathscr{F}_{s\to\omega}\Big(f\big(x+\frac{s}{2}\big)\overline{g\big(x-\frac{s}{2}\big)}\Big)\Big] \\
&= \int e^{2\pi i x t} f\big(x+\frac{\xi}{2}\big)\overline{g\big(x-\frac{\xi}{2}\big)}\,dx.
\end{aligned}
\tag{4.15}
$$

Now let us suppose that (4.13) holds for some tempered distribution σ; by taking the inverse Fourier transform and using (4.14) and (4.15), the following should be verified for every $f,g\in\mathscr{S}(\mathbb{R}^d)$:

$$
\begin{aligned}
&(2\tau)^{2d} e^{2\pi i(4\tau(1-\tau))t\xi}\int e^{-2\pi i(4\tau(1-\tau))tx} f(2\tau(\xi-x))\overline{g(-2\tau x)}\,dx \\
&\qquad \int e^{-2\pi i(4\tau(1-\tau))\xi s}\widehat{\overline{\phi}}(2\tau(t-s))\hat{\psi}(2\tau s)\,ds \\
&= \check{\sigma}(t,\xi)\int e^{2\pi i x t} f\big(x+\frac{\xi}{2}\big)\overline{g\big(x-\frac{\xi}{2}\big)}\,dx,
\end{aligned}
\tag{4.16}
$$

where $\check{\sigma}(t,\xi)$ is the inverse Fourier transform of σ. In particular, (4.16) should hold for f and g of the following type:

$$f(s)=e^{-\pi\lambda s^2},\quad g(s)=e^{-\pi\mu s^2},$$

for every $\lambda,\mu>0$. In this case we can compute explicitly the integrals involving f and g in (4.16) and we have:

$$\begin{aligned}
&\int e^{-2\pi i(4\tau(1-\tau))tx}e^{-\pi\lambda(2\tau\xi-2\tau x)^2}e^{-\pi\mu(-2\tau x)^2}\,dx=\\
&=e^{-4\pi\frac{\lambda\mu}{\lambda+\mu}\tau^2\xi^2}\int e^{-2\pi i(4\tau(1-\tau))tx}e^{-\pi\left(2(\lambda+\mu)^{1/2}\tau x-\frac{2\lambda\tau}{(\lambda+\mu)^{1/2}}\xi\right)^2}dx\\
&=(2\tau\sqrt{\lambda+\mu})^{-d}e^{-4\pi\frac{\lambda\mu}{\lambda+\mu}\tau^2\xi^2}e^{-2\pi i\frac{\lambda}{\lambda+\mu}4\tau(1-\tau)t\xi}\int e^{-2\pi i\frac{2(1-\tau)}{(\lambda+\mu)^{1/2}}ty}e^{-\pi y^2}\,dy\\
&=(2\tau\sqrt{\lambda+\mu})^{-d}e^{-2\pi i\frac{\lambda}{\lambda+\mu}4\tau(1-\tau)t\xi}e^{-4\pi\frac{\lambda\mu}{\lambda+\mu}\tau^2\xi^2}e^{-\pi\frac{4(1-\tau)^2}{\lambda+\mu}t^2}.
\end{aligned}\tag{4.17}$$

Similarly we obtain that

$$\int e^{2\pi ixt}f\left(x+\frac{\xi}{2}\right)\overline{g\left(x-\frac{\xi}{2}\right)}\,dx=(\sqrt{\lambda+\mu})^{-d}e^{-2\pi i\frac{\lambda}{\lambda+\mu}t\xi}e^{-\pi\frac{\lambda\mu}{\lambda+\mu}\xi^2}e^{-\pi\frac{1}{\lambda+\mu}t^2}.\tag{4.18}$$

Now, replacing (4.17) and (4.18) in (4.16) we have for $\check{\sigma}(t,\xi)$ the following expression

$$\begin{aligned}
\check{\sigma}(t,\xi)=(2\tau)^d e^{2\pi i(4\tau(1-\tau))t\xi-\pi it\xi-2\pi i\frac{\lambda}{\lambda+\mu}(4\tau(1-\tau)-1)t\xi}&\\
e^{-\pi\frac{4\lambda\mu\tau^2-\lambda\mu}{\lambda+\mu}\xi^2}e^{-\pi\frac{4(1-\tau)^2-1}{\lambda+\mu}t^2}\int e^{-2\pi i(4\tau(1-\tau))\xi s}\widehat{\overline{\phi}}\big(2\tau(t-s)\big)\,\hat{\psi}(2\tau s)\,ds.&
\end{aligned}\tag{4.19}$$

For $\tau\neq\frac{1}{2}$ we deduce then that $\check{\sigma}(t,\xi)$ necessarily depends on the two parameters λ and μ, and this is impossible since σ in (4.13) is independent of f and g. $\square$

Remark 13. *We also observe that in the case $\tau=1/2$ all terms in* (4.19) *involving the parameters λ and μ cancel, making σ independent of them, and confirming, as expected, that in this case the representation is in the Cohen class.*

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Modulation Spaces with $A_\infty^{\rm loc}$-Weights

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ABSTRACT

In this paper we describe the function space $M_{p,q}^{s,w}$ with $w \in A_\infty^{\rm loc}$ together with some related results of weighted modulation spaces.

RESUMEN

En este artículo describimos el espacio de la funciones $M_{p,q}^{s,w}$ con $w \in A_\infty^{\rm loc}$ junto con algunos resultados relacionados a espacios de modulación con peso.

Key words and phrases: *Modulation spaces, Exponential weights.*

Math. Subj. Class.: *41A17, 42B35.*

1 Modulation Spaces

Modulation spaces, which were initiated by Feichtinger in 1983 (see [5]), were investigated for the purpose of measuring smoothness of functions and distributions in a way other than Besov spaces. Besov spaces as well as Triebel-Lizorkin spaces are very close to Sobolev spaces and are used in partial differential equations. These spaces are defined by way of dilations. Feichtinger took full advantage of the group structure of $\mathbb{R}^n$. Recall that $\mathbb{R}^n$ carries the structure of a Lie group not with dilation but with addition. Therefore, it seems natural that we consider the translation.

The goal of the present paper is to combine the results in [17, 21]. The main results of [21] can be summarized as follows: Quite a few of the results of usual modulation spaces $M^s_{p,q}$ carries over to the $A^{\rm loc}_\infty$-weighted cases with $0 < p,q \le \infty$. In the present paper we shall establish the following results on modulation spaces. To describe the result, we make a setup.

Assume that $W : \mathbb{R}^n \to (0,\infty)$ is a measurable function with $A^{\rm loc}_\infty$ condition: There exists $1 < P < \infty$ such that W satisfies the $A^{\rm loc}_P$ condition

$$\sup_{Q:cube} \left(\frac{1}{|Q|} \int_Q W(x)\,dx \right) \left(\frac{1}{|Q|} \int_Q W(x)^{-\frac{1}{P-1}}\,dx \right)^{\frac{1}{P-1}} < \infty. \tag{1.1}$$

Suppose that the parameters p, q, s satisfy

$$0 < p < \infty,\ 0 < q < \infty,\ s \in \mathbb{R}. \tag{1.2}$$

Fix a window function $\varphi \in C^\infty_c$ so that it satisfies the non-degenerate condition:

$$\int_{\mathbb{R}^n} \varphi(x)\,dx \neq 0,\ \mathrm{supp}(\varphi) \subset [-1,1]^n. \tag{1.3}$$

We write $\varphi_{m,x}(z) = \exp(2\pi i m \cdot z)\varphi(z-x)$ for $m \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$. We define

$$\left\| f : M^{s,W}_{p,q} \right\|_g > \left(\sum_{m\in\mathbb{Z}^n} \langle m \rangle^q \left(\int_{\mathbb{R}^n} |\langle f, \varphi_{m,x} \rangle|^p W(x)\,dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \tag{1.4}$$

for $f \in C^\infty_c$, where we write $\langle x \rangle = \sqrt{1+|x|^2}$.

Theorem 1. *Assume* (1.1) *and* (1.2). *Then different choices of admissible φ satisfying* (1.3) *will yield equivalent norms. That is, if φ_1, φ_2 satisfy* (1.3), *then the norm equivalence*

$$\left\| f : M^{s,W}_{p,q} \right\|_{\varphi_1} \simeq \left\| f : M^{s,W}_{p,q} \right\|_{\varphi_2} \tag{1.5}$$

holds for $f \in C^\infty_c(\mathbb{R}^n)$.

In view of (1.5), we shall write $\left\| f : M^{s,W}_{p,q} \right\|$ instead of $\left\| f : M^{s,W}_{p,q} \right\|_g$.

As for this (new) modulation norm $\left\| f : M^{s,W}_{p,q} \right\|$, we have the following quantitiave information.

Lemma 1. *There exist $C > 0$ and $N \in \mathbb{N}$ depending only on W and p, q, s such that*

$$|\langle f, \psi\rangle| \le C \left\| f : M^{s,W}_{p,q} \right\| \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} e^{N|x|} |\partial^\alpha \psi(x)| \tag{1.6}$$

holds for all $\psi \in C_c^\infty$.

Denote by $M^{s,W}_{p,q}$ the (abstract) completion of C_c^∞ with $\left\| f : M^{s,W}_{p,q} \right\|_g$. In view of (1.6), we see that $M^{s,W}_{p,q}$ is a subset of $\mathscr{D}'$ satisfying

$$|\langle f, \varphi\rangle| \le C \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} e^{N|x|} |\partial^\alpha \varphi(x)| \tag{1.7}$$

holds for all $\varphi \in C_c^\infty$.

In the present paper we shall prove the molecular decomposition suitable for $M^{s,W}_{p,q}$.

Definition 1 (Molecule, Atom)**.** *Let $s \in \mathbb{R}$.*

1. *Suppose that $K, N \in \mathbb{N}$ are large enough and fixed. A C^K-function $\tau : \mathbb{R}^n \to \mathbb{C}$ is said to be an $(s; m, l)$-molecule, if it satisfies*

   $$|\partial^\alpha (e^{-im\cdot x}\tau(x))| \le \langle m\rangle^{-s} e^{-N|x-l|}, \ x \in \mathbb{R}^n$$

   for $|\alpha| \le K$.

2. *Suppose that $K, N \in \mathbb{N}$ are large enough and fixed. A C^K-function $\tau : \mathbb{R}^n \to \mathbb{C}$ is said to be an $(s; m, l)$-atom, if it satisfies*

   $$|\partial^\alpha (e^{-im\cdot x}\tau(x))| \le \langle m\rangle^{-s} \chi_{l+[-2,2]^n}, \ x \in \mathbb{R}^n$$

   for $|\alpha| \le K$.

3. *Also set*

   $$\mathscr{M}^s := \{\{\Psi^s_{ml}\}_{m,l\in\mathbb{Z}^n} : \ \textit{each } \Psi^s_{ml} \textit{ is an } (s; m, l)\textit{-molecule}\}$$
   $$\mathscr{A}^s := \{\{a^s_{ml}\}_{m,l\in\mathbb{Z}^n} : \ \textit{each } a^s_{ml} \textit{ is an } (s; m, l)\textit{-atom}\}.$$

Next, we introduce a sequence space $m_{p,q}$ to describe the condition of the coefficients of the molecular decomposition.

Definition 2 (Sequence space $m_{p,q}$)**.** *Let* $0 < p,q \leq \infty$. *Given a sequence* $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n}$, *define*

$$\|\lambda : m^W_{p,q}\| > \left[\sum_{m \in \mathbb{Z}^n} \left\{ \int_{\mathbb{R}^n} \left| \sum_{l \in \mathbb{Z}^n} \lambda_{ml} \chi_{l+[0,1]^n}(x) \right|^p W(x)\,dx \right\}^{\frac{q}{p}} \right]^{\frac{1}{q}}.$$

Here a natural modification is made when p *and/or* q *is infinite.* $m^W_{p,q}$ *is the set of doubly indexed sequences* $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n}$ *for which the quasi-norm* $\|\lambda : m^W_{p,q}\|$ *is finite.*

With these definitions in mind, we present a typical result in [21].

Theorem 2. *Assume* (1.1) *and* (1.2)*.*

1. *If* $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n} \in m^{s,W}_{p,q}$ *and* $\{\Psi^s_{ml}\}_{m,l \in \mathbb{Z}^n} \in \mathscr{M}^s$, *then*

$$f := \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot \Psi^s_{ml} \tag{1.8}$$

converges unconditionally in the topology of $M^{s,W}_{p,q}$.

2. *There exists* $\{a^s_{ml}\}_{m,l \in \mathbb{Z}^n} \in \mathscr{A}^s$ *such that any* $f \in M^{s,W}_{p,q}$ *admits the following decomposition*:

$$f = \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot a^s_{ml}, \tag{1.9}$$

where $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n}$ *satisfies*

$$\|\lambda : m^{s,W}_{p,q}\| \leq C \|f : M^{s,W}_{p,q}\| \tag{1.10}$$

for some $C > 0$ *independent of* f.

In the early 90's, more and more applications were found out. For example, time-frequency analysis, which is a branch of signal analysis, deals with the translation and the modulation, so that modulation spaces come into play naturally.

Also, it is remarkable that modulation spaces are applied effectively to the pseudo-differential operators by Sjöstrand, Tachizawa and many researchers [12, 14, 15, 19, 22, 23, 24, 25]. Modulation spaces are applicable to various partial differential equations. For example, Baoxiang and Chunyan used modulation spaces to investigate the KdV equation (see [3]). Recently modulation spaces can be applied even to the modeling of wireless channels and the quantum mechanics [2].

Now we describe the organization of this paper. In Section 2 we describe other weighted modulation spaces and compare them with ours. Section 3 is devoted to establishing Theorem 1 as well as Lemma 1. Section 4 is intended as the proof of Theorem 2. In Section 5 we consider the weighted modulation space $M^{s,W}_{p\infty}$. Finally in Section 6 we present some examples.

2 Various Weighted Modulation Spaces

Based on the standard notation of signal analysis, we adopt the following notations.

$$T_a f(x) := f(x-a),\ M_b f(x) := e^{ib\cdot x} f(x),\ a,b \in \mathbb{R}^n,\ f \in \mathscr{S}'.$$

Fix $\varphi \in C_c^\infty$ be a positinve non-zero function. Then define

$$\|f : M^s_{p,q}\| > \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\langle f, M_y T_x \varphi\rangle|^p \, dx \right)^{\frac{q}{p}} \langle y \rangle^{sq} \, dy \right)^{\frac{1}{q}}$$

for $s \in \mathbb{R}$ and $1 \le p,q \le \infty$. Denote by $M^s_{p,q}$ the set of all tempered distributions $f \in \mathscr{S}'$ for which the norm is finite. An important observation is that the function space $M^s_{p,q}$ does not depend on the specific choices of g. For more details we refer to [11, 18].

In general by weighted modulation norm we mean the following norm given by

$$\|f : M^v_{p,q}\| > \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\langle f, M_y T_x \varphi\rangle|^p v(x,y) \, dx \right)^{\frac{q}{p}} dy \right)^{\frac{1}{q}}.$$

Note that $M^s_{p,q}$ is recovered by setting $v(x,y) = \langle y \rangle^{sq}$. There are many important classes of weights.

Definition 3.

1. *A weight $v : \mathbb{R}^{2n} \to [0,\infty)$ is said to be a submultiplicative, if there exists a constant $C > 0$ such that $v(x+y) \le C\, v(x) v(y)$ for all $x,y \in \mathbb{R}^{2n}$.*

2. *A weight $v : \mathbb{R}^{2n} \to [0,\infty)$ is said to be subconvolutive, if $v^{-1} \in L^1(\mathbb{R}^{2n})$ and $v^{-1} * v^{-1} \le c\, v^{-1}$ for some constant $c > 0$.*

3. *A weight $v : \mathbb{R}^{2n} \to [0,\infty)$ is said to satisfy the Gelfand-Raikov-Shilov condition, if*

$$\lim_{n\to\infty} v(nx)^{\frac{1}{n}} = 1$$

for all $x \ne 0$.

4. *A weight $v : \mathbb{R}^{2n} \to [0,\infty)$ is said to satisfy the Beurling-Domar condition, if*

$$\sum_{j=1}^{\infty} \frac{\log v(nx)}{n} < \infty.$$

5. *A weight $v : \mathbb{R}^{2n} \to [0,\infty)$ is said to satisfy the logarithmic integral condition, if*

$$\int_{|x|\ge 1} \frac{\log v(x)}{|x|^{n+1}} \, dx < \infty.$$

Example 1.

1. *The function* $e^{\alpha|x|}$ *with* $\alpha \geq 0$ *is a submultiplicative weight. Similarly* $\langle x\rangle^{\alpha}$ *with* $\alpha \geq 0$ *is a submultiplicative weight.*
2. *The function* $\langle x\rangle^{n+\varepsilon}$ *is a subconvolutive weight.*

We refer to [7] for more details of the submultiplicative, moderate and subconvolutive weights not only on $\mathbb{R}^n$ but also on locally compact aberian groups.

Proposition 1. [13] *The Bourling-Domar condition is stronger than the Gelfand-Raikov-Shilov condition.*

Proof. This is just an easy consequence of the fact that the limit of a positive summable sequence is zero. □

In the present paper, we consider weights of the form

$$v(x,y) = W(x)\langle y\rangle^{s},$$

where $s \in \mathbb{R}$ and W belongs to the class $A_{\infty}^{\rm loc}$ described just below. As the example $W(x) = |x|^{\alpha}$, $\alpha > -n$ shows, it can happen that v fails the submultiplicative condition or subconvolutive condition. Another similar example shows that v does not necessarily satisfy the Bouring-Domar condition.

Before we go further, we recall the definition of $A_p^{\rm loc}$-weights. In the sequel by a "weight", we mean a non-negative measurable function $W \in L^1_{loc}$ satisfying $0 < W < \infty$ for a.e. and we define the local maximal operator $M^{\rm loc}$ by

$$M^{\rm loc} f(x) := \sup_{\substack{x\in Q\\ Q:cube, |Q|\leq 1}} \frac{1}{|Q|}\int_Q |f(y)|\,dy.$$

Let $1 \leq p < \infty$. Then we define

$$A_p^{\rm loc}(W) = \begin{cases} \displaystyle \operatorname*{ess.\,sup}_{x\in\mathbb{R}^n} \frac{M^{\rm loc}W(x)}{W(x)} & \text{if } p=1 \\ \displaystyle \sup_{\substack{Q:cube\\ |Q|\leq 1}} \left(\int_Q W(x)\frac{dx}{|Q|}\right)\cdot\left(\int_Q W(x)^{\frac{1}{1-p}}\frac{dx}{|Q|}\right)^{p-1} & \text{if } 1<p<\infty. \end{cases}$$

The quantity $A_p^{\rm loc}(W)$ is called the $A_p^{\rm loc}$-norm of W. Then it is easy to see that

$$A_p^{\rm loc}(W) \leq A_q^{\rm loc}(W),\ 1 \leq q \leq p < \infty.$$

The class A_p^{loc} of weights is the set of all weights W for which the norm $A_p^{\mathrm{loc}}(W)$ is finite. We also define

$$A_\infty^{\mathrm{loc}} := \bigcup_{1 \le p < \infty} A_p^{\mathrm{loc}}.$$

We remark that $|x|^{-n+\varepsilon} \in A_1^{\mathrm{loc}}$ for all $0 < \varepsilon < n$ and that $e^{\alpha|x|} \in A_1^{\mathrm{loc}}$ for all $\alpha \ge 0$.

Let W be a weight. Then we define

$$\|f : L_p^W\| > \left(\int_{\mathbb{R}^n} |f(x)|^p W(x)\,dx\right)^{\frac{1}{p}}, \, 1 \le p < \infty.$$

Here and below we assume that $W \in A_P^{\mathrm{loc}}$ with $1 \le P < \infty$ for the sake of definiteness.

3 Proof of Theorem 1

Now we prove Theorem 1 and Lemma 1. Before we prove Theorem 1, we first establish an auxiliary result (Proposition 2) and then we prove Theorem 1. Proposition 2 will be strengthened after we prove Lemma 1.

3.1 An auxiliary result on maximal operators

We write

$$p_N(\psi) > \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| \le N} \sup_{x \in \mathbb{R}^n} e^{N|x|} |\partial^\alpha \psi(x)|$$

for $\psi \in C_c^\infty$.

Proposition 2. *Let $k \in \mathbb{Z}$, $N > 0$ and $0 < \eta \le 1$. Then we have*

$$\sup_{\substack{\psi \in C_c^\infty \\ p_N(\psi) \le 1}} |M_k \psi * f(x)|^\eta \le c \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \frac{|M_l \varphi * f(x-y)|^\eta}{\langle k-l \rangle^{N\eta} e^{N\eta|y|}}\,dy \tag{3.11}$$

for all $f \in C_c^\infty$.

Proof. First let us consider the case when $\eta = 1$. Note that

$$\begin{aligned}
&\sum_{l \in \mathbb{Z}^n} \mathscr{F}\varphi(x+l)^2 \\
&= (2\pi)^{-\frac{n}{2}} \sum_{l \in \mathbb{Z}^n} \mathscr{F}[\varphi * \varphi](x+l) \\
&= (2\pi)^{-\frac{n}{2}} \sum_{m \in \mathbb{Z}^n} \left(\int_{\mathbb{R}^n} \sum_{l \in \mathbb{Z}^n} \mathscr{F}[\varphi * \varphi](y+l) \exp(-2\pi i y \cdot m)\,dy\right) \exp(2\pi i x \cdot m)
\end{aligned}$$

$$> \sum_{m\in\mathbb{Z}^n} \varphi*\varphi(-2\pi m)\exp(2\pi i x\cdot m) \equiv \varphi*\varphi(0)$$

from the Poisson summation formula. Consequently we obtain

$$M_k\psi * f = c_n \sum_{l\in\mathbb{Z}} M_k\psi * M_l\varphi * M_l\varphi * f. \tag{3.12}$$

Now we shall estimate each summand. First of all, a repeated integration by parts yields that for all $N>0$ there exists $c=c_N>0$ such that

$$|M_k\psi * M_l\varphi(y)| \le c\,\langle k-l\rangle^{-N} e^{-N|y|}.$$

As a consequence we obtain

$$|M_k\psi * M_l\varphi * M_l\varphi * f(x)| \le c\,\langle k-l\rangle^{-N}\int_{\mathbb{R}^n} e^{-N|y|}|M_l\varphi * f(x-y)|\,dy.$$

Inserting (3.12), we obtain the result when $\eta=1$. Namely we have proved

$$|M_k\psi * f(x)| \le c\sum_{l\in\mathbb{Z}}\langle k-l\rangle^{-N}\int_{\mathbb{R}^n} e^{-N|y|}|M_l\varphi * f(x-y)|\,dy \tag{3.13}$$

up to this point. Of course, the constant $c>0$ does depend on N. Now we pass to the case when $0<\eta<1$. We define

$$\mathcal{M}_{N,k}f(x) := \sup_{\substack{\psi\in C_c^\infty,\, p_N(\psi)\le 1\\ y\in\mathbb{R},\, l\in\mathbb{Z}}} \frac{|M_l\psi * f(x-y)|}{\langle k-l\rangle^N e^{N|y|}}.$$

Then from (3.13) we deduce

$$\begin{aligned}
\mathcal{M}_{N,k}f(x) &\le c\sup_{\substack{y\in\mathbb{R}\\ l\in\mathbb{Z}}}\left(\frac{1}{\langle k-l\rangle^N e^{N|y|}}\sum_{m\in\mathbb{Z}}\int\frac{|M_m\varphi * f(x-y-z)|}{\langle m-l\rangle^N e^{N|z|}}\,dy\right)\\
&\le c\sup_{y\in\mathbb{R}}\left(\sum_{m\in\mathbb{Z}}\int\frac{|M_m\varphi * f(x-y-z)|}{\langle m-k\rangle^N e^{N|y+z|}}\,dz\right)\\
&\le c\,\mathcal{M}_{N,k}f(x)^{1-\eta}\sum_{m\in\mathbb{Z}}\int\frac{|M_m\varphi * f(x-y)|^\eta}{\langle m-k\rangle^{N\eta}e^{N\eta|y|}}\,dy.
\end{aligned}$$

Here we have used the Peetre inequality $\langle a+b\rangle \le \sqrt{2}\langle a\rangle\cdot\langle b\rangle$. As a result, we obtain

$$|M_k\psi * f(x)|^\eta \le \mathcal{M}_{N,k}f(x)^\eta \le c\sum_{m\in\mathbb{Z}}\int\frac{|M_m\varphi * f(x-y)|^\eta}{\langle m-k\rangle^{N\eta}e^{N\eta|y|}}\,dy,$$

since $\mathcal{M}_{N,k}f(x)<\infty$. □

Proposition 3. *Let $W \in A_P^{\rm loc}$ and $F : \mathbb{R}^n \to [0,\infty)$ a measurable function. Then we have*

$$\left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} F(x-y)^\eta \frac{dy}{e^{B\eta|y|}} \right)^{\frac{p}{\eta}} W(x)\,dx \right\}^{\frac{1}{p}} \le C \left(\int_{\mathbb{R}^n} F(x)^p W(x)\,dx \right)^{\frac{1}{p}} \tag{3.14}$$

for all $p > P\eta$ and $B \gg 1$.

Proof. By replacing p/η with p, we can assume that $\eta = 1$ and $p > P$. Let $\ell \ge 1$. We denote $\chi_r = \dfrac{\chi_{(-r,r)^n}}{r^n}$. Then define $M^{\rm loc}_{\le \ell} f(x) > \sup\limits_{r \le \ell} \chi_r * |f|(x)$. Then there exists $\alpha > 0$ such that

$$\left(\int_{\mathbb{R}^n} M^{\rm loc}_{\le \ell} f(x)^p\,dx \right)^{\frac{1}{p}} \le e^{\alpha \ell} \left(\int_{\mathbb{R}^n} |f(x)|^p\,dx \right)^{\frac{1}{p}}. \tag{3.15}$$

Indeed, this inequality is true for $\ell = 1$ by the definition of $A_P^{\rm loc}$. Since $\chi_r * \chi_1 \ge \chi_{r+1}$ for $r \ge 1$, we have

$$M^{\rm loc}_{\le k} \le (M^{\rm loc}_{\le 1})^k.$$

As a consequece, we obtain (3.15).

Once we establish (3.15), (3.14) is an easy consequence of inequality

$$\begin{aligned} \int_{\mathbb{R}^n} F(x-y) e^{-B|y|}\,dy &\le \sum_{j=1}^\infty \int_{(-2^j,2^j)^n} F(x-y) e^{-2^{j-1}B}\,dy \\ &\le 2^n \sum_{j=1}^\infty e^{-2^{j-1}B} M^{\rm loc}_{\le 2^j} F(x). \end{aligned}$$

The proof is therefore complete. □

3.2 Proof of Theorem 1

Let $W \in A_\infty^{\rm loc}$ throughout. Then define

$$\|f_m : l_q(L_p^W)\| > \left(\sum_{m \in \mathbb{Z}^n} \|f_m : L_p^W\|^q \right)^{\frac{1}{q}}$$

for a family of measurable functions $\{f_m\}_{m \in \mathbb{Z}^n}$. Let $0 < p, q \le \infty$ and $s \in \mathbb{R}$. Then the modulation norm (1.4) can be written as

$$\|f : M^{s,W}_{p,q}\| > \left(\sum_{m \in \mathbb{Z}^n} \langle m \rangle^{qs} \| M_m \varphi * f : L_p^W \|^q \right)^{\frac{1}{q}}. \tag{3.16}$$

We are now in the position of establishing Theorem 1.

By Proposition 2 we have

$$|M_k\varphi_2 * f(x)|^\eta \le c\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\frac{|M_l\varphi_1 * f(x-y)|^\eta}{\langle k-l\rangle^{N\eta}e^{B\eta|y|}}\,dy.$$

If we invoke Proposition 3, we obtain

$$\|M_k\varphi_2 * f\|_{L^W_p} \le c\sum_{l\in\mathbb{Z}}\frac{1}{\langle k-l\rangle^{N\eta}}\|M_l\varphi_1 * f\|_{L^W_p}$$

if $\eta < P/p$, $N \gg 1$. Hence it follows that

$$\begin{aligned}\sum_{k\in\mathbb{Z}^n}\left(\langle k\rangle^s\|M_k\varphi_2 * f\|_{L^W_p}\right)^q &\le c\sum_{k\in\mathbb{Z}^n}\left(\sum_{l\in\mathbb{Z}}\frac{\langle k\rangle^s}{\langle k-l\rangle^{N\eta}}\|M_l\varphi_1 * f\|_{L^W_p}\right)^q\\ &\le c\sum_{l\in\mathbb{Z}^n}\left(\langle l\rangle^s\|M_l\varphi_2 * f\|_{L^W_p}\right)^q,\end{aligned}$$

which implies $\|f : M^{s,W}_{p,q}\|_{\varphi_2} \le c\|f : M^{s,W}_{p,q}\|_{\varphi_1}$. By symmetry Theorem 1 was proved completely.

3.3 Proof of Lemma 1

Instead of dealing with $\langle f,\psi\rangle$ directly, we have only to deal with $\psi * f(0)$, which is justified by the isomorphism $\psi\mapsto\psi(-\cdot)$. Proposition 3 and a normalization yield

$$|\psi * f(0)|^\eta \le c\,p_N(\psi)^\eta\sum_{l\in\mathbb{Z}}\int_{\mathbb{R}^n}\frac{|M_l\varphi * f(y)|^\eta}{\langle l\rangle^{N\eta}e^{N\eta|y|}}\,dy$$

with $0<\eta\ll\dfrac{\min(p,P,1)}{2}$.

$$\begin{aligned}\int_{\mathbb{R}^n}\frac{|M_l\varphi * f(y)|^\eta}{e^{N\eta|y|}}\,dy &> \int_{\mathbb{R}^n}\frac{|M_l\varphi * f(y)|^\eta W(y)^{\eta/p}}{e^{N\eta|y|}W(y)^{\eta/p}}\,dy\\ &\le (\|M_l\varphi * f\|_{L^W_p})^\eta\cdot\left(\int_{\mathbb{R}^n}\left(\frac{W(y)^{-\eta/p}}{e^{N\eta|y|}}\right)^{-p/(p-\eta)}dy\right)^{\frac{p-\eta}{\eta}}.\end{aligned}$$

Since $W^{-\frac{1}{P-1}}\in A^{\rm loc}_\infty$, we see that $W^{\frac{\eta/(p-\eta)}{\epsilon}}A^{\rm loc}_\infty$. Hence, if we choose $s\gg 1$, then we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}(e^{-N\eta|y|}W(y)^{-\eta/p})^{-p/(p-\eta)}\,dy\\ &\le\sum_{j=1}^\infty\int_{[-2^j,2^j]}e^{-2^{j-1}Np/(p-\eta)}W(y)^{\frac{\eta/(p-\eta)}{}}\,dy\\ &\le C_s\sum_{j=1}^\infty 2^{jn}e^{-2^{j-1}Np/(p-\eta)}M_{\le 2^j}[\chi_1](y)^sW(y)^{\frac{\eta/(p-\eta)}{}}\,dy\end{aligned}$$

$$< \infty.$$

As a consequence, Lemma 1 was proved.

We define $\mathscr{S}_e$ as the set of all C^∞-functions f for which the norm

$$p_N(\psi) > \sum_{\alpha\in\mathbb{Z}_+^n, |\alpha|\le N} \sup_{x\in\mathbb{R}^n} \exp(N|x|)|\partial^\alpha \psi(x)|$$

is finite. $\mathscr{S}_e'$ is defined as the topological dual of $\mathscr{S}_e$. We remark that $\mathscr{S}_e'$ is a special case of Gelfand-Shilov spaces (see [16]).

Proposition 4. *Proposition 3 remains vaild for $f \in \mathscr{S}_e'$.*

Proof. Keep to the same notation as Proposition 3. The proof does not undergo any major change until the end of the proof of Proposition 3. If $\mathscr{M}_{N,K}f(x)$ were finite, then we would obtain

$$|M_k\varphi * f(x)|^\eta \le \mathscr{M}_{N,K}f(x)^\eta \le c \sum_{m\in\mathbb{Z}} \int \frac{|M_m\gamma * f(x-y)|^\eta}{\langle m-k\rangle^{N\eta} e^{N\eta|y|}}\,dy. \tag{3.17}$$

However, this does not always work because $\mathscr{M}_{N,K}f(x)$ can be infinite. We shall show that (3.17) still holds for all $f \in \mathscr{S}_e'(\mathbb{R})$ even when $\mathscr{M}_{N,K}f(x) = \infty$. For this purpose let us assume the most right-hand side (3.17) is finite. Otherwise there is nothing to prove. Assuming that the most right-hand side (3.17) is finite, we shall establish $\mathscr{M}_{N,K}f(x) < \infty$. Since $f \in \mathscr{S}_e'(\mathbb{R})$, there exist $N_f > 0$ such that $\mathscr{M}_{N,K}f(x) < \infty$ for all $N \ge N_f$. As a consequence (3.17) holds for such N and N. From this we deduce

$$|M_k\varphi * f(x)|^\eta \le c \sum_{m\in\mathbb{Z}} \int \frac{|M_m\gamma * f(x-y)|^\eta}{\langle m-k\rangle^{N_f\eta} e^{N_f\eta|y|}}\,dy. \tag{3.18}$$

The constant in (3.17) being dependent implicitly on N, c in (3.17) must be dependent on f. To emphasize this dependence, let us write this constant as c_f. Then we have

$$\begin{aligned} |M_k\varphi * f(x)|^\eta &\le c_f \sum_{m\in\mathbb{Z}} \int \frac{|M_m\gamma * f(x-y)|^\eta}{\langle m-k\rangle^{N_f\eta} e^{N_f\eta|y|}}\,dy \\ &\le c_f \sum_{m\in\mathbb{Z}} \frac{1}{\langle m-k\rangle^{N\eta}} \int \frac{|M_m\gamma * f(x-y)|^\eta}{e^{N\eta|y|}}\,dy \end{aligned}$$

for all N with $N \le N_f$. As a consequence for all $N > 0$, there exists c_f such that

$$|M_k\varphi * f(x)|^\eta \le c_f \sum_{m\in\mathbb{Z}} \int \frac{|M_m\gamma * f(x-y)|^\eta}{\langle m-k\rangle^{N\eta} e^{N\eta|y|}}\,dy.$$

From the definition of the maximal operator $\mathscr{M}_{N,K}f(x)$, we have

$$\mathscr{M}_{N,K}f(x) \le c_f \sup_{y\in\mathbb{R}} \left(\sum_{m\in\mathbb{Z}} \int \frac{|M_m\gamma * f(x-y-z)|^\eta}{\langle k-l\rangle^{N\eta} \langle m-l\rangle^{N\eta} e^{N\eta(|y|+|z|)}}\,dz \right)$$

$$\leq c_f \sum_{m\in\mathbb{Z}} \int \frac{|M_m\gamma * f(x-z)|^\eta}{\langle k-m\rangle^{N\eta} e^{N\eta|z|}}\,dz$$
$$< \infty.$$

As a consequence (3.17) holds for all $f \in \mathscr{S}_e'(\mathbb{R})$. □

4 Proof of Theorem 2

A fundamental technique in harmonic analysis is to represent functions or distributions as a linear combination of functions of an elementary form. We shall investigate the structure of weighted modulation spaces.

We refer to [1, 4, 6, 8, 9, 10, 15, 20] for the definition of the molecules and atoms for different modulation spaces.

Now we prove Theorem 2.

1. Let $N \in \mathbb{N}$ be fixed. An integration by parts yields

$$\langle m\rangle^s \left|\sum_{l,m\in\mathbb{Z}^n} \lambda_{ml} M_k\varphi * \Psi^s_{ml}(x)\right|$$
$$\leq c \sum_{l,m\in\mathbb{Z}^n} \frac{|\lambda_{ml}|}{\langle k-m\rangle^N} \exp(-N|x-l|)$$
$$\leq c \sum_{j=1}^\infty \sum_{l\in\mathbb{Z}^n} \frac{e^{-Nj}}{\langle k-m\rangle^N} M^{\mathrm{loc}}_{\leq j}\left(\sum_{m\in\mathbb{Z}^n} \lambda_{ml}\chi_{Q_m}\right)$$

for some constant c depending only on N. As a result, we obtain the desired result by virtue of (3.15).

2. Note that $M_m * \varphi * M_m\varphi * \psi = c\psi$ for all $\psi \in \mathscr{S}_e$, since we have seen that $\sum_{m\in\mathbb{Z}^n} \mathscr{F}\varphi(\xi + m)^2 =: I \neq 0$. We set

$$a_{ml}(x) := \frac{1}{I}\int_{l+[0,1]^n} M_m\varphi(y) M_m\varphi * f(x-y)\,dy.$$

Then we have $f = \sum_{l,m\in\mathbb{Z}^n} a_{ml}$ in $\mathscr{S}_e'$. Since

$$M_{-m}a_{ml}(x) = \frac{1}{I}\int_{l+[0,1]^n} M_m\varphi(y)\langle f, \exp(-im\cdot(y+*))\varphi(x-y-*)\rangle\,dy,$$

we have $M_m[\partial^\alpha(M_{-m}a_{ml})](x) = \frac{1}{I}\int_{l+[0,1]^n} M_m\varphi(y) M_m[\partial^\alpha\varphi] * f(x-y)\,dy$. Therefore, if we define

$$\lambda_{ml} > \sup_{x\in l+[-2,2]^n} \sup_{|\alpha|\leq M} |\partial^\alpha(M_{-m}a_{ml})(x)|,$$

then, by Proposition 3, we have

$$\|\{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n} : m_{p,q}^{W}\| \le c\,\|f : M_{p,q}^{s,W}\|.$$

Hence, it follows that $f = \sum_{m,l\in\mathbb{Z}^n} \lambda_{ml}\cdot\frac{a_{ml}}{\lambda_{ml}}$ is an atomic decomposition of f. This is the desired result.

5 Weighted Modulation Space $M_{p,\infty}^{s,W}$

A minor modification of the results above will yield a theory of the function space $M_{p,\infty}^{s,W}$. We define the function space $M_{p,\infty}^{s,W}$ as follows:

Definition 4. *Let $0<p<\infty$, $0<q\le\infty$ and $s\in\mathbb{R}$. Assume that $W\in A_\infty^{\mathrm{loc}}$. Then define*

$$\|f : M_{p,q}^{s,W}\| > \left\{\sum_{l\in\mathbb{Z}^n}\langle m\rangle^{qs}\left(\int_{\mathbb{R}^n}|M_m\varphi * f(x)|^p W(x)\,dx\right)^{\frac{q}{p}}\right\}^{\frac{1}{q}}$$

for $f\in\mathscr{S}_e'$.

Lemma 2. *Let $0<p<\infty$, $s\in\mathbb{R}$, $W\in A_\infty^{\mathrm{loc}}$. If ε and q satisfy*

$$\varepsilon>0,\ 0<q<\infty,\ q\varepsilon>n.$$

then we have $M_{p,\infty}^{s,W}\hookrightarrow M_{p,q}^{s-\varepsilon,W}$.

Proof. This follows from a fundamental inequality

$$\left(\sum_{m\in\mathbb{Z}^n}\langle m\rangle^{-q\varepsilon}|a_m|^q\right)^{\frac{1}{q}} \le \sup_{m\in\mathbb{Z}^n}|a_m|\left(\sum_{m\in\mathbb{Z}^n}\langle m\rangle^{-q\varepsilon}\right)^{\frac{1}{q}}$$

which holds for all complex sequences $\{a_m\}_{m\in\mathbb{Z}^n}$. □

The atomic decomposition theorem can be formulated as follows:

Theorem 3. *Let $0<p<\infty$, $0<q\le\infty$ and $s\in\mathbb{R}$. Assume that $W\in A_\infty^{\mathrm{loc}}$.*

1. *The function space $M_{p,q}^{s,W}$ does not depend on the choice of specific φ satisfying* (1.3).
2. *If $\lambda=\{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n}\in m_{p,q}^{s,W}$ and $\{\Psi_{ml}^s\}_{m,l\in\mathbb{Z}^n}\in\mathscr{M}_0^s$, then*

$$f := \sum_{m,l\in\mathbb{Z}^n}\lambda_{ml}\cdot\Psi_{ml}^s \tag{5.19}$$

converges unconditionally in the topology of $\mathscr{S}_e'$.

3. *There exists* $\{a^s_{ml}\}_{m,l\in\mathbb{Z}^n} \in \mathscr{A}^s$ *such that any* $f \in M^{s,W}_{p,q}$ *admits the following decomposition*:

$$f = \sum_{m,l\in\mathbb{Z}^n} \lambda_{ml} \cdot a^s_{ml}, \tag{5.20}$$

where $\lambda = \{\lambda_{ml}\}_{m,l\in\mathbb{Z}^n}$ *satisfies*

$$\|\lambda : m^{s,W}_{p,q}\| \le C \|f : M^{s,W}_{p,q}\| \tag{5.21}$$

for some $C > 0$ *independent of* f.

Proof. Almost all the proofs remains unchanged except for the convergence in (5.19). This will be established by Lemma 2. □

6 Examples

Here we shall present some examples of weights.

Example 2. *A weight* $W_a(\xi) = \exp(a|\xi|)$, $a \in \mathbb{R}$ *belongs to the class of our admissible weights. It is interesting that* $M^{s,W_a}_{p,q}$ *is much larger than* $M^s_{p,q} = M^{s,W_0}_{p,q}$ *for* $a < 0$.

Example 3. *If we define* $W(x) = (1+|x|^2)^{\frac{a}{2}}$, *then* $M^{s,W}_{2,2}$ *is the weighted Sobolev space.*

Proposition 5. *Let* $0 < p < \infty, 0 < q \le \infty$ *and* $s \in \mathbb{R}$. *If we define* $W(x) = (1+|x|^2)^{\frac{a}{2}}$, *then* $M^{s,W}_{p,q} \subset \mathscr{S}'$.

Proof. In analogy with Proposition 2, we can prove

$$\sup_{\substack{\psi\in C^\infty_c \\ q_N(\psi)\le 1}} |M_k\psi * f(x)|^\eta \le c \sum_{l\in\mathbb{Z}} \int_{\mathbb{R}^n} \frac{|M_l\varphi * f(x-y)|^\eta}{\langle k-l\rangle^{N\eta}\langle y\rangle^{N\eta}}\,dy \tag{6.22}$$

for all $f \in C^\infty_c$, where $q_N(\psi) = \sum_{|\alpha|\le N} \sup_{x\in\mathbb{R}^n} \langle x\rangle^N |\partial^\alpha\psi(x)|$. Therefore, we can proceed as in the proof of Lemma 1. □

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Analytic Continuation and Applications of Eigenvalues of Daubechies' Localization Operator

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ABSTRACT

In this paper we introduce generating functions of eigenvalues of Daubechies' localization operator, study their analytic properties and give analytic continuation of these eigenvalues. Making use of generating functions, we establish a reconstruction formula of symbol functions of Daubechies' localization operator with rotational invariant symbols.

RESUMEN

Introducimos funciones generadas por los autovalores del operador de localización de Daubechies, estudiamos sus propiedades analíticas y damos continuación analítica de los autovalores. Haciendo uso de las funciones generadas, establecemos la fórmula de reconstrucción de funciones símbolo del operador de localización de Daubechies con símbolos rotacional invariante.

Key words and phrases: *Hermite functions, Daubechies (localization) operator, Borel transform, asymptotic expansion.*

Math. Subj. Class.: *33, 44, 46F.*

1 Introduction

The Daubechies (localization) operator was introduced by Ingrid Daubechies in [4], where she mainly treated localization operators with rotational invariant symbols. In particular, she expressed eigenvalues as Mellin transforms of symbol functions. She also proved that Hermite functions are eigenfunctions of localization operators with rotational invariant symbols. So far, the theory of localization operators has been studied by several researchers in various fields ([2], [5], [7], [10], [11], [12]).

In this paper we will study analytic properties of generating functions of eigenvalues of Daubechies' localization operators. We will also give an analytic continuation of eigenvalues of Daubechies' localization operator. Making use of generating functions, we will establish the reconstruction formula of symbol functions of Daubechies' localization operators with rotational invariant symbol.

For the simplicity, we will confine ourselves to the 1-dimensional case in this paper. In section 2 we will introduce Daubechies' localization operator. In section 3 we will give the analytic continuation of eigenvalues of Daubechies' localization operator. In section 4 we will define the generating function of eigenvalues. In final section 5 we will establish the reconstruction formulas for rotational invariant symbol function.

2 Daubechies' Localization Operator

According to [4], we define the **localization operator** P_F as follows.

Definition 1 ([4]). *Daubechies' localization operator is*

$$P_F(f)(x) = (2\pi)^{-1} \int\int_{\mathbb{R}^2} F(p,q)\phi_{p,q}(x) < \phi_{p,q}, f > dp dq, \tag{1}$$

where $F(p,q) \in L^1(\mathbb{R}^2)$*,* $f(x) \in L^2(\mathbb{R})$*,*

$$\phi_{p,q}(x) = \pi^{-1/4} e^{ipx} e^{-(x-q)^2/2},$$

and $< \phi_{p,q}, f >$ *denotes the inner product*

$$\int_{\mathbb{R}} \overline{\phi_{p,q}(x)} f(x) dx.$$

The function $F(p,q)$ is called the **symbol function** of the operator P_F.

Daubechies obtained the following results.

Proposition 1 ([4]). *Suppose that $F(p,q) \in L^1(\mathbb{R}^2)$. Then*

(i) If $F(p,q) \geq 0$, then P_F is a positive operator.

(ii) P_F is bounded operator. that is,

$$\|P_F(f)\|_{L^2} \leq (2\pi)^{-1/2}\|f\|_{L^2}\|F\|_{L^1}, \quad (f \in L^2(\mathbb{R})).$$

(iii) P_F is a trace class operator.

Proposition 2 ([4]). *Suppose*

$$F(p,q) = \tilde{F}(r^2), \qquad \textit{where} \quad r^2 = p^2 + q^2.$$

Then

(i) The Hermite functions $h_m(x)$ are eigenfunctions of the operator P_F:

$$P_F(h_m)(x) = \lambda_m h_m(x), \qquad m \in \mathbb{N}.$$

(ii) Secondly,

$$\lambda_m = \frac{1}{m!}\int_0^\infty e^{-s}s^m\tilde{F}(2s)ds, \qquad m \in \mathbb{N},$$

where the Hermite functions $h_m(x)$ are defined by

$$h_m(x) = (-1)^m(2^m m!\sqrt{\pi})^{-1/2}\exp(x^2/2)\frac{d^m}{dx^m}\exp(-x^2), \qquad m \in \mathbb{N}.$$

For details on Hermite functions, we refer the reader to [6, 7, 10].

In what follows we assume that

(i) $F(p,q) \in L^1(\mathbb{R}^2)$.

(ii) $F(p,q)$ is rotational invariant, that is,

$$F(p,q) = \tilde{F}(r^2), \qquad \text{where} \quad r^2 = p^2 + q^2.$$

3 Analytic Continuation of Eigenvalues of Daubechies Operator

In this section we consider the analytic continuation of the eigenvalues $\{\lambda_m\}_{m=0}^\infty$. By Proposition 2 we have

$$\lambda_m = \frac{1}{m!}\int_0^\infty e^{-s}s^m\tilde{F}(2s)ds.$$

We put

$$\lambda(z) = \frac{1}{\Gamma(z+1)} \int_0^\infty e^{-s} s^z \tilde{F}(2s) ds, \quad z \in \mathbb{C}, \mathrm{Re}(z) > 0.$$

where $\Gamma(z)$ is Euler's Gamma function.

Then we have the following proposition.

Proposition 3. *$\lambda(z)$ have following properties:*

(i) $\lambda(z)$ is holomorphic in the right half plane $Re(z) > 0$.

(ii) There exists a positive constant C such that

$$|\lambda(z)| \leq \frac{C}{\sqrt{|z|}} e^{\frac{\pi}{2}|y|}, \qquad z = x + iy \in \mathbb{C}, x > 0.$$

(iii) $\lambda(z)$ interpolates the eigenvalues $\{\lambda_m\}_{m=0}^\infty$, that is,

$$\lambda(m) = \lambda_m, \qquad m \in \mathbb{N}.$$

(iv) There exists a positive constant C such that

$$|\lambda_m| \leq \frac{C}{\sqrt{|m|}}, \qquad m \in \mathbb{N}.$$

Proof. The proof is as follows.

(i) We can prove the holomorphicity of $\lambda(z)$ by Morea's theorem and Lebesgue's dominated convergence theorem.

(iii) is obvious.

(iv) follows from (ii) and (iii).

So we will prove statement (ii). By Stirling's formula,

$$\Gamma(z+1) \sim z^z e^{-z} \sqrt{2\pi z}, \qquad \mathrm{Re}(z) > 0$$

and $e^{-s}s^x \le e^{-x}x^x$ for $s \ge 0$, we have

$$\begin{aligned}
|\lambda(z)| &= \left|\frac{1}{\Gamma(z+1)}\int_0^\infty e^{-s}s^z\tilde{F}(2s)\,ds\right| \\
&\le \frac{1}{|\Gamma(z+1)|}\int_0^\infty e^{-s}|s^z|\,|\tilde{F}(2s)|\,ds \\
&\le \frac{C}{|z^z e^{-z}|\sqrt{2\pi|z|}}\int_0^\infty e^{-s}s^x|\tilde{F}(2s)|\,ds \\
&\le \frac{Ce^{y\arg(z)}}{x^x e^{-x}\sqrt{2\pi|z|}}\int_0^\infty e^{-x}x^x|\tilde{F}(2s)|\,ds \\
&\le \frac{C}{\sqrt{2\pi|z|}}e^{\frac{\pi}{2}|y|}\int_0^\infty |\tilde{F}(2s)|\,ds \\
&\le \frac{C'}{\sqrt{|z|}}e^{\frac{\pi}{2}|y|}.
\end{aligned}$$

Remark 1. *The function $\lambda(z)$ is the unique analytic continuation of eigenvalues $\{\lambda_m\}_{m=0}^\infty$ because of (ii) in Proproposition 3 and Carlson's theorem [3].*

4 Generating Functions of Eigenvalues of Daubechies Operator

In this section we introduce two generating functions $\Lambda(w)$ and $G(t)$ of the eigenvalues $\{\lambda_m\}_{m=0}^\infty$. We begin with $\Lambda(w)$. Put

$$\Lambda(w) = \sum_{m=0}^\infty \lambda_m w^m, \quad (|w| < 1).$$

Due to (iv) in Proposition 3, the right-hand side is a convergent series if $|w| < 1$.

We will show some analytic properties of $\Lambda(w)$.

Proposition 4. *Suppose that $\{\lambda_m\}_{m=0}^\infty$ are eigenvalues of P_F. Then*

(i) The function $\Lambda(w)$ is given by the integral

$$\Lambda(w) = \int_0^\infty e^{-s(1-w)}\tilde{F}(2s)ds, \qquad \mathrm{Re}(w) < 1.$$

(ii) $\Lambda(w)$ is holomorphic in the left-half plane $\{w \in \mathbb{C} : \mathrm{Re}(w) < 1\}$ and is bounded in its closure $\{w \in \mathbb{C} : \mathrm{Re}(w) \le 1\}$.

(iii) $\Lambda(iv) \in C_0(\mathbb{R})$ for $v \in \mathbb{R}$, that is, $\Lambda(iv) \in C(\mathbb{R})$ and $\lim_{|v|\to\infty}\Lambda(iv) = 0$.

Proof. We prove the three parts.

(i) By (ii) in Proposition 2, we have

$$\begin{aligned}\Lambda(w) &= \sum_{m=0}^{\infty} \lambda_m w^m \\ &= \sum_{m=0}^{\infty} \frac{w^m}{m!} \int_0^{\infty} e^{-s} s^m \tilde{F}(2s)\,ds \\ &= \int_0^{\infty} e^{-s} \tilde{F}(2s) \sum_{m=0}^{\infty} \frac{(ws)^m}{m!}\,ds \\ &= \int_0^{\infty} e^{-s(1-w)} \tilde{F}(2s)\,ds.\end{aligned}$$

(ii) For $\mathrm{Re}(w) \le 1$, we have

$$|\Lambda(w)| \le \int_0^{\infty} |e^{-s(1-w)}||\tilde{F}(2s)|\,ds \le \int_0^{\infty} |\tilde{F}(2s)|\,ds = ||\tilde{F}||_{L^1}.$$

(iii) $\Lambda(iv)$ is the Fourier transform of the L^1 function $e^{-s}\tilde{F}(2s)$, for $s \ge 0$. Hence it belongs to $C_0(\mathbb{R}^n)$ by the Riemann–Lebesgue theorem [9].

Proposition 5. *Suppose that $F(p,q)$ is positive. If*

$$\limsup_{m\to\infty} {\lambda_m}^{1/m} = 1,$$

then $w = 1$ is a singular point of $\Lambda(w)$.

Proof. Since $F(p,q)$ is positive, then P_F is a positive operator by Proposition 1. Therefore, all the eigenvalues of P_F are nonnegative. By the Cauchy–Hadamard formula, the radius of convergence of the power series

$$\sum_{m=0}^{\infty} \lambda_m w^m$$

is 1. By Vivanti's theorem, $w = 1$ is a singular point of $\Lambda(w)$.

Proposition 6. *Suppose the support of $\tilde{F}(2s)$ is contained in $[0,a]$. Then there exists a positive constant C such that*

(i) $|\lambda_m| \le C\dfrac{a^m}{m!}, \qquad m \in \mathbb{N}.$

(ii) $\Lambda(w)$ is an entire function of exponential type.

Proof. We prove the two parts of the proposition.

(i) Since the support of $\tilde{F}(2s)$ is contained in the closed interval $[0,a]$, by (ii) in Proposition 2, we have

$$\begin{aligned}\lambda_m &= \frac{1}{m!}\int_0^a e^{-s}\tilde{F}(2s)s^m\,ds\\ &\le \frac{a^m}{m!}\int_0^a |\tilde{F}(2s)|\,ds.\end{aligned}$$

Therefore, the inequality $|\lambda_m| \le C\dfrac{a^m}{m!}$ is valid.

(ii) Since

$$\begin{aligned}|\Lambda(w)| &\le \int_0^a |\tilde{F}(2s)|e^{-s(1-u)}\,ds\\ &\le e^{a|u|}\int_0^a |\tilde{F}(2s)|ds,\end{aligned}$$

we have

$$|\Lambda(w)| \le Ce^{a|u|}, \qquad w = u + iv \in \mathbb{C}.$$

Now we consider following formal power series:

$$\sum_{m=0}^{\infty} m!\lambda_m t^{-m-1}.$$

In general, the series on right-hand side is divergent. But this formal power series is an asymptotic expansion of the Hilbert transform of $\tilde{F}(2s)e^{-s}$. Namely, if we put

$$G(t) = \int_0^{\infty} \frac{\tilde{F}(2s)e^{-s}}{t-s}ds, \qquad t \in \mathbb{C}\setminus[0,\infty],$$

then $G(t)$ has following properties.

Proposition 7. *For the function $G(t)$ we have*

(i) $G(t)$ is Laplace transform of $\Lambda(w)$.

(ii) $\sum_{m=0}^{\infty} m!\lambda_m t^{-m-1}$ is an asymptotic expansion of $G(t)$.

Proof.

(i) By (ii) in Proposition 4, $\Lambda(w)$ is bounded in left-half plane. So we can consider the Laplace transform of $\Lambda(w)$ along the negative real axis:

$$\begin{aligned}\int_0^{-\infty}\Lambda(w)e^{-tw}\,dw &= \int_0^{-\infty}\Big\{\int_0^{\infty} e^{-s(1-w)}\tilde{F}(2s)\,ds\Big\}e^{-tw}\,dw\\ &= \int_0^{\infty}\tilde{F}(2s)e^{-s}\Big\{\int_0^{-\infty} e^{w(s-t)}dw\Big\}ds\\ &= \int_0^{\infty}\frac{\tilde{F}(2s)e^{-s}}{t-s}\,ds\\ &= G(t), \qquad \text{for Re } t<0.\end{aligned}$$

(ii) Secondly,

$$\begin{aligned}G(t) &= \int_0^{\infty}\frac{\tilde{F}(2s)e^{-s}}{t-s}\,ds\\ &= \frac{1}{t}\int_0^{\infty}\frac{\tilde{F}(2s)e^{-s}}{1-st^{-1}}\,ds\\ &= \frac{1}{t}\int_0^{\infty}\tilde{F}(2s)e^{-s}\Big\{\sum_{m=0}^{N}(st^{-1})^m+\frac{(st^{-1})^{N+1}}{1-st^{-1}}\Big\}ds\\ &= \sum_{m=0}^{N} m!\lambda_m t^{-m-1}+\frac{1}{t^{N+1}}\int_0^{\infty}\frac{\tilde{F}(2s)e^{-s}s^{N+1}}{t-s}\,ds.\end{aligned}$$

Hence if $|t|\geq R$ and $0<\delta\leq \arg(t)\leq 2\pi-\delta$, then we have

$$\begin{aligned}|G(t)-\sum_{m=0}^{N} m!\lambda_m t^{-m-1}| &\leq \frac{(N+1)!}{R\sin\delta}\frac{\lambda_{N+1}}{|t|^{N+1}}\\ &\leq C\frac{N!\sqrt{N+1}}{R\sin\delta|t|^{N+1}}.\end{aligned}$$

Proposition 8. *Suppose that support of $\tilde{F}(2s)$ is contained in $[0,a]$. Then*

(i) $G(t)$ is holomorphic in $\mathbb{C}\setminus[0,a]$.

(ii) $\sum_{m=0}^{\infty} m!\lambda_m t^{-m-1}$ converges in $|t|>a$.

Proof.

(i) From the assumption on the support of $\tilde{F}(2s)$, we have

$$G(t)=\int_0^{a}\frac{\tilde{F}(2s)e^{-s}}{t-s}ds.$$

So $G(t)$ is holomorphic in $\mathbb{C}\setminus[0,a]$.

(i) By (i) in Proposition 6,

$$|\lambda_m| \leq C\frac{a^m}{m!}, \qquad m \in \mathbb{N}.$$

Hence

$$\sum_{m=0}^{\infty} m!\lambda_m t^{-m-1}$$

converges if $|t| > a$.

Remark 2. *The function* $\Lambda(w)$ *is the Borel transform of* $G(t)$*. For details on the Borel and Hilbert transforms, we refer the reader to [1, 8, 9].*

5 Reconstruction of Symbol Functions

In this section we establish our main results.

Theorem 1. *The function*

$$\tilde{F}(2s) = (2\pi)^{-1}e^s\mathfrak{F}(\Lambda(iv))(s),$$

is valid in distribution sense, where

$$\mathfrak{F}(\Lambda(iv)) = \int_{-\infty}^{\infty} e^{-isv)}\Lambda(iv)dv$$

is Fourier transform of $\Lambda(iv)$*.*

Proof. By (i) in Proposition 4, we have

$$\begin{aligned}\Lambda(iv) &= \int_0^{\infty} e^{-s(1-iv)}\tilde{F}(2s)ds\\ &= \int_0^{\infty} e^{isv}e^{-s}\tilde{F}(2s)ds.\end{aligned}$$

This means that $\Lambda(iv)$ is the inverse Fourier transform of $e^{-s}\tilde{F}(2s)$. Since $\tilde{F}(2s)$ is an L^1-function, then $e^{-s}\tilde{F}(2s)$ is a tempered distribution. Hence, as tempered distribution, we have

$$\tilde{F}(2s) = e^s\mathfrak{F}(\Lambda(iv))(s).$$

Theorem 2. *The function* $\tilde{F}(2s)$ *is given by the formula*

$$\tilde{F}(2s) = e^s \lim_{t\to 0}\frac{-1}{2\pi i}\left(G(s+it) - G(s-it)\right).$$

Proof. It is well known that the boundary value

$$\lim_{t\to 0}\frac{-1}{2\pi i}\left[G(s+it)-G(s-it)\right]$$

is the inverse map of the Hilbert transform [8]. Since $G(t)$ is Hilbert transform of $e^{-s}\tilde{F}(2s)$, we have

$$\lim_{t\to 0}\frac{-1}{2\pi i}\left(G(s+it)-G(s-it)\right)=e^{-s}\tilde{F}(2s).$$

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Strichartz Estimates for the Schrödinger Equation

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ABSTRACT

The objective of this paper is to report on recent progress on Strichartz estimates for the Schrödinger equation and to present the state-of-the-art. These estimates have been obtained in Lebesgue spaces, Sobolev spaces and, recently, in Wiener amalgam and modulation spaces. We present and compare the different technicalities. Then, we illustrate applications to well-posedness.

RESUMEN

El objetivo de este trabajo es reportar los progresos recientes sobre estimativas de Strichartz para la ecuación de Schrödinger y presentar el estado de arte. Estas estimativas han sido obtenidas en espacios de Lebesgue, espacios de Sobolev, y recientemente, en espacios de Wiener amalgamados y de modulación. Presentamos y comparamos los diferentes aspectos técnicos envueltos. Ilustramos los resultados con aplicaciones a buena colocación.

Key words and phrases: *Dispersive estimates, Strichartz estimates, Wiener amalgam spaces, Modulation spaces, Schrödinger equation.*

Math. Subj. Class.: *42B35,35B65, 35J10, 35B40.*

1 Introduction

In this note, we focus on the Cauchy problem for Schrödinger equations. To begin with, the Cauchy problem for the free Schrödinger equation reads as follows

$$\begin{cases} i\partial_t u + \Delta u = 0 \\ u(0,x) = u_0(x), \end{cases} \tag{1}$$

with $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, $d \geq 1$. In terms of the Fourier transform, we can write the solution as follows

$$u(t,x) = \left(e^{it\Delta}u_0\right)(x) := \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} e^{-4\pi^2 it|\xi|^2} \widehat{u_0}(\xi)d\xi, \tag{2}$$

where the Fourier multiplier $e^{it\Delta}$ is known as *Schrödinger propagator*. The corresponding inhomogeneous equation is

$$\begin{cases} i\partial_t u + \Delta u = F(t,x) \\ u(0,x) = u_0(x), \end{cases} \tag{3}$$

with $t > 0$ and $x \in \mathbb{R}^d$, $d \geq 1$. By Duhamel's principle and (2), the integral version of (3) has the form

$$u(t,x) = e^{it\Delta}u_0(\cdot) + \int_0^t e^{i(t-s)\Delta}F(s,\cdot)ds. \tag{4}$$

The study of space-time integrability properties of the solution to (2) and (4) has been pursued by many authors in the last thirty years. The matter of fact is given by the Strichartz estimates, that have become a fundamental and amazing tool for the study of PDE's. They have been studied in the framework of different function/distribution spaces, like Lebesgue, Sobolev, Wiener amalgam and modulation spaces and have found applications to well-posedness and scattering theory for nonlinear Schrödinger equations [3, 8, 9, 11, 19, 24, 26, 36, 38, 39, 49].

In this paper we exhibit these problems. First, in Section 3, we introduce the dispersive estimates and show how can be carried out for these different spaces. The classical L^p dispersive estimates read as follows

$$\|e^{it\Delta}u_0\|_{L^r_x} \lesssim |t|^{-d(\frac{1}{2}-\frac{1}{r})}\|u_0\|_{L^{r'}_x}, \quad 2 \leq r \leq \infty, \quad \frac{1}{r} + \frac{1}{r'} = 1.$$

Section 4 is devoted to the study of Strichartz estimates. The nature of these estimates is highlighted and the results among different kinds of spaces are compared with each others. Historically, the L^p spaces [19, 24, 26, 39, 49] were the first to be looked at. The celebrated homogeneous Strichartz estimates for the solution $u(t,x) = \left(e^{it\Delta}u_0\right)(x)$ read

$$\|e^{it\Delta}u_0\|_{L^q_t L^r_x} \lesssim \|u_0\|_{L^2_x}, \tag{5}$$

for $q \geq 2$, $r \geq 2$, with $2/q + d/r = d/2$, $(q,r,d) \neq (2,\infty,2)$, i.e., for (q,r) *Schrödinger admissible* (see Definition 4.1). Here, as usual, we set

$$\|F\|_{L_t^q L_x^r} = \left(\int \|F(t,\cdot)\|_{L_x^r}^q \, dt \right)^{1/q}.$$

In the sequel, the estimates for Sobolev spaces were essentially derived from the Lebesgue ones. Recently, several authors ([1, 2, 6, 7, 8, 44, 45]) have turned their attention to fixed time and space-time estimates for the Schrödinger propagator between spaces widely used in time-frequency analysis, known as Wiener amalgam spaces and modulation spaces. The first appearance of amalgam spaces can be traced to Wiener in his development of the theory of generalized harmonic analysis [46, 47, 48] (see [22] for more details). In this setting, Cordero and Nicola [6, 7, 8] have discovered that the pattern to obtain dispersive and Strichartz estimates is similar to that of Lebesgue spaces. The main idea is to show that the fundamental solution K_t (see (20) below) lies in the Wiener amalgam space $W(L^p, L^q)$ (see Section 2 for the definition) which generalizes the classical L^p space and, consequently, provides a different information between the local and global behavior of the solutions. Beside the similar arguments, we point out also some differences, mainly in proving the sharpness of dispersive estimates and Strichartz estimates. Indeed, dilation arguments in Wiener amalgam and modulation spaces don't work as in the classical L^p spaces.

Modulation spaces were introduced by Feitchinger in 1980 and then were also redefined by Wang [44] using isometric decompositions. The two different definitions allow to look at the problem in two different manners. As a result, in [45], a beautiful use of interpolation theory on modulation spaces allows to combine the estimates obtained by means of the classical definition in [1, 2] and the isometric definition in [44, 45], to obtain more general fixed time estimates in this framework. In order to control the growth of singularity at $t = 0$, we usually have the restriction $d(1/2 - 1/p) \leqslant 1$; cf. [5, 26]. By using the isometric decomposition in the frequency space, as in [44, 45], one can remove the singularity at $t = 0$ and preserve the decay at $t = \infty$ in certain modulation spaces.

The Strichartz estimates can be applied, e.g., to the well-posedness of non-linear Schrödinger equations or of linear Schrödinger equations with time-dependent potentials. We shall show examples in the last Section 5.

Notation. We define $|x|^2 = x \cdot x$, for $x \in \mathbb{R}^d$, where $x \cdot y = xy$ is the inner product on $\mathbb{R}^d$. The space of smooth functions with compact support is denoted by $\mathscr{C}_0^\infty(\mathbb{R}^d)$, the Schwartz class by $\mathscr{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathscr{S}'(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\xi) = \mathscr{F} f(\xi) = \int f(t) e^{-2\pi i t \xi} dt$. Translation and modulation operators (*time and frequency shifts*) are defined, respectively, by

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_\xi f(t) = e^{2\pi i \xi t} f(t).$$

We have the formulas $(T_x f)\widehat{} = M_{-x}\hat{f}$, $(M_\xi f)\widehat{} = T_\xi \hat{f}$, and $M_\xi T_x = e^{2\pi i x\xi} T_x M_\xi$. The notation $A \lesssim B$ means $A \le cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ means $c^{-1}A \le B \le cA$, for some $c \ge 1$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous embedding of the linear space B_1 into B_2.

2 Function Spaces and Preliminaries

In this section we present the function/distribution spaces we work with, and the properties used in our study.

2.1 Lorentz spaces

([34, 35]). We recall that the Lorentz space $L^{p,q}$ on $\mathbb{R}^d$ is defined as the space of tempered distributions f such that

$$\|f\|^*_{pq} = \left(\frac{q}{p}\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\right)^{1/q} < \infty,$$

when $1 \le p < \infty$, $1 \le q < \infty$, and

$$\|f\|^*_{pq} = \sup_{t>0} t^{1/p} f^*(t) < \infty$$

when $1 \le p \le \infty$, $q = \infty$. Here, as usual, $\lambda(s) = |\{|f| > s\}|$ denotes the distribution function of f and $f^*(t) = \inf\{s : \lambda(s) \le t\}$.

One has $L^{p,q_1} \hookrightarrow L^{p,q_2}$ if $q_1 \le q_2$, and $L^{p,p} = L^p$. Moreover, for $1 < p < \infty$ and $1 \le q \le \infty$, $L^{p,q}$ is a normed space and its norm $\|\cdot\|_{L^{p,q}}$ is equivalent to the above quasi-norm $\|\cdot\|^*_{pq}$.

The function $|x|^{-\alpha}$ lives in $L^{d/\alpha,\infty}$, $0 < \alpha < d$ but observe that this function doesn't live in any L^p, $1 \leqslant p \leqslant \infty$. We now recall the following classical Hardy-Littlewood-Sobolev fractional integration theorem (see e.g. [33, Theorem 1, pag 119] and [34]), which will be used in the sequel

Proposition 2.1. Let $d \geqslant 1$, $0 < \alpha < d$ and $1 < p < q < \infty$ such that

$$\frac{1}{q} = \frac{1}{p} - \frac{d-\alpha}{d}. \tag{6}$$

Then the following estimate

$$|||\cdot|^{-\alpha} * f||_q \lesssim ||f||_p \tag{7}$$

holds for all $f \in L^p(\mathbb{R}^d)$.

Potential and Sobolev spaces. For $s \in \mathbb{R}$, we define the Fourier multipliers $\langle\Delta\rangle^s f = \mathscr{F}^{-1}((1+|\cdot|^2)^{s/2}\hat{f})$, and $|\Delta|^s f = \mathscr{F}^{-1}(|\cdot|^s\hat{f})$. Then, for $1 \le p \le \infty$, the potential space [4] is defined by

$$W_s^p = \{f \in \mathscr{S}', \langle\Delta\rangle^s f \in L^p\}$$

with norm $\|f\|_{W_s^p} = \|\langle\Delta\rangle^s f\|_{L^p}$. The homogeneous potential space [4] is defined by

$$\dot{W}_s^p = \{f \in \mathscr{S}', |\langle\Delta\rangle|^s f \in L^p\}$$

with norm $\|f\|_{\dot{W}_s^p} = \||\Delta|^s f\|_{L^p}$.

For $p = 2$ the previous spaces are called Sobolev spaces H_s^p and homogeneous Sobolev spaces $\dot{H}_s^p$, respectively.

2.2 Wiener amalgam spaces

([12, 14, 15, 16, 17]). Let $g \in \mathscr{C}_0^\infty$ be a test function that satisfies $\|g\|_{L^2} = 1$. We will refer to g as a window function. For $1 \le p \le \infty$, recall the $\mathscr{F}L^p$ spaces, defined by

$$\mathscr{F}L^p(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d) : \exists h \in L^p(\mathbb{R}^d), \hat{h} = f\};$$

they are Banach spaces equipped with the norm

$$\|f\|_{\mathscr{F}L^p} = \|h\|_{L^p}, \quad \text{with} \, \hat{h} = f.$$

In the same way, for $1 < p < \infty$, $1 \le q \le \infty$, the Banach spaces $\mathscr{F}L^{p,q}$ are defined by

$$\mathscr{F}L^{p,q}(\mathbb{R}^d) = \{f \in \mathscr{S}'(\mathbb{R}^d) : \exists h \in L^{p,q}(\mathbb{R}^d), \hat{h} = f\};$$

equipped with the norm

$$\|f\|_{\mathscr{F}L^{p,q}} = \|h\|_{L^{p,q}}, \quad \text{with} \, \hat{h} = f.$$

Let B one of the following Banach spaces: $L^p, \mathscr{F}L^p$, $1 \le p \le \infty$, $\mathscr{F}L^{p,q}$, $1 < p < \infty$, $1 \le q \le \infty$, valued in a Banach space, or also spaces obtained from these by real or complex interpolation. Let C be the L^p space, $1 \le p \le \infty$, scalar-valued. For any given function f which is locally in B (i.e. $gf \in B$, $\forall g \in \mathscr{C}_0^\infty$), we set $f_B(x) = \|fT_x g\|_B$.

The *Wiener amalgam space* $W(B,C)$ with local component B and global component C is defined as the space of all functions f locally in B such that $f_B \in C$. Endowed with the norm $\|f\|_{W(B,C)} = \|f_B\|_C$, $W(B,C)$ is a Banach space. Moreover, different choices of $g \in \mathscr{C}_0^\infty$ generate the same space and yield equivalent norms.

If $B = \mathscr{F}L^1$ (the Fourier algebra), the space of admissible windows for the Wiener amalgam spaces $W(\mathscr{F}L^1, C)$ can be enlarged to the so-called Feichtinger algebra $W(\mathscr{F}L^1, L^1)$. Recall that the Schwartz class $\mathscr{S}$ is dense in $W(\mathscr{F}L^1, L^1)$.

We use the following definition of mixed Wiener amalgam norms. Given a measurable function F of the two variables (t,x) we set

$$\|F\|_{W(L^{q_1},L^{q_2})_t W(\mathscr{F}L^{r_1},L^{r_2})_x} = \| \|F(t,\cdot)\|_{W(\mathscr{F}L^{r_1},L^{r_2})_x}\|_{W(L^{q_1},L^{q_2})_t}.$$

Observe that [6]

$$\|F\|_{W(L^{q_1},L^{q_2})_t W(\mathscr{F}L^{r_1},L^{r_2})_x} = \|F\|_{W\left(L_t^{q_1}(W(\mathscr{F}L_x^{r_1},L_x^{r_2})),L_t^{q_2}\right)}.$$

The following properties of Wiener amalgam spaces will be frequently used in the sequel.

Lemma 2.1. *Let B_i, C_i, $i=1,2,3$, be Banach spaces such that $W(B_i,C_i)$ are well defined. Then,*

(i) Convolution. *If $B_1 * B_2 \hookrightarrow B_3$ and $C_1 * C_2 \hookrightarrow C_3$, we have*

$$W(B_1,C_1) * W(B_2,C_2) \hookrightarrow W(B_3,C_3). \tag{8}$$

In particular, for every $1 \le p,q \le \infty$, we have

$$\|f * u\|_{W(\mathscr{F}L^p,L^q)} \le \|f\|_{W(\mathscr{F}L^\infty,L^1)} \|u\|_{W(\mathscr{F}L^p,L^q)}. \tag{9}$$

(ii) Inclusions. *If $B_1 \hookrightarrow B_2$ and $C_1 \hookrightarrow C_2$,*

$$W(B_1,C_1) \hookrightarrow W(B_2,C_2).$$

Moreover, the inclusion of B_1 into B_2 need only hold "locally" and the inclusion of C_1 into C_2 "globally". In particular, for $1 \le p_i,q_i \le \infty$, $i=1,2$, we have

$$p_1 \ge p_2 \text{and } q_1 \le q_2 \Longrightarrow W(L^{p_1},L^{q_1}) \hookrightarrow W(L^{p_2},L^{q_2}). \tag{10}$$

(iii) Complex interpolation. *For $0<\theta<1$, we have*

$$[W(B_1,C_1),W(B_2,C_2)]_{[\theta]} = W\left([B_1,B_2]_{[\theta]},[C_1,C_2]_{[\theta]}\right),$$

if C_1 or C_2 has absolutely continuous norm.

(iv) Duality. *If B',C' are the topological dual spaces of the Banach spaces B,C respectively, and the space of test functions $\mathscr{C}_0^\infty$ is dense in both B and C, then*

$$W(B,C)' = W(B',C'). \tag{11}$$

The proof of all these results can be found in ([12, 14, 15, 22]).

Finally, let us recall the following lemma [8, Lemma 6.1], that will be used in the last Section 5.

Lemma 2.2. *Let* $1 \leq p,q,r \leq \infty$. *If*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r'}, \tag{12}$$

then

$$W(\mathscr{F}L^{p'},L^p)(\mathbb{R}^d)\cdot W(\mathscr{F}L^{q'},L^q)(\mathbb{R}^d)\subset W(\mathscr{F}L^r,L^{r'})(\mathbb{R}^d) \tag{13}$$

with norm inequality $\|fh\|_{W(\mathscr{F}L^r,L^{r'})}\lesssim \|f\|_{W(\mathscr{F}L^{p'},L^p)}\|h\|_{W(\mathscr{F}L^{q'},L^q)}$.

2.3 Modulation spaces

([13, 21]). Let $g\in\mathscr{S}(\mathbb{R}^d)$ be a non-zero window function and consider the so-called short-time Fourier transform (STFT) $V_g f$ of a function/tempered distribution f with respect to the the window g:

$$V_g f(x,\xi)=\langle f,M_\xi T_x g\rangle=\int e^{-2\pi i\xi y}f(y)\overline{g(y-x)}\,dy,$$

i.e., the Fourier transform $\mathscr{F}$ applied to $f\overline{T_x g}$.

For $s\in\mathbb{R}$, we consider the weight function $\langle x\rangle^s=(1+|x|^2)^{s/2}, x\in\mathbb{R}^d$. If $1\leq p,q\leq\infty$, $s\in\mathbb{R}$, the *modulation space* $\mathcal{M}^{p,q}_s(\mathbb{R}^d)$ is defined as the closure of the Schwartz class with respect to the norm

$$\|f\|_{\mathcal{M}^{p,q}_s}=\left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_g f(x,\xi)|^p\,dx\right)^{q/p}\langle\xi\rangle^{sq}\,d\xi\right)^{1/q}$$

(with obvious modifications when $p=\infty$ or $q=\infty$).

Among the properties of modulation spaces, we record that they are Banach spaces whose definition is independent of the choice of the window $g\in\mathscr{S}(\mathbb{R}^d)$, $\mathcal{M}^{2,2}=L^2$, $(\mathcal{M}^{p,q}_s)'=\mathcal{M}^{p',q'}_{-s}$, whenever $p,q<\infty$.

Another definition of these spaces uses the unite-cube decomposition of the frequency space, we address interested readers to [44].

Finally we recall the behaviour of modulation spaces with respect to complex interpolation (see [14, Corollary 2.3]).

Proposition 2.2. Let $1\leq p_1,p_2,q_1,q_2\leq\infty$, with $q_2<\infty$. If T is a linear operator such that, for $i=1,2$,

$$\|Tf\|_{M^{p_i,q_i}}\leq A_i\|f\|_{M^{p_i,q_i}}\quad \forall f\in M^{p_i,q_i},$$

then

$$\|Tf\|_{M^{p,q}}\leq CA_1^{1-\theta}A_2^{\theta}\|f\|_{M^{p,q}}\quad \forall f\in M^{p,q},$$

where $1/p=(1-\theta)/p_1+\theta/p_2$, $1/q=(1-\theta)/q_1+\theta/q_2$, $0<\theta<1$ and C is independent of T.

We observe that definition and properties of modulation spaces refer to the case $p,q\geqslant 1$. For the quasi-Banach case $0<p,q<1$ see, e.g., [2, 44, 45].

2.4 T^*T method

[19, 20] The T^*T method is an abstract tool of Harmonic Analysis, discovered by Tomas in 1975. This method allows to know the continuity of a linear operator T (and thus of its adjoint T^*), simply by the boundedness of the composition operator T^*T.

For any vector space D, we denote by D_a^* its algebraic dual, by $\mathscr{L}_a(D,X)$ the space of linear maps from D to some other vector space X, and by $\langle \varphi, f\rangle_D$ the pairing between D_a^* and D ($f \in D$, $\varphi \in D_a^*$), taken to be linear in f and antilinear in φ.

Lemma 2.3. *Let $\mathscr{H}$ be a Hilbert space, X a Banach space, X^* the dual of X, and D a vector space densely contained in X. Let $T \in \mathscr{L}_a(D,\mathscr{H})$ and $T^* \in \mathscr{L}_a(\mathscr{H},D_a^*)$ be its adjoint, defined by*

$$\langle T^*h, f\rangle_D = \langle h, Tf\rangle, \qquad \forall f \in D, \quad \forall h \in \mathscr{H},$$

where $\langle,\rangle$ is the inner product in $\mathscr{H}$ (antilinear in the first argument). Then the following three conditions are equivalent.
(1) There exists $a, 0 \leqslant a < \infty$ such that for all $f \in D$

$$\|Tf\|_{\mathscr{H}} \leqslant a\|f\|_X; \tag{14}$$

(2) Let $h \in \mathscr{H}$. Then T^*h can be extended to a continuous linear functional on X, and there exists $a, 0 \leqslant a < \infty$, such that for all $h \in \mathscr{H}$

$$\|T^*h\|_{X^*} \leqslant a\|h\|_{\mathscr{H}}. \tag{15}$$

(3) Let $f \in X$. Then T^*Tf can be extended to a continuous linear functional on X, and there exists $a, 0 \leqslant a < \infty$, such that for all $f \in D$,

$$\|T^*Tf\|_{X^*} \leqslant a^2\|f\|_X. \tag{16}$$

The constant a is the same in all the three cases. If one of (all) those conditions is (are) satisfied, the operators T and T^*T extend by continuity to bounded operators from X to $\mathscr{H}$ and from X to X^*, respectively.

Proof. From the fact that D is densely contained in X, it follows that X^* is a subspace of D_a^*.
(1) $\Rightarrow$ (2). Let $h \in \mathscr{H}$. Then, for all $f \in D$

$$|\langle T^*h, f\rangle_D| = |\langle h, Tf\rangle| \leqslant ||h||_{\mathscr{H}}\, ||Tf||_{\mathscr{H}} \leqslant a\, ||h||_{\mathscr{H}}\, ||f||_X.$$

(2) $\Rightarrow$ (1). Let $f \in D$. Then, for all $h \in \mathscr{H}$

$$|\langle h, Tf\rangle| = |\langle T^*h, f\rangle_D| \leqslant ||T^*h||_{X^*}\, ||f||_X \leqslant a\, ||h||_{\mathscr{H}}\, ||f||_X$$

Clearly (1) and (2) imply (3), and therefore (1) or (2) implies (3).
(3) $\Rightarrow$ (1). Let $f \in D$. Then

$$||Tf||^2 = |\langle Tf, Tf\rangle| = |\langle T^*Tf, f\rangle_D| \leqslant ||T^*Tf||_{X^*}\,||f||_X \leqslant a^2\,||f||_X^2.$$

Since D is a dense subspace of X, we see that T can be extended to a bounded linear functional from X to $\mathscr{H}$. ◻

The following corollary is extremely useful.

Corollary 2.3. *Let $\mathscr{H}, \mathscr{D}$ and two triplets (X_i, T_i, a_i), $i = 1,2$, satisfy the conditions of Lemma 2.3. Then for all choices of $i, j = 1,2$, $\mathscr{R}(T_i^* T_j) \subset X_i^*$ and for all $f \in D$,*

$$\|T_i^* T_j f\|_{X_i^*} \leqslant a_i a_j \|f\|_{X_j}. \tag{17}$$

In particular, $T_i^* T_j$ extends by continuity to a bounded operator from X_j to X_i^*, and (17) *holds for all $f \in X_j$.*

Ginibre and Velo [19] applied Lemma 2.3 and Corollary 2.3 to the bounded operator $T : L^1(I, \mathscr{H}) \to \mathscr{H}$, defined by

$$Tf = \int_I U(-t)f(t)dt, \tag{18}$$

where I is an interval of $\mathbb{R}$ (possibly $\mathbb{R}$ itself) and U a unitary strongly continuous one parameter group in $\mathscr{H}$. Then its adjoint T^* is the operator

$$T^*h(t) = U(t)h$$

from $\mathscr{H}$ to $L^\infty(I, \mathscr{H})$, where the duality is defined by the scalar products in $\mathscr{H}$ and in $L^2(I, \mathscr{H})$, such that T^*T is the bounded operator from $L^1(I, \mathscr{H})$ to $L^\infty(I, \mathscr{H})$ given by

$$T^*Tf = \int_I U(t-t')f(t')dt'.$$

Clearly the conditions of Lemma 2.3 are satisfied with $X = L^1(I, \mathscr{H})$, the operator T defined in (18), the constant $a = 1$, and $\mathscr{D}$ any dense subspace of X.

Let us introduce the retarded operator $(T^*T)_R$, defined by

$$(T^*T)_R f(t) = (U_R *_t f)(t) = \int_I U_R(t-t')f(t')dt'$$

where $U_R(t) = \chi_+(t)U(t) := \chi_{[0,\infty)}(t)U(t)$.

We recall that a space X of distributions in space-time is said to be *time cut-off stable* if the multiplication by the characteristic function χ_J, of an interval J in time, is a bounded operator in X with norm uniformly bounded with respect to J. The spaces under our consideration are of the type $X = L_t^q(I, Y)$, where Y is a space of distribution in the space variable and for which that property obviously holds.

Lemma 2.4. *Let $\mathscr{H}$ an Hilbert space, let I be an interval of $\mathbb{R}$, let $X \subset \mathscr{S}'(I \times \mathbb{R}^d)$ be a Banach space, let X be time cut-off stable, and let the conditions of Lemma 2.3 hold for the operator T defined in* (18). *Then the operator $(T^*T)_R$ is (strictly speaking extends to) a bounded operator from $L^1_t(I,\mathscr{H})$ to X^* and from X to $L^\infty_t(I,\mathscr{H})$.*

Proof. We recall the proof for sake of clarity. It is enough to demonstrate the second property, from which the first one follows by duality. Let $f \in D$. Then, for each t

$$\begin{aligned}\|(T^*T)_R f(t)\|_{\mathscr{H}} = \|T\chi_+(t-\cdot)f\|_{\mathscr{H}} &\leqslant a \sup_t \{\|\chi_+(t-\cdot)\|_{\mathscr{B}(X)}\}\|f\|_X \\ &\leq Ca\|f\|_X,\end{aligned}$$

by the unitary of U, the estimate (14) of Lemma 2.3, and the time cut-off stability of X. ◻

3 Fixed Time Estimates

In this section we study estimates for the solution $u(t,x)$ to the Cauchy problem (1), for fixed t. Since multiplication on the Fourier transform side intertwines with convolution on the space side, formula (2) can be rewritten as

$$u(t,x) = (K_t * u_0)(x), \tag{19}$$

where K_t is the inverse Fourier transform of the multiplier $e^{-4\pi^2 it|\xi|^2}$, given by

$$K_t(x) = \frac{1}{(4\pi it)^{d/2}} e^{i|x|^2/(4t)}. \tag{20}$$

First, we establish the estimates for Lebesgue spaces. Since $e^{it\Delta}$ is a unitary operator, we obtain the L^2 *conservation law*

$$\|e^{it\Delta}u_0\|_{L^2(\mathbb{R}^d)} = \|u_0\|_{L^2(\mathbb{R}^d)}. \tag{21}$$

Furthermore, since $K_t \in L^\infty$ with $\|K_t\|_\infty \asymp t^{-d/2}$, applying Young inequality to the fundamental solution (19) we obtain the L^1 *dispersive estimate*

$$\|e^{it\Delta}u_0\|_{L^\infty(\mathbb{R}^d)} \lesssim |t|^{-d/2}\|u_0\|_{L^1(\mathbb{R}^d)}. \tag{22}$$

This shows that if the initial data u_0 has a suitable integrability in space, then the evolution will have a power-type decay in time. Using the Riesz-Thorin theorem (see, e.g., [35]), we can interpolate (21) and (22) to obtain the important L^p *fixed time estimates*

$$\|e^{it\Delta}u_0\|_{L^r(\mathbb{R}^d)} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)}\|u_0\|_{L^{r'}(\mathbb{R}^d)} \tag{23}$$

for all $2 \leqslant r \leqslant \infty$, with $1/r + 1/r' = 1$. These estimates represent the complete range of L^p to L^q fixed time estimates available. In this setting, the necessary conditions are usually obtained by scaling conditions (see, for example, [39, Exercise 2.35], and [29] for the interpretation in terms of Gaussian curvature of the characteristic manifold). The following proposition ([50, page 45]) is an example of this technique in the case $p = q'$.

Proposition 3.1. Let $1 \leqslant r \leqslant \infty$ and $\alpha \in \mathbb{R}$ such that

$$\|e^{it\Delta}u_0\|_{L^r(\mathbb{R}^d)} \leqslant Ct^\alpha \|u_0\|_{L^{r'}(\mathbb{R}^d)}, \tag{24}$$

for all $u_0 \in S(\mathbb{R}^d)$, $t \neq 0$ and some C independent of t and u_0. Then $\alpha = -d(\frac{1}{2} - \frac{1}{r})$, $r' \leqslant r$ (and thus $2 \leqslant r \leqslant \infty$).

Proof. We can rescale the initial data u_0 by a factor λ and use (24) for

$$v(x) := u_0(\lambda x), \qquad \lambda > 0, \quad u_0 \in \mathscr{S}(\mathbb{R}^d).$$

The corresponding solution with $v(x)$ as initial data is $u(\lambda^2 t, \lambda x)$, where $u(t,x) = e^{it\Delta}u_0$. Therefore, by (24) and the scaling property

$$\|f(\lambda\cdot)\|_r = \lambda^{-d/r}\|f(\cdot)\|_r$$

one has

$$\lambda^{-d/r}\|u(\lambda^2 t, \cdot)\|_{L^r(\mathbb{R}^d)} \leqslant Ct^\alpha \lambda^{-d/r'}\|u_0\|_{L^{r'}(\mathbb{R}^d)},$$

for all $\lambda > 0$, $t \neq 0$ and $u_0 \in S(\mathbb{R}^d)$. Choosing $t = \lambda^{-2}$, we obtain

$$\|u(1,\cdot)\|_{L^r(\mathbb{R}^d)} \leqslant C\lambda^{-2\alpha - \frac{d}{r'} + \frac{d}{r}}\|u_0\|_{L^{r'}(\mathbb{R}^d)},$$

for all $\lambda > 0$ and $u_0 \in S(\mathbb{R}^d)$. Since $\|u(1,\cdot)\|_{L^r(\mathbb{R}^d)}$ and $\|u_0\|_{L^{r'}(\mathbb{R}^d)}$ are two positive constants, we have

$$\begin{aligned} &\text{for } \lambda \to \infty, \qquad -2\alpha - \frac{d}{r'} + \frac{d}{r} \geqslant 0, \\ &\text{for } \lambda \to 0, \qquad -2\alpha - \frac{d}{r'} + \frac{d}{r} \leqslant 0 \end{aligned}$$

and then we obtain the necessary condition for α. Moreover, since $e^{it\Delta}$ is invariant under translation, by [23, Theorem 1.1] we obtain $r' \leqslant r$, i.e., $2 \leqslant r \leqslant \infty$. By standard density argument we attain the desired result. ∎

For $s \in \mathbb{R}$, consider the Fourier multiplier $\langle\Delta\rangle^s$, defined by $\langle\Delta\rangle^s f = \mathscr{F}^{-1}(\langle\cdot\rangle^s \hat{f})$. Then, from (23) and the commutativity property of Fourier multipliers, one immediately obtains the $W^{s,r}$ *fixed time estimates*

$$\|e^{it\Delta}u_0\|_{W^{s,r}(\mathbb{R}^d)} \lesssim |t|^{-d\left(\frac{1}{2}-\frac{1}{r}\right)}\|u_0\|_{W^{s,r'}(\mathbb{R}^d)} \tag{25}$$

for all $s \in \mathbb{R}, 2 \leqslant r \leqslant \infty$, $1/r + 1/r' = 1$. Finally, we note that the conservation law (21) can be rephrased in this setting as the H^s *conservation law*

$$\|e^{it\Delta}u_0\|_{H^s(\mathbb{R}^d)} = \|u_0\|_{H^s(\mathbb{R}^d)}. \tag{26}$$

The Schrödinger propagator does not preserve any $W^{s,r}$ norm other than the H^s norm.

Now, we focus on Wiener amalgam spaces. K_t in (20) lives in $W(\mathscr{F}L^1, L^\infty) \subset L^\infty$, see [1, 6, 44]. This is the finest Wiener amalgam space-norm for K_t which, consequently, gives the worst behavior in the time variable. It is also possible to improve the latter, at the expense of a rougher x-norm, see [8]. Indeed, since $K_t \in W(\mathscr{F}L^p, L^\infty)$ with norm (see [8, Corollary 3.1])

$$\|K_t\|_{W(\mathscr{F}L^p,L^\infty)} \asymp |t|^{-d/p}(1+t^2)^{(d/2)(1/p-1/2)}, \tag{27}$$

from the fundamental solution (19) and the convolution relations for Wiener amalgam spaces in Lemma 2.1(i), it turns out, for $2 \le q \le \infty$, the $W(\mathscr{F}L^p, L^q)$ *dispersive estimates*

$$\|e^{it\Delta}u_0\|_{W(\mathscr{F}L^{q'},L^\infty)} \lesssim |t|^{d(2/q-1)}(1+t^2)^{d(1/4-1/q)}\|u_0\|_{W(\mathscr{F}L^q,L^1)}. \tag{28}$$

As well as for Lebesgue spaces, we can use complex interpolation between the dispersive estimates (28) and the L^2 conservation law ($L^2 = W(\mathscr{F}L^2, L^2)$) to obtain the following $W(\mathscr{F}L^p, L^q)$ *fixed time estimates*, that combine [6, Theorem 3.5] and [8, Theorem 3.3].

Theorem 3.2. *For* $2 \le q, r, s \le \infty$ *such that*

$$\frac{1}{s} = \frac{1}{r} + \frac{2}{q}\left(\frac{1}{2} - \frac{1}{r}\right),$$

we have

$$\|e^{it\Delta}u_0\|_{W(\mathscr{F}L^{s'},L^r)} \lesssim |t|^{d\left(\frac{2}{q}-1\right)\left(1-\frac{2}{r}\right)}(1+t^2)^{d\left(\frac{1}{4}-\frac{1}{q}\right)\left(1-\frac{2}{r}\right)}\|u_0\|_{W(\mathscr{F}L^s,L^{r'})} \tag{29}$$

In particular, for $s = 2$,

$$\|e^{it\Delta}u_0\|_{W(L^2,L^r)} \lesssim (1+t^2)^{-\frac{d}{2}\left(\frac{1}{2}-\frac{1}{r}\right)}\|u_0\|_{W(L^2,L^{r'})}, \tag{30}$$

and, for $s = r$,

$$\|e^{it\Delta}u_0\|_{W(\mathscr{F}L^{r'},L^r)} \lesssim (|t|^{-2}+|t|^{-1})^{d\left(\frac{1}{2}-\frac{1}{r}\right)}\|u_0\|_{W(\mathscr{F}L^r,L^{r'})}. \tag{31}$$

Proof. Let us sketch the proof for the sake of readers. Estimate (29) follow by complex interpolation between estimate (28), which corresponds to $r = \infty$, and (21), which corresponds to $r = 2$.

Indeed, $L^2 = W(\mathscr{F}L^2, L^2) = W(L^2, L^2)$. Using Lemma 2.1(iii), with $\theta = 2/r$ (observe that $0 < 2/r < 1$), and $1/s' = (1-2/r)/q' + (2/r)/2$, so that relation (29) holds, we obtain

$$\left[W(\mathscr{F}L^{q'}, L^\infty), W(\mathscr{F}L^2, L^2)\right]_{[\theta]} = W\left([\mathscr{F}L^{q'}, \mathscr{F}L^2]_{[\theta]}, [L^\infty, L^2]_{[\theta]}\right)$$

$$= W(\mathscr{F}L^{s'}, L^r)$$

and

$$\begin{aligned}\left[W(\mathscr{F}L^q, L^1), W(\mathscr{F}L^2, L^2)\right]_{[\theta]} &= W\left([\mathscr{F}L^q, \mathscr{F}L^2]_{[\theta]}, [L^1, L^2]_{[\theta]}\right)\\ &= W(\mathscr{F}L^s, L^{r'}).\end{aligned}$$

This yields the desired estimate (29). □

Let us compare the previous results with the classical L^p estimates. For $2 \leqslant r \leqslant \infty$, $\mathscr{F}L^{r'} \hookrightarrow L^r$, and the inclusion relations for Wiener amalgam spaces (Lemma 2.1 (ii)) yield $W(\mathscr{F}L^{r'}, L^r) \hookrightarrow W(L^r, L^r) = L^r$ and $L^{r'} = W(L^{r'}, L^{r'}) \hookrightarrow W(\mathscr{F}L^r, L^{r'})$. Thereby the estimate (31) is an improvement of (23) for every fixed time $t \neq 0$, and also uniformly for $|t| > c > 0$. Moreover, in [8] Cordero and Nicola proved that the range $r \geq 2$ in (31) is sharp, and the same for the decay $t^{-d\left(\frac{1}{2}-\frac{1}{r}\right)}$ at infinity and the bound $t^{-2d\left(\frac{1}{2}-\frac{1}{r}\right)}$, when $t \to 0$.

Modulation spaces are new settings inherited by time-frequency analysis where the fixed time estimates recently have been studied, see [1, 2, 44, 45]. Here, instead of using the representation of the solution $u(t,x)$ in (19), the solution is written in the form of Fourier multiplier $e^{it\Delta}u_0$ as in (2), see [1, 2]. Indeed, a sufficient condition for the boundedness of a Fourier multiplier on modulation spaces is that its symbol is in $W(\mathscr{F}L^1, l^\infty)$ ([1, Lemma 8]). Moreover, the Schrödinger symbol $\sigma = e^{-it|\xi|^2}$ lives in $W(\mathscr{F}L^1, l^\infty)$ and its norm is

$$\begin{aligned}\|\sigma\|_{W(\mathscr{F}L^1, l^\infty)} &= \sup_x \int_{\mathbb{R}^d} |V_g\sigma(x,\omega)|\, d\omega\\ &\asymp (1+t^2)^{-d/4} \int_{\mathbb{R}^d} e^{-\frac{\pi}{t^2+1}|\omega|^2}\, d\omega \asymp (1+t^2)^{d/4},\end{aligned}$$

where $g(\xi) = e^{-\pi|\xi|^2}$. Then, by [2, Lemma 2] (also for $s = 0$ [1, Corollary 18]) one has that $e^{it\Delta}$ extends to a bounded operator on $\mathscr{M}^{p,q}_s$, i.e., the $\mathscr{M}^{p,q}_s$ *fixed time estimates*

$$\|u(t,x)\|_{\mathscr{M}^{p,q}_s} \lesssim (1+|t|)^{d/2} \|u_0\|_{\mathscr{M}^{p,q}_s}, \tag{32}$$

for all $s \geqslant 0$ and $1 \leqslant p, q \leqslant \infty$. In particular, modulation space properties are preserved by the time evolution of the Schrödinger equation, in strong contrast with the case of Lebesgue spaces. Observe that (32), in the case $s = 0$, was also obtained using isometric decompositions in [44]. Later, Wang, Zaho, Guo in [45] obtain the following fixed time estimates

$$\|u(t,x)\|_{\mathscr{M}^{p,q}_s} \lesssim (1+|t|)^{-d(1/2-1/p)} \|u_0\|_{\mathscr{M}^{p',q}_s}, \tag{33}$$

for all $s \in \mathbb{R}$, $2 \leqslant p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$. Comparing (23) with (32) and (33), we see that the singularity at $t = 0$ contained in (23) has been removed in (32) and (33) and the decay rate in (33) when $t = \infty$ is the same one as in (23). The estimate (33) also indicates that $e^{it\Delta}$ is uniformly bounded on $\mathcal{M}^{2,q}$. The complex interpolation between the case $p=2$ in (33), and $p=\infty$ in (32) yields

$$\|u(t,x)\|_{\mathcal{M}^{p,q}_s} \lesssim (1+|t|)^{d(1/2-1/p)}\|u_0\|_{\mathcal{M}^{p,q}_s}, \tag{34}$$

for all $2\leqslant p\leqslant\infty$, $s\geqslant 0$. However, it is still not clear whether the growth order on time in the right-hand side of (34) is optimal.

4 Strichartz Estimates

In many applications, especially in the study of well-posedness of PDE's, it is useful to have estimates for the solution both in time and space variables. In this direction, the main result is represented by the *Strichartz estimates*. First, let us introduce the following definitions.

Definition 4.1. Following [26], we say that the exponent pair (q,r) is *Schrödinger-admissible* if $d\geqslant 1$ and

$$2\leqslant q,r\leqslant\infty,\quad \frac{1}{q}=\frac{d}{2}\Big(\frac{1}{2}-\frac{1}{r}\Big),\quad (q,r,d)\neq(2,\infty,2).$$

Definition 4.2. Following [18], we say that the exponent pair (q,r) is *Schrödinger-acceptable* if

$$1\leqslant q<\infty,\quad 2\leqslant r\leqslant\infty,\quad \frac{1}{q}<d\Big(\frac{1}{2}-\frac{1}{r}\Big),\ \text{or}\quad (q,r)=(\infty,2).$$

The original version of Strichartz estimates in L^p spaces, closely related to restriction problem of Fourier transform to surfaces, was elaborated by Robert Strichartz [36] in 1977(who, in turn, had precursors in [31, 41]). In 1995 a brilliant idea of Ginibre and Velo [20] was the use of the T^*T Method (Lemma 2.3) to detach the couple (q,r) from (q',r') (see also [49]). The study of the endpoint case $(q,r)=(2,2d/(d-2))$ is treated in [26], where Keel and Tao prove the estimate also for the endpoint when $d\geq 3$ (for $d=2$, the endpoint is $(q,r)=(2,\infty)$ and the estimate is false). We shall give a standard proof of the L^p *Stichartz estimates* in the non-endpoint cases [10, 50] (see also [39] where the following theorem is proved using an abstract lemma, the *Christ-Kiselev Lemma*, which is very useful in establishing retarded Strichartz estimates).

Theorem 4.3. *For any Schrödinger-admissible couples (q,r) and $(\tilde q,\tilde r)$ one has the homogeneous Strichartz estimates*

$$\|e^{it\Delta}u_0\|_{L^q_tL^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|u_0\|_{L^2_x(\mathbb{R}^d)}, \tag{35}$$

the dual homogeneous Strichartz estimates

$$\Big\|\int_{\mathbb{R}} e^{-is\Delta}F(s,\cdot)\,ds\Big\|_{L^2_x(\mathbb{R}^d)} \lesssim \|F\|_{L^{\tilde q'}_t L^{\tilde r'}_x(\mathbb{R}\times\mathbb{R}^d)}, \tag{36}$$

and the inhomogenous (retarded) Strichartz estimates

$$\left\| \int_{s<t} e^{i(t-s)\Delta} F(s,\cdot)\, ds \right\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)}. \tag{37}$$

Proof. We shall only prove this theorem in the non-endpoint case, when $q \neq 2$, addressing the interested reader to [26] for the whole study. We use the T^*T *method* as follows. Let (q,r) be Schrödinger admissible and consider the linear operator $T : L^1_t L^2_x \longrightarrow L^2_x$, defined as

$$T(F) = \int_{\mathbb{R}} e^{-is\Delta} F(s,\cdot)\, ds.$$

Its adjoint $T^* : L^2_x \longrightarrow L^\infty_t L^2_x$ is the Schrödinger propagator (2)

$$T^*(u) = e^{it\Delta} u.$$

Applying Minkowski's inequality, the fixed time estimate (23) and (6), we obtain the diagonal untruncated estimates

$$\begin{aligned} \left\| \int_{\mathbb{R}} e^{i(t-s)\Delta} F(s,\cdot)\, ds \right\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} &\leqslant \left\| \int_{\mathbb{R}} \|e^{i(t-s)\Delta} F(s,\cdot)\|_{L^r_x(\mathbb{R}^d)}\, ds \right\|_{L^q_t(\mathbb{R})} \\ &\lesssim \left\| \|F\|_{L^{r'}_x(\mathbb{R}^d)} * \frac{1}{|t|^{d(\frac{1}{2}-\frac{1}{r})}} \right\|_{L^q_t(\mathbb{R})} \lesssim \|F\|_{L^{q'}_t L^{r'}_x(\mathbb{R}\times\mathbb{R}^d)}, \end{aligned}$$

whenever $2 < q,r \leqslant \infty$ are such that $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$, and for any Schwartz function $F \in \mathscr{S}(\mathbb{R}\times\mathbb{R}^d)$. Then, using Lemma 2.3, one obtains the homogeneous Strichartz estimates (35) and the corresponding dual homogeneous Strichartz estimates (36). Corollary 2.3 applied to the previous two estimates yields the non-diagonal untruncate estimates:

$$\left\| \int_{\mathbb{R}} e^{i(t-s)\Delta} F(s,\cdot)\, ds \right\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} \leqslant \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x(\mathbb{R}\times\mathbb{R}^d)}.$$

By untruncated diagonal estimates one obtains the diagonal ones for the truncated operator, noting that

$$\begin{aligned} \left\| \int_{-\infty}^{t} e^{i(t-s)\Delta} F(s,\cdot)\, ds \right\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} &\leqslant \left\| \int_{-\infty}^{t} \|e^{i(t-s)\Delta} F(s,\cdot)\|_{L^r_x(\mathbb{R}^d)}\, ds \right\|_{L^q_t(\mathbb{R})} \\ &\leqslant \left\| \int_{\mathbb{R}} \|e^{i(t-s)\Delta} F(s,\cdot)\|_{L^r_x(\mathbb{R}^d)}\, ds \right\|_{L^q_t(\mathbb{R})} \lesssim \|F\|_{L^{q'}_t L^{r'}_x(\mathbb{R}\times\mathbb{R}^d)}. \end{aligned}$$

Moreover, using Lemma 2.4, with $X = L^{q'}_t L^{r'}_x$ and the truncated operator $(T^*T)_R F(t) = \int_0^t e^{i(t-s)\Delta} F(s)\, ds$, one obtains

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s,\cdot) ds \right\|_{L^\infty_t L^2_x(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L^{q'}_t L^{r'}_x}, \tag{38}$$

for all admissible pairs (q,r). Then, by complex interpolation between this estimate and the diagonal truncated ones above one gets the non-diagonal truncate estimates (37), for any couple $(q,r), (\tilde{q},\tilde{r})$ Schrödinger admissible. $\square$

The estimates are known to fail at the endpoint $(q,r,d) = (2,\infty,2)$, see [28], where Smith constructed a counterexample using the Brownian motion, although the homogeneous estimates can be saved assuming spherical symmetry [27, 32, 38]. The exponents in the homogeneous estimates are optimal ([39, Exercise 2.42]); some additional estimates are instead available in the inhomogeneous case (see, for example, [30]). Indeed, Kato [25] proved that inhomogeneous estimates (37) hold true when the pairs (q,r) and $(\tilde{q},\tilde{r})$ are Schrödinger acceptable and satisfy the scaling condition $1/q+1/\tilde{q} = d/2(1-1/r-1/\tilde{r})$ in the range $1/r, 1/\tilde{r} > (d-2)/(2d)$. Afterwards, for $d > 2$, Foschi [18] improved this result by looking for the optimal range of Lebesgue exponents for which inhomogeneous Strichartz estimates hold (results almost equivalent have recently obtained by Vilela [43]). Actually, this range is larger than the one given by admissible exponents for homogeneous estimates, as was shown by the following result [18, Proposition 24].

Proposition 4.4. If v is the solution to (3), with zero initial data and inhomogeneous term F supported on $\mathbb{R}\times\mathbb{R}^d$, then we have the estimate

$$\|v\|_{L_t^q L_x^r(\mathbb{R}\times\mathbb{R}^d)} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(\mathbb{R}\times\mathbb{R}^d)} \tag{39}$$

whenever $(q,r),(\tilde{q},\tilde{r})$ are Schrödinger acceptable pairs which satisfy the scaling condition

$$\frac{1}{q}+\frac{1}{\tilde{q}} = \frac{d}{2}\Big(1-\frac{1}{r}-\frac{1}{\tilde{r}}\Big),$$

and either the conditions

$$\frac{1}{q}+\frac{1}{\tilde{q}} < 1, \quad \frac{d-2}{r} \leqslant \frac{d}{\tilde{r}}, \quad \frac{d-2}{\tilde{r}} \leqslant \frac{d}{r}$$

or the conditions

$$\frac{1}{q}+\frac{1}{\tilde{q}} = 1, \quad \frac{d-2}{r} < \frac{d}{\tilde{r}}, \quad \frac{d-2}{\tilde{r}} < \frac{d}{r}, \quad \frac{1}{r} \leqslant \frac{1}{q}, \quad \frac{1}{\tilde{r}} \leqslant \frac{1}{\tilde{q}}.$$

For a discussion about the sharpness of this proposition we refer to [18], where explicit counterexamples are constructed to show the necessary conditions for inhomogeneous Strichartz estimates.

Since the Schrödinger operator $e^{it\Delta}$ commutes with Fourier multipliers like $|\Delta|^s$ or $\langle\Delta\rangle^s$, it is easy to obtain Strichartz estimates for potential and Sobolev spaces. In particular, if I is an interval containing the origin and $u: I\times\mathbb{R}^d \to \mathbb{C}$ is the solution to the inhomogeneous Schrödinger equation with initial data $u_0 \in \dot{H}_x^s(\mathbb{R}^d)$, given by the Duhamel formula (4), then, applying $|\Delta|^s$ to both sides of the equation and using the estimate of Theorem 4.3, one obtains

$$\|u\|_{L_t^q \dot{W}_x^{s,r}(I\times\mathbb{R}^d)} \lesssim \|u_0\|_{\dot{H}_x^s(\mathbb{R}^d)} + \|F\|_{L_t^{\tilde{q}'} \dot{W}_x^{s,\tilde{r}'}(I\times\mathbb{R}^d)}$$

for all Schrödinger admissible couples (q,r) and $(\tilde{q},\tilde{r})$. In particular, if one considers the homogeneous case (i.e. $F=0$), the Sobolev embedding $\dot{W}_x^{s,r} \hookrightarrow L_x^{r_1}$, $0<s<d/2$ and $1/r_1 = 1/r - s/d$, yields the $\dot{H}^s$ *Stichartz estimates*

$$\|u\|_{L_t^q L_x^{r_1}(I\times\mathbb{R}^d)} \lesssim \|u_0\|_{\dot{H}_x^s(\mathbb{R}^d)}, \quad \frac{2}{q}+\frac{d}{r_1}+s=\frac{d}{2}.$$

Since $s>0$ one has

$$\frac{2}{q}+\frac{n}{r_1}<\frac{n}{2},$$

hence, for any fixed value of s, the new Schrödinger admissible couple (q,r_1) lies on a parallel line below the corresponding case $s=0$.

Strichartz estimates in Wiener amalgam spaces enable us to control the local regularity and decay at infinity of the solution *separately*. For comparison, the classical estimates (35) can be rephrased in terms of Wiener amalgam spaces as follows:

$$\|e^{it\Delta}u_0\|_{W(L^q,L^q)_t W(L^r,L^r)_x} \lesssim \|u_0\|_{L_x^2}. \tag{40}$$

In this framework, Cordero and Nicola perform these estimates mainly in two directions. First, in [6], for $q \geqslant 4$ they modify the classical estimate (40) by (conveniently) moving local regularity from the time variable to the space variable. Indeed, $\mathscr{F}L^{r'} \subset L^r$ if $r \geqslant 2$, but the bound in (31) is worse than the one in (23), as $t \to 0$; consequently one has

$$\|e^{it\Delta}u_0\|_{W(L^{q/2},L^q)_t W(\mathscr{F}L^{r'},L^r)} \lesssim \|u_0\|_{L_x^2}, \tag{41}$$

for $4<q\leqslant\infty$, $2\leqslant r\leqslant\infty$, with (q,r) Schrödinger admissible. When $q=4$ the same estimate holds with the Lorentz space $L^{r',2}$ in place of $L^{r'}$. Dual homogeneous and retarded estimates hold as well. Thereby, the solution averages locally in time by the $L^{q/2}$ norm, which is rougher than the L^q norm in (35) or, equivalently, in (40), but it displays an $\mathscr{F}L^{r'}$ behavior locally in space, which is better than L^r. In [8] it is shown the sharpness of these Strichartz estimates, except for the threshold $q \geq 4$, which seems quite hard to obtain. Secondly, in [8], a converse approach is performed, by showing that it is possible to move local regularity in (35) from the space variable to the time variable. As a result, new estimates involving the Wiener amalgam spaces $W(L^p,L^q)$, that generalize (35), are obtained, i.e., the following [8, Theorem 1.1].

Theorem 4.5. *Let* $1\leq q_1,r_1\leq\infty$, $2\leq q_2,r_2\leq\infty$ *such that* $r_1\leq r_2$,

$$\frac{2}{q_1}+\frac{d}{r_1}\geq\frac{d}{2}, \tag{42}$$

$$\frac{2}{q_2}+\frac{d}{r_2}\leq\frac{d}{2}, \tag{43}$$

$(r_1,d)\neq(\infty,2)$, $(r_2,d)\neq(\infty,2)$ and, if $d\geq 3$, $r_1\leq 2d/(d-2)$. Assume the same for $\tilde q_1,\tilde q_2,\tilde r_1,\tilde r_2$. Then, we have the homogeneous Strichartz estimates

$$\|e^{it\Delta}u_0\|_{W(L^{q_1},L^{q_2})_t W(L^{r_1},L^{r_2})_x}\lesssim \|u_0\|_{L^2_x}, \tag{44}$$

the dual homogeneous Strichartz estimates

$$\|\int e^{-is\Delta}F(s)\,ds\|_{L^2}\lesssim \|F\|_{W(L^{\tilde q_1'},L^{\tilde q_2'})_t W(L^{\tilde r_1'},L^{\tilde r_2'})_x}, \tag{45}$$

and the retarded Strichartz estimates

$$\|\int_{s<t} e^{i(t-s)\Delta}F(s)\,ds\|_{W(L^{q_1},L^{q_2})_t W(L^{r_1},L^{r_2})_x}\lesssim \|F\|_{W(L^{\tilde q_1'},L^{\tilde q_2'})_t W(L^{\tilde r_1'},L^{\tilde r_2'})_x}. \tag{46}$$

This outcome is achieved by first establishing the estimates for the particular case $q_1=\tilde q_1=\infty$, $r_1=\tilde r_1=2$, and then by complex interpolation with the classical ones (35).

Figure 1 illustrates the range of exponents for the homogeneous estimates when $d\geq 3$. Notice that, if $q_1\leq q_2$, these estimates follow immediately from (40) and the inclusion relations of Wiener amalgam spaces. So, the issue consists in the cases $q_1>q_2$. Since there are no relations between the pairs (q_1,r_1) and (q_2,r_2) other than $r_1\leq r_2$, these estimates tell us, in a sense, that the analysis of the local regularity of the Schrödinger propagator is quite independent of its decay at infinity.

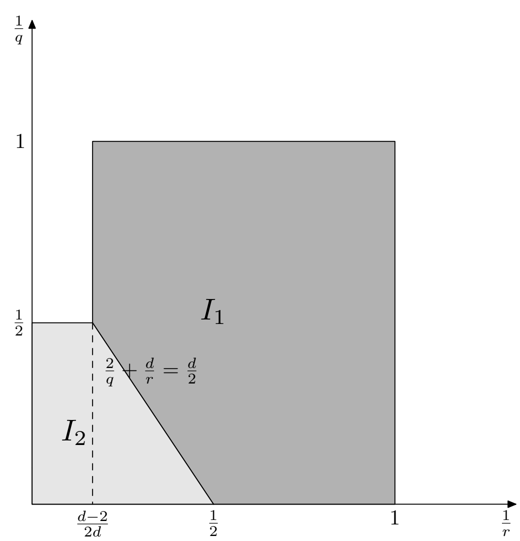

Figure 1: When $d\geq 3$, (44) holds for all pairs $(1/q_1,1/r_1)\in I_1$, $(1/q_2,1/r_2)\in I_2$, with $1/r_2\leq 1/r_1$.

In [8] it is proved that, for $d \geq 3$, all the constraints on the range of exponents in Theorem 4.5 are necessary, except for $r_1 \leq r_2$, $r_1 \leq 2d/(d-2)$, which is still left open. However, the following weaker result holds [8, Proposition 5.3]:

Assume $r_1 > r_2$ and $t \neq 0$. Then the propagator $e^{it\Delta}$ does not map $W(L^{r'_1}, L^{r'_2})$ continuously into $W(L^{r_1}, L^{r_2})$.

shows that the estimates (44) for exponents $r_1 > r_2$, if true, cannot be obtained from fixed-time estimates and orthogonality arguments. The arguments employed for the necessary conditions differ from the classical setting of Lebesgue spaces, because the general scaling consideration does not work directly. Indeed, the known bounds for the norm of the dilation operator $f(x) \longmapsto f(\lambda x)$ between Wiener amalgam spaces ([37, 40]), yield constraints which are weaker than the desired ones. So, the necessary conditions are obtained considering families of rescaled Gaussians as initial data, for which the action of the operator $e^{it\Delta}$ and the involved norms can be computed explicitly, see [8].

We end up this section with recalling Stricharz estimates for modulation spaces. The main result in this framework is due to Wang and Hudzik [45]. They use the same arguments as in Keel and Tao [26], who point out that the ranges of exponents (q,r) in (23) could most likely be not optimal. In fact, Keel and Tao show that if the semigroup $e^{it\Delta}$ satisfies the estimate

$$\|e^{it\Delta}u_0\|_{L^p} \lesssim (1+|t|)^{-d(1/2-1/p)}\|u_0\|_{L^{p'}} \tag{47}$$

then (35), (36) and (37) hold if one substitutes q and $\tilde{q}$ by any $\gamma \geq \max(q,2)$ and $\tilde{\gamma} \geqslant \max(\tilde{q},2)$, respectively. Since the estimate (34) is similar to (47), they optimize (35), (36) and (37) in the function spaces $M_s^{p,q}$ to cover the exponents (γ, q) and $(\tilde{\gamma}, \tilde{q})$ satisfying $\gamma \geqslant \max(q,2)$ and $\tilde{\gamma} \geqslant \max(\tilde{q},2)$. Since the precise formulation of these results requires the introduction of other function spaces, we refer interested readers to [45, Section 3].

5 Applications

We start by focusing on the Cauchy problem for the nonlinear Schrödinger equation (NLS)

$$\begin{cases} i\partial_t u + \Delta u + N(u) = 0 \\ u(0,x) = u_0(x). \end{cases} \tag{48}$$

The nonlinearity N considered will be either power-like

$$p_k(u) = \lambda |u|^{2k} u, \quad k \in \mathbb{N},\ \lambda \in \mathbb{R}$$

or exponential-like

$$e_\rho(u) = \lambda(e^{\rho|u|^2} - 1)u, \quad \lambda, \rho \in \mathbb{R}.$$

Both nonlinearities are smooth. The corresponding equations having power-like nonlinearities p_k are sometimes referred to as algebraic nonlinear Schrödinger equations. The sign of the coefficient λ determines the defocusing, absent, or focusing character of the nonlinearity.

We shall study the well-posedness of (48), in different spaces. Recall that the problem (48) is locally well-posed in e.g. $H^s_x(\mathbb{R}^d)$ if, for any $u_0^* \in H^s_x(\mathbb{R}^d)$, there exists a time $T>0$ and an open ball B in $H^s_x(\mathbb{R}^d)$ containing u_0^* and a subset of $C^0_t H^s_x([T,T]\times\mathbb{R}^d)$ such that for each $u_0 \in B$ there exists a unique solution $u \in X$ to the equation (48) and the map $u_0 \mapsto u$ is continuous from B (with the H^s_x topology) to X (with the $C^0_t H^s_x([T,T]\times\mathbb{R}^d)$ topology).

A fundamental tool in well-posedness theory is the *contraction theorem*. Let us first work abstractly, viewing (48) as an instance of the more general

$$u = u_{\text{lin}} + DN(u) \tag{49}$$

where $u_{\text{lin}} := e^{it\Delta}u_0$ is the linear solution, N is the nonlinearity and D is the Duhamel operator

$$DF(t,x) := \int_0^t e^{i(t-s)\Delta} F(s,\cdot)\,ds.$$

The following abstract tool [39, Proposition 1.38] then allows us to find the desired contraction map.

Proposition 5.1 (Abstract iteration argument)**.** Let $\mathcal{N},\mathcal{T}$ be two Banach spaces. Let $D: \mathcal{N}\to\mathcal{T}$ be a bounded linear operator with the bound

$$\|DF\|_{\mathcal{T}} \leqslant C_0\|F\|_{\mathcal{N}} \tag{50}$$

for all $F\in\mathcal{N}$ and some constant $C_0>0$, and let $N:\mathcal{S}\to\mathcal{N}$, with $N(0)=0$, be a nonlinear operator which is Lipschitz continuous and obeys the bounds

$$\|N(u)-N(v)\|_{\mathcal{N}} \leqslant \frac{1}{2C_0}\|u-v\|_{\mathcal{T}} \tag{51}$$

for all u,v in the ball $B_\epsilon := \{u\in\mathcal{S} : \|u\|_{\mathcal{T}}\leqslant\epsilon\}$, for some $\epsilon>0$. Then, for all $u_{\text{lin}}\in B_{\epsilon/2}$, there exists a unique solution $u\in B_\epsilon$ to the equation (49), with Lipschitz map $u_{\text{lin}}\mapsto u$ with constant at most 2. That is, we have

$$\|u\|_{\mathcal{T}} \leqslant 2\|u_{\text{lin}}\|_{\mathcal{T}} \tag{52}$$

Proof. Observe that for $v=0$ the estimate (51) becomes

$$\|N(u)\|_{\mathcal{N}} \leqslant \frac{1}{2C_0}\|u\|_{\mathcal{T}} \tag{53}$$

(since $N(0)=0$ by hypothesis). Then, fix $u_{\text{lin}}\in B_{\epsilon/2}$, and consider the map

$$\phi(u) := u_{\text{lin}} + DN(u).$$

Using (50) and (53) one has

$$||\phi(u)||_{\mathscr{T}} = ||u_{\text{lin}} + DN(u)||_{\mathscr{T}} \leqslant \frac{\epsilon}{2} + \frac{C_0}{2C_0}\epsilon = \epsilon$$

for all $u \in B_\epsilon$, i.e., ϕ maps the ball B_ϵ into B_ϵ. Moreover, ϕ is a contraction on B_ϵ, indeed by (50) and (51) one has

$$\begin{aligned} ||\phi(u) - \phi(v)||_{\mathscr{T}} &= ||DN(u) - DN(v)||_{\mathscr{T}} \leqslant C_0 ||N(u) - N(v)||_{\mathscr{N}} \\ &\leqslant C_0 \frac{1}{2C_0} ||u - v||_{\mathscr{T}} = \frac{1}{2} ||u - v||_{\mathscr{T}}, \end{aligned}$$

for all $u, v \in B_\epsilon$. Then, the contraction theorem asserts that there exists a unique fixed point u for ϕ and moreover the map $u_{\text{lin}} \mapsto u$ is Lipschitz with constant at most 2, that is (52). □

Proposition 5.1 is the main ingredient of the results in [1, 7, 8, 9, 39, 45].

First, consider the NLS (48) with $N = p_k$, with the initial data u_0 in the Sobolev space $H^s_x(\mathbb{R}^d)$. To study this Cauchy problem it is convenient to introduce a single space S^s that recaptures all the Strichartz norms at a certain regularity $H^s_x(\mathbb{R}^d)$ simultaneously. For sake of simplicity, we reduce to the case $s = 0$ which corresponds to the case L^2_x, introducing the *Strichartz space* $S^0(I \times \mathbb{R}^d)$, for any time interval I, defined as the closure of Schwartz class $\mathscr{S}$ with respect to the norm

$$||u||_{\mathscr{S}^0(I\times\mathbb{R}^d)} := \sup_A ||u||_{L^q_t L^r_x(I\times\mathbb{R}^d)},$$

where the set A is given by $A := \{(\infty, 2), (q, r)\}$, with (q, r) Schrödinger admissible. We define also the space $N^0(I \times \mathbb{R}^d) := L^{q'}_t L^{r'}_x$. Then, using Proposition 5.1 and the L^p Strichartz estimates of Theorem 4.3 one can prove the following [39, Proposition 3.15]

Theorem 5.2 (L^2_x subcritical solution)**.** *Let k be subcritical for L^2_x (that is, $0 < k < \frac{2}{d}$) and let $\mu = \pm 1$. Then the NLS* (48) *is locally well-posed in L^2_x in a subcritical sense. Indeed, for any $R > 0$ there exists a time $T > 0$ such that for all u_0 in the ball $B_R := \{u_0 \in L^2_x(\mathbb{R}^d) : ||u_0||_{L^2_x} < R\}$ there exists a unique solution u in L^2_x of* (48) *in the space $\mathscr{S}^0([-T, T]\times\mathbb{R}^d) \subset C^0_t L^2_x([-T, T]\times\mathbb{R}^d)$. Moreover, the map $u_0 \mapsto u$, from B_R to $\mathscr{S}^0([-T, T] \times \mathbb{R}^d)$, is Lipschitz continuous.*

For results in the framework of modulation spaces we address to [2, 44, 45]. In particular, we examine [2]. The main result, obtained only with the $M^{p,q}_s$ dispersive estimates (32), is the following.

Theorem 5.3. *Assume that $u_0 \in M^{p,1}_s(\mathbb{R}^d)$ and $N \in \{p_k, e_\rho\}$. Then, there exists $T = T(\|u_0\|_{M^{p,1}_s})$ such that* (48) *has a unique solution $u \in C^0 M^{p,1}_s([0, T] \times \mathbb{R}^d)$. Moreover, if $T < \infty$, then $\limsup_{t\to T} \|u(t, \cdot)\| = \infty$.*

Proof. The proof is simply an application of the abstract iteration argument. Let us write it for the nonlinearity $N = p_k$. We choose the spaces $\mathscr{T} := C^0 M_s^{p,1}([0,T]\times\mathbb{R})$, $\mathscr{N} := M_s^{p,1}$, and the Duhamel operator

$$D := \int_0^t e^{i(t-s)\Delta}\cdot ds.$$

Then, it is sufficient to prove (50) and (51) in this setting. Then, by the Minkowsky integral inequality, $M_s^{p,q}$ dispersive estimates (32) and [2, Corollary 2] one has

$$\begin{aligned}\Big\| \int_0^t e^{i(t-\tau)\Delta}(p_k(u))(\tau)d\tau \Big\|_{M_s^{p,1}} &\leqslant \int_0^t \|e^{i(t-\tau)\Delta}(p_k(u))(\tau)\|_{M_s^{p,1}} d\tau \\ &\leqslant c_1 T C_T \sup_{t\in[0,T]} \|p_k(u)(t)\|_{M_s^{p,1}} \\ &\leqslant c_1 c_2 C_T T \|u(t)\|^{2k+1}_{M_s^{p,1}}\end{aligned}$$

where $C_T = \sup_{t\in[0,T)}(1+|t|)^{d/2}$. Choosing $T>0$ such that $c_1c_2C_TT \leqslant C_0$, it follows (50) and by

$$p_k(u)(\tau) - p_k(v)(\tau) = \lambda(u-v)|u|^{2k}(\tau) + \lambda v(|u|^{2k} - |v|^{2k})(\tau),$$

it follows (51). ◻

For Wiener amalgam spaces there are no results for the NLS. In [8] there is a result concerning linear Schrödinger equations with time-dependent potentials. Indeed, in [8] the well-posedness is proved in L^2 of the following Cauchy problem, for all $d \geq 1$,

$$\begin{cases} i\partial_t u + \Delta u = V(t,x)u, & t\in[0,T] = I_T,\ x\in\mathbb{R}^d, \\ u(0,x) = u_0(x), \end{cases} \tag{54}$$

and for the class of potentials

$$V \in L^\alpha(I_T; W(\mathscr{F}L^{p'}, L^p)_x), \quad \frac{1}{\alpha} + \frac{d}{p} \leq 1,\ 1\leq\alpha<\infty.\ d<p\leq\infty. \tag{55}$$

Theorem 5.4. *Consider the class of potentials* (55)*. Then, for all* (q,r) *such that* $2/q + d/r = d/2$, $q>4, r\geq 2$*, the Cauchy problem* (54) *has a unique solution*

(i) $u \in \mathscr{C}(I_T; L^2(\mathbb{R})) \cap L^{q/2}(I_T; W(\mathscr{F}L^{r'}, L^r))$, *if* $d = 1$;

(ii) $u \in \mathscr{C}(I_T; L^2(\mathbb{R}^d)) \cap L^{q/2}(I_T; W(\mathscr{F}L^{r'}, L^r)) \cap$
$\cap L^2(I_T; W(\mathscr{F}L^{2d/(d+1),2}, L^{2d/(d-1)}))$, *if* $d > 1$.

Proof. It is enough to prove the case $d = 1$. Indeed, for $d \geq 2$, condition (55) implies $p > 2$, so that $\mathscr{F}L^{p'} \hookrightarrow L^p$ and the inclusion relations of Wiener amalgam spaces yield $W(\mathscr{F}L^{p'}, L^p) \hookrightarrow$

$W(L^p, L^p) = L^p$. Hence our class of potentials is a subclass of those of [6, Theorem 6.1], for which the quoted theorem provides the desired result.

We now turn to the case $d = 1$. The proof follows the ones of [9, Theorem 1.1, Remark 1.3] and [6, Theorem 6.1] (see also [49]).

First of all, since the interval I_T is bounded, by Hölder's inequality and by taking p large, we may assume $1/\alpha + d/p = 1$.

We choose a small time interval $J = [0, \delta]$ and set, for $q \geq 2$, $q \neq 4$, $r \geq 1$,

$$Z_{q/2,r} = L^{q/2}(J; W(\mathscr{F}L^{r'}, L^r)_x).$$

Now, fix an admissible pair (q_0, r_0) with r_0 arbitrarily large (hence $(1/q_0, 1/r_0)$ is arbitrarily close to $(1/4, 0)$) and set $Z = \mathscr{C}(J; L^2) \cap Z_{q_0/2, r_0}$, with the norm $\|v\|_Z = \max\{\|v\|_{\mathscr{C}(J;L^2)}, \|v\|_{Z_{q_0/2,r_0}}\}$. We have $Z \subset Z_{q/2,r}$ for all admissible pairs (q, r) obtained by interpolation between $(\infty, 2)$ and (q_0, r_0). Hence, by the arbitrary of (q_0, r_0) it suffices to prove that Φ defines a contraction in Z.

Consider now the integral formulation of the Cauchy problem, namely $u = \Phi(v)$, where

$$\Phi(v) = e^{it\Delta}u_0 + \int_0^t e^{i(t-s)\Delta}V(s)v(s)\,ds.$$

By the homogeneous and retarded Strichartz estimates in [6, Theorems 1.1, 1.2] the following inequalities hold:

$$\|\Phi(v)\|_{Z_{q/2,r}} \leq C_0\|u_0\|_{L^2} + C_0\|Vv\|_{Z_{(\tilde{q}/2)', \tilde{r}'}}, \tag{56}$$

for all admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$, $q > 4, \tilde{q} > 4$.

Consider now the case $1 \leq \alpha < 2$. We choose $((\tilde{q}/2)', \tilde{r}) = (\alpha, 2p/(p+2))$. Since $v \in L^\infty(J; L^2)$, applying (13) for $q = 2$ we get

$$\|Vv\|_{W(\mathscr{F}L^{\tilde{r}}, L^{\tilde{r}'})} \lesssim \|V\|_{W(\mathscr{F}L^{p'}, L^p)}\|v\|_{L^2},$$

whereas Hölder's Inequality in the time-variable gives

$$\|Vv\|_{Z_{(\tilde{q}/2)', \tilde{r}'}} \lesssim \|V\|_{L^\alpha(J; W(\mathscr{F}L^{p'}, L^p))}\|v\|_{L^\infty(J;L^2)}.$$

The estimate (56) then becomes

$$\|\Phi(v)\|_{Z_{q/2,r}} \leq C_0\|u_0\|_{L^2} + C_0\|V\|_{L^\alpha(J; W(\mathscr{F}L^{p'}, L^p))}\|v\|_{L^\infty(J;L^2)}. \tag{57}$$

By taking $(q, r) = (\infty, 2)$ or $(q, r) = (q_0, r_0)$ one deduces that $\Phi : Z \to Z$ (the fact that $\Phi(u)$ is *continuous* in t when valued in L^2_x follows from a classical limiting argument [9, Theorem 1.1, Remark 1.3]). Also, if J is small enough, $C_0\|V\|_{L^\alpha_t L^p_x} < 1/2$, and Φ is a contraction. This gives a unique solution in J. By iterating this argument a finite number of times one obtains a solution in $[0,T]$.

The case $2 \le \alpha < \infty$ is similar. □

This result generalizes [6, Theorem 6.1], by treating the one dimensional case as well and allowing the potentials to belong to Wiener amalgam spaces with respect to the space variable x. Other results on Schrödinger equations with potentials in $L_t^p L_x^q$ can be found in [9].

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On the Weyl Transform with Symbol in the Gel'fand-Shilov Space and its Dual Space

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ABSTRACT

In this paper, we claim two subjects. One is that the Weyl transform with symbol in the Gel'fand-Shilov space $\mathscr{S}_r^r$, $r \geq \frac{1}{2}$, is a trace class operator. The other one is that the Weyl transform with symbol in the generalized function $(\mathscr{S}_r^r)'$, $r \geq \frac{1}{2}$, is a continuous linear transformation from the Gel'fand-Shilov space $\mathscr{S}_r^r$ to $(\mathscr{S}_r^r)'$. As $r > 1$, Z. Lozanov-Crvenković and D. Perišić have proved in [6] this result. Our second claim includes their result.

RESUMEN

En este artículo afirmamos dos asuntos. El primero es que la transformada de Weyl con símbolo en el espacio de Gel'fand-Shilov $\mathscr{S}_r^r$, $r \geq \frac{1}{2}$, es un operador de clase trazo. El segundo asunto es que la transformación de Weyl con símbolo en las funciones generalizadas $(\mathscr{S}_r^r)'$, $r \geq \frac{1}{2}$, es una transformación lineal continua del espacio Gel'fand-Shilov $\mathscr{S}_r^r$ to $(\mathscr{S}_r^r)'$. Como $r > 1$, Z. Lozanov-Crvenković y D. Perišić probaron en [6] este resultado. Nuestro resultado incluye su resultado.

Key words and phrases: *Weyl transform, Gel'fand-Shilov space, Fourier-Wigner transform, trace class operator, Schwartz's kernel theorem.*

Math. Subj. Class.: *46F05; 46F15; 81R15; 81S40.*

1 Introduction

The subject of this article is to show the properties, as operators, of the Weyl transform with the symbol in the Gel'fand-Shilov space $\mathscr{S}_r^r$, $r \ge 1/2$, and its dual space $(\mathscr{S}_r^r)'$, $r \ge 1/2$.

The Weyl transform was first considered by Hermann Weyl arising in quantum mechanics in [14] and the properties of the Weyl transform as operators have been studied by many mathematicians. See for instance, [1], [6], [9], [11], [12], [13], [15] and others. These investigations are mainly to consider the correspondence between the functional space, in which the symbol belongs, and the operator class, in which the Weyl transform belongs.

There exist two remarkable results about these considerations: first, that the Weyl transform with the symbol in Schwartz class is a trace class operator in [13]; and secondly, that the Weyl transform with the symbol in $(\mathscr{S}_r^r)'$, $r > 1$, is a continuous and linear maps from $\mathscr{S}_r^r(\mathbb{R}^d)$ to $(\mathscr{S}_r^r)'(\mathbb{R}^d)$ in [6]. Our discussion is principally aimed at slightly developing these two results. They depend on two areas of study: first, the correspondence between the Weyl transform with the symbol in $\mathscr{S}_r^r$, $r \ge 1/2$, and the sequence space with some exponential decrease, and secondly, the study of the Schwartz's kernel theorem for $(\mathscr{S}_r^r)'$, $r \ge 1/2$. We consider these subjects in detail.

The plan of the paper is as follows. In the next section we introduce some properties of the Gel'fand-Shilov space. In section 3 we treat the Weyl transform with symbol in $\mathscr{S}_r^r$. In section 4 we show the Schwartz's kernel theorem for $(\mathscr{S}_r^r)'$ and the property of the Weyl transform with symbol in generalized functions. Through this article we always treat the index $r \ge 1/2$.

2 The Gel'fand-Shilov Space $\mathscr{S}_r^r$ and its Dual $(\mathscr{S}_r^r)'$

First of all, we give some notations. We use a multi-index $\alpha \in \mathbb{Z}_+^d$, namely, $\alpha = (\alpha_1 \cdots \alpha_d)$, where $\alpha_i \in \mathbb{Z}$ and $\alpha_i \ge 0$. So, for $x \in \mathbb{R}^d$, $x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$ and $\partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}$, where $\partial_{x_j}^{\alpha_j} = (\frac{\partial}{\partial x_j})^{\alpha_j}$.

Definition 1 ([4]). *Let* $A, B \in (0,\infty)^d$. For $r = (r_1, \cdots, r_d)$ and $r_i \ge 0$, $1 \le i \le d$,
$\mathscr{S}_{r,A}^{r,B}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \forall \delta \in (0,\infty)^d,\ \forall \rho \in (0,\infty)^d,\ \exists C_{\delta\rho} \ge 0$ *s.t.*

$$|x^k \partial_x^q \varphi(x)| \le C_{\delta\rho}(A+\delta)^k (B+\rho)^q k^{kr} q^{qr}\ ,\ \forall k, q \in \mathbb{Z}_+^d\},$$

where

$$(A+\delta)^k = (A_1+\delta_1)^{k_1} \cdots (A_d+\delta_d)^{k_d},$$
$$(B+\rho)^q = (B_1+\rho_1)^{q_1} \cdots (B_d+\rho_d)^{q_d}.$$

The space $\mathscr{S}^{r,B}_{r,A}(\mathbb{R}^d)$ is a Fréchet space with the semi-norms

$$\|\varphi\|^{\delta\rho} = \sup_{x,k,q} \frac{|x^k \partial_x^q \varphi(x)|}{(A+\delta)^k (B+\rho)^q k^{kr} q^{qr}} \ , \ \delta_i, \rho_i = 1, \frac{1}{2}, \frac{1}{3} \cdots .$$

The space $\mathscr{S}^r_r(\mathbb{R}^d)$ is given by the inductive limit

$$\mathscr{S}^r_r(\mathbb{R}^d) = \varinjlim_{A,B\to\infty} \mathscr{S}^{r,B}_{r,A}(\mathbb{R}^d).$$

The Gel'fand-Shilov space is the subspace of the Schwartz class $\mathscr{S}(\mathbb{R}^d)$.

Let $a \in (0,\infty)^d$ be $a = \frac{r}{eA^{\frac{1}{r}}}$. For any $a, B \in (0,\infty)^d$, we define the space $\mathscr{S}^{r,B}_{r,a}(\mathbb{R}^d)$ by
$\mathscr{S}^{r,B}_{r,a}(\mathbb{R}^d) = \{\varphi \in C^\infty(\mathbb{R}^d) \mid \forall \delta, \rho \in (0,\infty)^d,\ \exists C_{\delta\rho} > 0\ s.t.\ |\partial_x^q \varphi(x)| \le C_{\delta\rho}(B+\rho)^q q^{qr} e^{-a_\delta |x|^{\frac{1}{r}}},$
$\forall k, q \in \mathbb{Z}^d_+\}$,
where $a_\delta = \frac{r}{e(A+\delta)^{\frac{1}{r}}}$ and

$$\|\varphi\|_{\delta\rho} = \sup_{x,\beta} \frac{|\partial_x^\beta \varphi(x)|}{(B+\rho)^\beta \beta^{\beta r} e^{-a_\delta |x|^{\frac{1}{r}}}}.$$

The Gel'fand-Shilov spaces $\mathscr{S}^r_r(\mathbb{R}^d)$ enjoy the following properties [4]:

Proposition 1. *Let $\{\varphi_j\}$ be a sequence in $\mathscr{S}^r_r(\mathbb{R}^d)$. Then we obtain*

$$\varphi_j \longrightarrow 0 \ as\ j \longrightarrow +\infty\ in\ \mathscr{S}^r_r$$

if and only if there are positive constants B and a such that

$$\sup_{x,\beta} \frac{|\partial_x^\beta \varphi_j(x)|}{B^\beta \beta^{\beta r} e^{-a|x|^{\frac{1}{r}}}} \longrightarrow 0\ as\ j\ \longrightarrow +\infty.$$

Proposition 2. (i) $\mathscr{S}^r_r \equiv \{0\}$, $0 < r < \frac{1}{2}$.

(ii) *For $r_1 < r_2$, $\mathscr{S}^{r_1}_{r_1}$ is included in $\mathscr{S}^{r_2}_{r_2}$ and $\mathscr{S}^{r_1}_{r_1}$ is dense in $\mathscr{S}^{r_2}_{r_2}$.*

(iii) *Let $\hat{\mathscr{S}}^r_r$ be the image of the Fourier transform of $\mathscr{S}^r_r$. Then $\hat{\mathscr{S}}^r_r = \mathscr{S}^r_r$.*

Remark 1. As $r = 1$, the Gel'fand-Shilov space $\mathscr{S}^1_1(\mathbb{R}^d)$ is known to be isomorphism to the space of test functions of the Fourier-hyperfunctions [7].

We define the Hermite functions $\{h_n(x)\}_{n=0,1,2\cdots}$ on $\mathbb{R}^1$ by

$$h_n(x) = (2^n n!)^{-\frac{1}{2}} \pi^{-\frac{1}{4}} (-1)^n e^{\frac{x^2}{2}} \left(\frac{d}{dx}\right)^n e^{-x^2}.$$

It is known that the set $\{h_n(x)\}_{n=0,1,2,\cdots}$ is a complete orthonormal system in $L^2(\mathbb{R}^1)$. That is, for any f in $L^2(\mathbb{R}^1)$,

$$f(x) = \sum_{n=0}^{\infty} a_n h_n(x) \text{ in } L^2(\mathbb{R}^1),$$

where $a_n = (f, h_n) = \int_{\mathbb{R}^1} f(x) h_n(x) dx$. This expansion is called the Hermite expansions and $\{a_n\}_{n=0,1,2\cdots}$ is called the Hermite coefficients. For d-dimensions, the Hermite functions on $\mathbb{R}^d$ is defined by

$$h_\alpha(x) = h_{\alpha_1}(x_1) \otimes \cdots \otimes h_{\alpha_d}(x_d), \quad \alpha \in \mathbb{Z}_+^d\,,\ x \in \mathbb{R}^d.$$

The set $\{h_\alpha(x)\}_{\alpha \in \mathbb{Z}_+^d}$ is also a complete orthonormal system in $L^2(\mathbb{R}^d)$.

Proposition 3 ([17]). *Let $\phi \in \mathscr{S}_r^r(\mathbb{R}^d)$, $r \geq \frac{1}{2}$. Then there exist some constants $C > 0$ and $L \in (0,\infty)^d$ such that*

$$\phi = \sum_{|\alpha|=0}^{\infty} \left(\phi, h_\alpha\right) h_\alpha \text{ and } |\left(\phi, h_\alpha\right)| \leq C \exp(-L\alpha^{\frac{1}{2r}}).$$

Conversely, if $|a_\alpha| \leq C \exp(-L\alpha^{\frac{1}{2r}})$ for some constants $C > 0$ and $L \in (0,\infty)^d$, then the series $\sum_{|\alpha|=0}^{\infty} a_\alpha h_\alpha(x)$ converges to a function in $\mathscr{S}_r^r(\mathbb{R}^d)$, where $h_\alpha(x)$ is the Hermite function.

Definition 2. We denote by $(\mathscr{S}_r^r)'(\mathbb{R}^d)$ the dual space of the Gel'fand-Shilov space $\mathscr{S}_r^r(\mathbb{R}^d)$.

3 The Weyl Transform with Symbol in $\mathscr{S}_r^r$

As quantization from classical mechanics to quantum mechanics, H. Weyl introduced the operator $\mathscr{W}(F)$ as follows: for any $F \in \mathscr{S}(\mathbb{R}^{2d})$,

$$\mathscr{W}(F)\varphi(\xi) = \iint_{\mathbb{R}^{2d}} F(x,y)[\pi(x,y)\varphi](\xi) dx dy,\ \varphi \in L^2(\mathbb{R}^d), \tag{3.1}$$

where $[\pi(x,y)\varphi](\xi) = e^{i(x\cdot\xi + \frac{1}{2}x\cdot y)}\varphi(\xi + y)$. We call this transform $\mathscr{W}(F)$ the Weyl transform with symbol F. The Weyl transform $\mathscr{W}(F)$ is also expressed by the following matrix element: for any φ, $\psi \in L^2(\mathbb{R}^d)$,

$$\begin{aligned}(\mathscr{W}(F)\varphi, \psi) &= \iint_{\mathbb{R}^{2d}} F(x,y)(\pi(x,y)\varphi, \psi) dx dy \\ &= \iint_{\mathbb{R}^{2d}} F(x,y) V(\varphi,\psi)(x,y) dx dy,\end{aligned}$$

where $V(\varphi,\psi)(x,y)$ is the Fourier-Wigner transform of φ and ψ defined by

$$V(\varphi,\psi)(x,y) = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{ix\cdot p} \varphi(p + \frac{y}{2}) \overline{\psi(p - \frac{y}{2})} dp.$$

The Fourier-Wigner transform has the following property, see for example [5]. To be definite, we shall repeat here the proof.

Proposition 4. *Let $\varphi,\psi\in\mathscr{S}_r^r(\mathbb{R}^d)$, $r\geq\frac{1}{2}$. Then $V(\varphi,\psi)\in\mathscr{S}_r^r(\mathbb{R}^{2d})$.*

Proof. It follows from Proposition 2 (iii) that a partial Fourier transform of the first variables is a continuous map from $\mathscr{S}_r^r$ to $\mathscr{S}_r^r$, so it suffices to show that if φ,ψ are in $\mathscr{S}_r^r(\mathbb{R}^d)$, then $\varphi(p+\frac{y}{2})\bar{\psi}(p-\frac{y}{2})$ is in $\mathscr{S}_r^r(\mathbb{R}^{2d})$. Suppose $\varphi,\psi\in\mathscr{S}_r^r(\mathbb{R}^d)$. Since

$$p^{\alpha}=\sum_{|k|=0}^{\alpha}\binom{\alpha}{k}\left(p+\frac{y}{2}\right)^{k}\left(p-\frac{y}{2}\right)^{\alpha-k}\ \text{ and }\ y^{\beta}=\sum_{|l|=0}^{\beta}\binom{\beta}{l}\left(p+\frac{y}{2}\right)^{l}(-1)^{|\beta-l|}\left(p-\frac{y}{2}\right)^{\beta-l},$$

we have that

$$\begin{aligned}p^{\alpha}y^{\beta}\partial_p^{\gamma}\partial_y^{\delta}\varphi(p+\frac{y}{2})\bar{\psi}(p-\frac{y}{2})=&\sum_{k,l,m,n}^{\alpha,\beta,\gamma,\delta}\binom{\alpha}{k}\binom{\beta}{l}\binom{\gamma}{m}\binom{\delta}{n}(-1)^{|\beta-l|}\left(p+\frac{y}{2}\right)^{k+l}\left(p-\frac{y}{2}\right)^{\alpha+\beta-k-l}\\&\times\partial_p^{m}\partial_y^{n}\varphi(p+\frac{y}{2})\partial_p^{\gamma-m}\partial_y^{\delta-n}\bar{\psi}(p-\frac{y}{2}).\end{aligned}\tag{3.2}$$

Set $u=p+\dfrac{y}{2}$ and $v=p-\dfrac{y}{2}$, then

$$(3.2)=\sum_{k,l,m,n}^{\alpha,\beta,\gamma,\delta}\binom{\alpha}{k}\binom{\beta}{l}\binom{\gamma}{m}\binom{\delta}{n}(-1)^{|\beta+\delta-l-n|}\left(\frac{1}{2}\right)^{|\delta|}u^{k+l}v^{\alpha+\beta-k-l}\partial_u^{m+n}\partial_v^{\gamma-m+\delta-n}\varphi(u)\bar{\psi}(v).$$

So we obtain that for any $\alpha,\ \beta\,\gamma,\ \delta\in\mathbb{Z}_+^d$,

$$\begin{aligned}|p^{\alpha}y^{\beta}\partial_p^{\gamma}\partial_y^{\delta}\varphi(p+\frac{y}{2})\bar{\psi}(p-\frac{y}{2})|\leq&\sum_{k,l,m,n}^{\alpha,\beta,\gamma,\delta}\binom{\alpha}{k}\binom{\beta}{l}\binom{\gamma}{m}\binom{\delta}{n}|u^{k+l}\partial_u^{m+n}\varphi(u)||v^{\alpha+\beta-k-l}\partial_v^{\gamma-m+\delta-n}\bar{\psi}(v)|\\\leq&\,C_1C_2\sum_{k,l,m,n}^{\alpha,\beta,\gamma,\delta}\binom{\alpha}{k}\binom{\beta}{l}\binom{\gamma}{m}\binom{\delta}{n}A_1^{k+l}A_2^{\alpha+\beta-k-l}B_1^{m+n}B_2^{\gamma-m+\delta-n}\\&\times(k+l)^{(k+l)r}(\alpha+\beta-k-l)^{(\alpha+\beta-k-l)r}(m+n)^{(m+n)r}\\&(\gamma-m+\delta-n)^{(\gamma-m+\delta-n)r}\end{aligned}\tag{3.3}$$

for suitable constants $A_1,A_2,B_1,B_2\in(0,\infty)^d$ and $C_1,C_2>0$. Thus we have that

$$(3.3)\leq C_3A_3^{\alpha+\beta}B_3^{\gamma+\delta}(\alpha+\beta)^{(\alpha+\beta)r}(\gamma+\delta)^{(\gamma+\delta)r}\tag{3.4}$$

for some constants $A_3,B_3\in(0,\infty)^d$ and $C_3>0$. Since

$$(\alpha+\beta)^{(\alpha+\beta)r}\leq e^{\alpha r}e^{\beta r}\alpha^{\alpha r}\beta^{\beta r}\ \text{ and }\ (\gamma+\delta)^{(\gamma+\delta)r}\leq e^{\gamma r}e^{\delta r}\gamma^{\gamma r}\delta^{\delta r},$$

if we put $A_4 = A_3 e^r$ and $B_4 = B_3 e^r$, then we have

$$(3.4) \le C_3 A_4^{\alpha+\beta} B_4^{\gamma+\delta} \alpha^{\alpha r} \beta^{\beta r} \gamma^{\gamma r} \delta^{\delta r}.$$

Hence we obtain that there exist constants $A_4,\ B_4 \in (0,\infty)^d$ and $C_3 > 0$ such that

$$|p^\alpha y^\beta \partial_p^\gamma \partial_y^\delta \varphi(p+\frac{y}{2})\bar{\psi}(p-\frac{y}{2})| \le C_3 A_4^{\alpha+\beta} B_4^{\gamma+\delta} \alpha^{\alpha r} \beta^{\beta r} \gamma^{\gamma r} \delta^{\delta r}$$

for any α, β, γ and $\delta \in \mathbb{Z}_+^d$. This completes the proof of Proposition 4. □

A straightforward computation with (3.1) shows that if $F(x,y) \in \mathscr{S}(\mathbb{R}^{2d})$, then we have

$$\mathscr{W}(F)\varphi(p) = \int_{\mathbb{R}^d} K(p,p')\varphi(p')dp',\ \varphi \in L^2(\mathbb{R}^d), \tag{3.5}$$

where the kernel $K(p,p') = \mathscr{F}_1^{-1}F(\frac{p+p'}{2}, p'-p)$. Here $\mathscr{F}_1^{-1}F$ denotes the inverse Fourier transform of F in the first variables.

The Weyl transform has the following fundamental properties, see for example [15].

Proposition 5. (i) *If the symbol F is in $L^1(\mathbb{R}^{2d})$, then the Weyl transform $\mathscr{W}(F)$ is a bounded operator on $L^2(\mathbb{R}^d)$,*

(ii) *Let the symbol F be in $L^2(\mathbb{R}^{2d})$. Then the Weyl transform $\mathscr{W}(F)$ is the Hilbert Schmidt operator on $L^2(\mathbb{R}^d)$. Conversely let Φ be the Hilbert-Schmidt operator. Then there exists $F \in L^2(\mathbb{R}^d)$ such that*

$$\Phi = \mathscr{W}(F).$$

We obtain the following result concerning on the property of the Weyl transform $\mathscr{W}(F)$ with the symbol F in $\mathscr{S}_r^r(\mathbb{R}^d)$, $r \ge 1/2$:

Theorem 1. *Let $\mathscr{W}(\mathscr{S}_r^r(\mathbb{R}^{2d}))$ be the set of all the Weyl transforms with the symbol in the Gel'fand-Shilov space $\mathscr{S}_r^r(\mathbb{R}^{2d})$. Then*

$\mathscr{W}(\mathscr{S}_r^r(\mathbb{R}^{2d})) = \{R \in \mathscr{B}(L^2(\mathbb{R}^d))\ |\exists a, a' \in (0,\infty)^d\ ,\ \exists C > 0$ *such that*
$|(Rh_\alpha, h_\beta)| \le Ce^{-a|\alpha|^{\frac{1}{2r}}} e^{-a'|\beta|^{\frac{1}{2r}}},\ \forall \alpha, \beta \in \mathbb{Z}_+^d\}$,

where $\mathscr{B}(L^2(\mathbb{R}^d))$ is the set of all bounded operators on $L^2(\mathbb{R}^d)$ and h_α, h_β are the Hermite functions.

Proof. Let $\mathscr{G} = \{R \in \mathscr{B}(L^2(\mathbb{R}^d))\ |\exists a, a' \in (0,\infty)^d\ ,\ \exists C > 0$ such that $|(Rh_\alpha, h_\beta)| \le Ce^{-a|\alpha|^{\frac{1}{2r}}} e^{-a'|\beta|^{\frac{1}{2r}}},\ \forall \alpha, \beta \in \mathbb{Z}_+^d\}$. By (3.5) and Proposition 2 (iii), it is apparent that the symbol $F \in$

$\mathscr{S}_r^r(\mathbb{R}^{2d})$ if and only if the kernel $K \in \mathscr{S}_r^r(\mathbb{R}^{2d})$. By proposition 3 and Fubini's theorem, we have that

$$\begin{aligned}
|(\mathscr{W}(F)h_\alpha, h_\beta)| &\le \left|\left(\int_{\mathbb{R}^d} K(p,p')h_\alpha(p')dp', h_\beta(p)\right)\right| \\
&= \left|\iint_{\mathbb{R}^{2d}} K(p,p')h_\alpha(p')h_\beta(p)dp'dp\right| \\
&= \left|(K, h_\alpha \otimes h_\beta)\right| \\
&\le Ce^{-a|\alpha|^{\frac{1}{2r}}} e^{-a'|\beta|^{\frac{1}{2r}}}
\end{aligned}$$

for some constants $a, a' \in (0,\infty)^d$ and $C > 0$. Therefore $\mathscr{W}(F) \in \mathscr{G}$. Conversely, let $R_1 \in \mathscr{G}$. Then

$$\begin{aligned}
\sum_{|\alpha|=0}^{\infty} \|R_1 h_\alpha\|^2_{L^2(\mathbb{R}^d)} &= \sum_{|\alpha|=0}^{\infty} (R_1 h_\alpha, R_1 h_\alpha) \\
&\le \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} |(R_1 h_\alpha, h_\beta)||(h_\beta, R_1 h_\alpha)| \\
&= \sum_{|\alpha|=0}^{\infty} \sum_{|\beta|=0}^{\infty} e^{-2a|\alpha|^{\frac{1}{2r}}} e^{-2a'|\beta|^{\frac{1}{2r}}} < +\infty.
\end{aligned}$$

Hence R_1 is the Hilbert-Schmidt operator. Therefore it follows from Proposition 5 that there exists $G \in L^2(\mathbb{R}^{2d})$ such that $\mathscr{W}(G) = R_1$. Then from (3.5) there exists $C > 0$ and $a, a' \in (0,\infty)^d$ such that

$$|(R_1 h_\alpha, h_\beta)| = |(\mathscr{W}(G)h_\alpha, h_\beta)| = |(K, h_\alpha \otimes h_\beta)| \le Ce^{-a|\alpha|^{\frac{1}{2r}}} e^{-a'|\beta|^{\frac{1}{2r}}}.$$

By proposition 3, we obtain that $K \in \mathscr{S}_r^r(\mathbb{R}^{2d})$ and so is G. $\square$

Corollary 1. *If $F, G \in \mathscr{S}_r^r(\mathbb{R}^{2d})$, then there exists $H \in \mathscr{S}_r^r(\mathbb{R}^{2d})$ such that $\mathscr{W}(H) = \mathscr{W}(F)\mathscr{W}(G)$*

Proof. Let $F, G \in \mathscr{S}_r^r(\mathbb{R}^{2d})$. Then we have that

$$\begin{aligned}
|(\mathscr{W}(F)\mathscr{W}(G)h_\alpha, h_\beta)| &= |(\mathscr{W}(G)h_\alpha, \mathscr{W}(F)^* h_\beta)| \\
&\le \sum_\gamma |(\mathscr{W}(G)h_\alpha, h_\gamma)||(h_\gamma, \mathscr{W}(F)^* h_\beta)| \\
&= \sum_\gamma |(\mathscr{W}(G)h_\alpha, h_\gamma)||(\mathscr{W}(F)h_\gamma, h_\beta)| \\
&\le Ce^{-a|\alpha|^{\frac{1}{2r}}} e^{-b|\beta|^{\frac{1}{2r}}} \sum_\gamma e^{-(a'+b')|\gamma|^{\frac{1}{2r}}} \\
&= C'e^{-a|\alpha|^{\frac{1}{2r}}} e^{-b|\beta|^{\frac{1}{2r}}}
\end{aligned}$$

for suitable constants $a, a', b, b' \in (0,\infty)^d$ and $C, C' > 0$. Hence we obtain that $\mathscr{W}(F)\mathscr{W}(G) \in \mathscr{G}$. Therefore it follows from Theorem 1 that H is in $\mathscr{S}_r^r(\mathbb{R}^{2d})$ such that $\mathscr{W}(H) = \mathscr{W}(F)\mathscr{W}(G)$. □

Remark 2. It is known that $\mathscr{W}(F)\mathscr{W}(G) = \mathscr{W}(F *_{\frac{1}{4}} G)$, where

$$(F *_{\frac{1}{4}} G)(x,y) = \iint_{\mathbb{R}^{2d}} F(x-\xi, y-\eta)G(\xi,\eta)e^{\frac{i}{4}\times 2(y\cdot\xi - x\cdot\eta)}d\xi d\eta.$$

So if $F, G \in \mathscr{S}_r^r(\mathbb{R}^{2d})$, then $(F *_{\frac{1}{4}} G) \in \mathscr{S}_r^r(\mathbb{R}^{2d})$ from Corollary 1.

Definition 3. We denote by S_1 the family of all trace class operators defined as follows: for $A \in \mathscr{B}(L^2(\mathbb{R}^d))$, there exists an orthonormal basis $\{v_k\}$ of $L^2(\mathbb{R}^d)$ such that

$$\sum_k \|Av_k\|_{L^2(\mathbb{R}^d)} < \infty.$$

Proposition 6. *Let $A \in \mathscr{B}(L^2(\mathbb{R}^d))$. If $\{v_j\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ then*

$$\sum_j \|Av_j\|_{L^2(\mathbb{R}^d)} \le \sum_{j,k} |\left(Av_j, v_k\right)|.$$

Proof. It is obvious as $Av_j = 0$, so it suffices to show as $Av_j \neq 0$. Let w_j be an unit vector for any index j. Then we have

$$\begin{aligned}
\sum_j |\left(A^* w_j, v_j\right)| &= \sum_j |\left(w_j, Av_j\right)| \\
&= \sum_j \left| \sum_k \left(w_j, v_k\right)\left(v_k, Av_j\right) \right| \\
&= \sum_j \left| \sum_k \left(A^* v_k, v_j\right)\left(w_j, v_k\right) \right| \\
&\le \sum_{j,k} |\left(A^* v_k, v_j\right)| \\
&= \sum_{j,k} |\left(Av_j, v_k\right)|. \qquad (3.6)
\end{aligned}$$

Choose $w_j = \dfrac{Av_j}{\|Av_j\|}$. The inequality now follows. Indeed from (3.6) we have

$$\sum_j |\left(w_j, Av_j\right)| = \sum_j \left| \left(\frac{Av_j}{\|Av_j\|}, Av_j \right) \right| = \sum_j \frac{1}{\|Av_j\|} |\left(Av_j, Av_j\right)|^2 = \sum_j \|Av_j\| \le \sum_{j,k} |\left(Av_j, v_k\right)|. \qquad \square$$

We obtain the following result from Theorem 1 and Proposition 6:

Corollary 2. *The Weyl transform $\mathscr{W}(F)$ with symbol in $\mathscr{S}_r^r(\mathbb{R}^{2d})$ is of the trace class S_1.*

Remark 3. Since the Gel'fand-Shilov classes are included in the Schwartz class, the preceding Corollary 2 can be seen also as a consequence of the results of A. Voros [13], proving that the Weyl transforms with symbol in the Schwartz class are trace operators.

4 On the Weyl Transform with Symbol in $(\mathscr{S}_r^r)'$

We first show the Schwartz's kernel theorem for $(\mathscr{S}_r^r)'$, $r \geq \frac{1}{2}$, and give the property of the Weyl transform with the symbol in $(\mathscr{S}_r^r)'$, $r \geq \frac{1}{2}$, as a corollary of the Schwartz's kernel theorem. S. -Y. Chung, D. Kim and E. G. Lee proved the Schwartz kernel Theorem for $(\mathscr{S}_1^1)'$ in [2] and Z. Lozanov-Crvenković and D. Perišić gave the Schwartz kernel theorem for $(\mathscr{S}_r^r)'$ as $r > 1$ in [6]. Our result includes their results.

We prove the following Schwartz's kernel theorem for $(\mathscr{S}_r^r)'$, $r \geq \frac{1}{2}$, along the idea in [2]:

Theorem 2. *Let k be a continuous and linear operator from $\mathscr{S}_r^r(\mathbb{R}_{x_2}^{d_2})$ to $\left(\mathscr{S}_r^r\right)'(\mathbb{R}_{x_1}^{d_1})$, $r \geq \frac{1}{2}$. Then there exists K in $\left(\mathscr{S}_r^r\right)'(\mathbb{R}_{x_1}^{d_1} \times \mathbb{R}_{x_2}^{d_2})$, $r \geq \frac{1}{2}$, such that*

$$\langle k\psi, \varphi\rangle = \langle K, \varphi \otimes \psi\rangle,$$

where φ is in $\mathscr{S}_r^r(\mathbb{R}_{x_1}^{d_1})$, $r \geq \frac{1}{2}$, and ψ is in $\mathscr{S}_r^r(\mathbb{R}_{x_2}^{d_2})$, $r \geq \frac{1}{2}$.

To prove the Theorem 2, we begin from some preparations. We define the heat kernel $E(x,t)$ by

$$E(x,t) = \left(\frac{1}{\sqrt{4\pi t}}\right)^d e^{-\frac{|x|^2}{4t}}, \ (x,t) \in \mathbb{R}^d \times (0,\infty).$$

The heat kernel enjoys the following properties:

· $E(x,t) \in \mathscr{S}(\mathbb{R}_x^d)$,

· $\displaystyle\int_{\mathbb{R}^d} E(x,t)dx = 1$,

and

· $\left(\dfrac{\partial}{\partial t} - \Delta\right)E(x,t) = 0$, in $\mathbb{R}^d \times (0,\infty)$.

Moreover we obtain the following estimate on the heat kernel $E(x,t)$:

Proposition 7 ([16])**.** *For any $\alpha \in \mathbb{Z}_+^d$, we have*

$$|\partial_x^\alpha E(x,t)| \leq E(x,t)(\alpha!)^{\frac{1}{2}}(2t)^{-|\alpha|}(1+|x|)^\alpha, \ x \in \mathbb{R}^d, \ 0 < t \leq \frac{1}{2}.$$

From this estimate, we immediately obtain the following properties:

Proposition 8. $E(x,t) \in \mathscr{S}_r^r(\mathbb{R}_x^d)$, $r \geq \frac{1}{2}$.

Proposition 9. *Let $E(x,t)$ is in $\mathscr{S}_{r,a}^{r,B}$ for any a, $B > 0$. Then for every $T > 0$ and $\varepsilon > 0$, there is a constant $C > 0$ such that*

$$\|E(x-\cdot,t)\|_{\delta\rho} \leq C\exp[\varepsilon(|x|^{\frac{1}{r}} + (1/t)^{1/(2r-1)})], \ x \in \mathbb{R}^d, \ 0 < t < T. \ \left(r > \frac{1}{2}\right)$$

In the case where $r = 1/2$, we have the following inequality:

$$\|E(x-\cdot,t)\|_{\delta\rho} \le C_{\varepsilon,t} e^{\varepsilon|x|^2},\ x \in \mathbb{R}^d,\ t>0,\ \left(r=\frac{1}{2}\right).$$

Moreover we need the several propositions, which are the result of C. Dong and T. Matsuzawa [3], to prove Theorem 2 as follows:

Proposition 10 ([3])**.** *Let $\varphi(x) \in \mathscr{S}_{r,a}^{r,B}(\mathbb{R}^d)$, $r \ge 1/2$. Then*

$$U(x,t) \equiv \int_{\mathbb{R}^d} E(x-y,t)\varphi(y)dy \in \mathscr{S}_{r,a}^{r,B}(\mathbb{R}^d)\ ,\ t>0$$

and

$$U(x,t) \to \varphi(x) \text{ in } \in \mathscr{S}_{r,a}^{r,B}(\mathbb{R}^d) \text{ as } t \to 0.$$

Proposition 11 ([3])**.** *If Every C^∞-function $U(x,t)$ defined in $\mathbb{R}_+^{d+1} = \{(x,t) \mid x \in \mathbb{R}^d\ ,\ t>0\}$ satisfies the conditions:*

$$\left(\frac{\partial}{\partial t} - \Delta\right) U(x,t) = 0\ ,\ \text{in } \mathbb{R}_+^{d+1},$$

and for every $T>0$ and $\varepsilon>0$, there is a constant $C>0$ such that

$$|U(x,t)| \le C \exp[\varepsilon(|x|^{\frac{1}{r}} + (1/t)^{1/(2r-1)})],\ x \in \mathbb{R}^d,\ 0<t<T.\ \left(r>\frac{1}{2}\right)$$

In the case where $r = 1/2$, $U(x,t)$ has the following inequality:

$$|U(x,t)| \le C_{\varepsilon,t} e^{\varepsilon|x|^2},\ x \in \mathbb{R}^d,\ t>0.\ \left(r=\frac{1}{2}\right)$$

Then $U(x,t)$ can be expressed in the form $U(x,t) = \langle u_y\ ,\ E(x-y,t)\rangle$ with unique element $u \in \left(\mathscr{S}_r^r\right)'(\mathbb{R}^d)$.

Proof of Theorem 2. We show the proof of theorem 2 as $r > \frac{1}{2}$. Since k is continuous, the bilinear form $\mathbb{B}$ on $\mathscr{S}_{r,a}^{r,B}(\mathbb{R}^{d_1}) \times \mathscr{S}_{r,a'}^{r,B'}(\mathbb{R}^{d_2})$, for any $a,B \in (0,\infty)^{d_1}$ and $a',B' \in (0,\infty)^{d_2}$,
$\mathbb{B}(\varphi,\psi) = \langle k\psi,\varphi\rangle\ ,\ \varphi \in \mathscr{S}_{r,a}^{r,B}(\mathbb{R}^{d_1})\ ,\ \psi \in \mathscr{S}_{r,a'}^{r,B'}(\mathbb{R}^{d_2})$
is separately continuous. Since $\mathscr{S}_{r,a}^{r,B}(\mathbb{R}^{d_1})$ and $\mathscr{S}_{r,a'}^{r,B'}(\mathbb{R}^{d_2})$ is Fréchet space, $\mathbb{B}$ is continuous. Hence we obtain that there exists a constant $C_{a,a',B,B'} > 0$ such that

$$|\langle k\psi,\varphi\rangle| \le C_{a,a',B,B'} \|\varphi\|_{\delta\rho} \|\psi\|_{\delta'\rho'} \qquad (\sharp).$$

Set for $(x_1,x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ and $t>0$,

$$K_t(x_1,x_2) = \langle kE(x_2-\cdot,t),\ E(x_1-\cdot,t)\rangle\,.$$

Now we show K_t converges in $\left(\mathcal{S}_r^r\right)'(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ as $t\to 0$. By (♯) and Proposition 9, for any $\varepsilon,\ \varepsilon'>0$, there exists a constant $C_{\varepsilon,\varepsilon'}>0$ such that

$$|K_t(x_1,x_2)|\le C_{\varepsilon,\varepsilon'}\exp[\varepsilon(|x_1|^{\frac{1}{r}}+(1/t)^{1/(2r-1)})]\exp[\varepsilon'(|x_2|^{\frac{1}{r}}+(1/t)^{1/(2r-1)})].$$

Moreover we obtain

$$\left(\frac{\partial}{\partial t}-\Delta\right)K_t(x_1,x_2)=0\ .$$

Therefore, by Proposition 11, there exists $K_0\in\left(\mathcal{S}_r^r\right)'(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$ such that $K_0=\lim_{t\to0}K_t$ in $\left(\mathcal{S}_r^r\right)'(\mathbb{R}^{d_1}\times\mathbb{R}^{d_2})$.

For $\varphi\in\mathcal{S}_r^r(\mathbb{R}^d_{x_1})$, $\psi\in\mathcal{S}_r^r(\mathbb{R}^d_{x_2})$,

$$\begin{aligned}\langle K_t\ ,\ \varphi\otimes\psi\rangle &= \iint_{\mathbb{R}^{d_1+d_2}} K_t(x_1,x_2)\varphi(x_1)\psi(x_2)dx_1dx_2\\ &= \iint_{\mathbb{R}^{d_1+d_2}} \langle kE(x_2-y_2,t)\psi(x_2),\ E(x_1-y_1,t)\varphi(x_1)\rangle\, dx_1dx_2.\end{aligned}$$

Since the Riemann sum of an integral converges in $\mathcal{S}_r^r$, we obtain

$$\langle K_t\ ,\ \varphi\otimes\psi\rangle=\langle k\int_{\mathbb{R}^{d_2}}E(x_2-y_2,t)\psi(x_2)\ ,\ \int_{\mathbb{R}^{d_1}}E(x_1-y_1,t)\varphi(x_1)\rangle.$$

Therefore, by Proposition 10, we obtain

$$\langle K_0,\varphi\otimes\psi\rangle=\langle k\psi,\varphi\rangle\ ,$$

as $t\to0$. □

Similarly, we can also show the proof of Theorem 2 as $r=\frac{1}{2}$.

Remark 4. Z. Lozanov-Crvenković and D. Perišić also proved the Schwartz kernel theorem for the spaces of tempered ultradistributions in [6] by means of the Hermite expansions.

We define the Weyl transform with symbol $T\in(\mathcal{S}_r^r)'$ by

$$\langle\mathcal{W}(T)\varphi,\psi\rangle=\langle T,V(\varphi,\overline{\psi})\rangle,\ \varphi,\psi\in\mathcal{S}_r^r(\mathbb{R}^d),$$

where $V(\varphi,\bar\psi)$ is the Fourier-Wigner transform of φ and $\bar\psi$. It follows from Proposition 4 that this definition is well defined. M. Cappiello, T. Gramchev and L. Rodino also showed this subject in [1]. We obtain the following result from Theorem 2.

Corollary 3. *The map $\mathcal{W}$ from $\mathcal{S}(\mathbb{R}^{2d})$ to the space of bounded operators on $L^2(\mathbb{R}^d)$, defined by*

$$\mathcal{W}(F)\varphi(\xi)=\iint_{\mathbb{R}^{2d}}F(x,y)[\pi(x,y)\varphi](\xi)dxdy\ ,\ \varphi\in L^2(\mathbb{R}^d),$$

extends uniquely to a bijection from $(\mathcal{S}_r^r)'(\mathbb{R}^{2d})$, $r\ge1/2$, to the space of continuous linear maps from $\mathcal{S}_r^r(\mathbb{R}^d)$, $r\ge1/2$, to $(\mathcal{S}_r^r)'(\mathbb{R}^d)$, $r\ge1/2$.

Proof. Let k be a continuous linear map from $\mathscr{S}_r^r(\mathbb{R}^d)$ to $(\mathscr{S}_r^r)'(\mathbb{R}^d)$. By Theorem 2, for any k, there exists $K \in (\mathscr{S}_r^r)'(\mathbb{R}^{2d})$ such that

$$\langle k\varphi, \psi\rangle = \langle K, \varphi \otimes \psi\rangle \ , \ \varphi, \psi \in \mathscr{S}_r^r(\mathbb{R}^d).$$

So we have

$$\begin{aligned}\langle k\varphi, \psi\rangle &= \langle K, \varphi \otimes \psi\rangle \\ &= \langle \mathscr{F}_1 \mathbf{S}K, V(\varphi, \bar{\psi})\rangle, \end{aligned} \tag{4.1}$$

where $\mathscr{F}_1$ is the Fourier transform of the first variable and $\mathbf{S}$ is defined by $\mathbf{S}h(a,b) = h(a + \frac{b}{2}, a - \frac{b}{2})$. Set $T = \mathscr{F}_1\mathbf{S}K$,

$$\begin{aligned}(4.1) &= \langle T, V(\varphi, \overline{\psi})\rangle \\ &= \langle \mathscr{W}(T)\varphi, \psi\rangle. \end{aligned}$$

Since $\mathscr{F}_1\mathbf{S}K \in (\mathscr{S}_r^r)'(\mathbb{R}^{2d})$, for any k, there exists $T \in (\mathscr{S}_r^r)'(\mathbb{R}^{2d})$ such that $k = \mathscr{W}(T)$. $\square$

Remark 5. Z. Lozanov-Crvenković and D. Perišić gave the similar result for $(\mathscr{S}_r^r)'$ as $r > 1$ in [6].

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