ISSN 0716-7776

VOLUME 13, Nº 02
2011

Cubo

A Mathematical Journal

Universidad de La Frontera
Facultad de Ingeniería, Ciencias y Administración
Departamento de Matemática y Estadística
Temuco - Chile

Universidade Federal de Pernambuco
Centro de Ciências Exatas e da Natureza
Departamento de Matemática
Recife - Brazil

CUBO

A Mathematical Journal

The Department of Mathematics of the Universidad de La Frontera Temuco-Chile and the Department of Mathematics of the Universidade Federal de Pernambuco, Recife-Brazil, edit **CUBO,** A Mathematical Journal. **CUBO** appears in three issues per year and is indexed in ZentralBlatt Math., Mathematical Reviews, MathSciNet and Latin Index. **CUBO** publishes original research papers, preferably not longer than 20 printed pages, that contain substantial mathematical results in all branches of pure and applied mathematics.

CUBO (ISSN 0716-7776) Editorial Office: Departamento de Matemática, Universidade Federal de Pernambuco, CEP 50740-540, Recife-PE., Brazil, cch@dmat.ufpe.br.

CUBO
A Mathematical Journal

EDITORIAL BOARD

CUBO
A MATHEMATICAL JOURNAL
Universidad de La Frontera
Universidade Federal de Pernambuco
Volume 13/Nº02 – JUNE 2011

SUMMARY

CUBO A Mathematical Journal
Vol.13, N$^{\underline{o}}$02, (1–35). June 2011

Homogeneous Besov Spaces associated with the spherical mean operator

L.T.Rachdi and A.Rouz
Department of Mathematics,
Faculty of Sciences of Tunis,
2092 El Manar 2 Tunis, Tunisia.
email: ahlemrouz@yahoo.fr

ABSTRACT

We define and study homogeneous Besov spaces associated with the spherical mean operator. We establish some results of completeness, continuous embeddings and density of subspaces. Next, we define a discrete equivalent norm on this space and we give other properties.

RESUMEN

Definimos y estudiamos los espacios homogneos Besov asociados con el operador esférico medio. Se establecen algunos resultados de la exhaustividad, de inclusiones continuas y de la densidad de subespacios. A continuación, se define una norma equivalente discreta en este espacio y se dan otras propiedades.

Keywords and phrases:: Spherical mean operator, Besov space, Banach space, Fourier transform.

Mathematics Subject Classification: 46E35 , 44A35.

1 Introduction

For a continuous function f on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, the spherical mean operator $\mathscr{R}$ is defined as

$$\mathscr{R}(f)(r,x)=\int_{S^n} f(r\eta, x+r\xi)d\sigma_n(\eta,\xi);\quad (r,x)\in\mathbb{R}\times\mathbb{R}^n,$$

where S^n is the unit sphere, i.e. $S^n=\{(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n\ ;\ \eta^2+|\xi|^2=1\}$ and σ_n is the surface measure on S^n normalized to have total measure one.

The dual of the spherical mean operator ${}^t\mathscr{R}$ is defined by

$${}^t\mathscr{R}(g)(r,x)=\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\int_{\mathbb{R}^n} g(\sqrt{r^2+|x-y|^2},y)dy,$$

where dy is the Lebesgue measure on $\mathbb{R}^n$.

The spherical mean operator $\mathscr{R}$ and its dual ${}^t\mathscr{R}$ play an important role and have many applications, for example, in image processing of so-called synthetic aperture radar (SAR) data [14, 15], or in the linearized inverse scattering problem in acoustics [9].
Many aspects of such operator have been studied [1, 3, 6, 18, 21]. In particular, in [18] the first author with the others associated to the spherical mean operator the Fourier transform defined by

$$\forall(\mu,\lambda)\in\Gamma,\quad \mathscr{F}(f)(\mu,\lambda)=\int_{\mathbb{R}^n}\int_0^{+\infty} f(r,x)\ \varphi_{\mu,\lambda}(r,x)\,d\nu_n(r,x),$$

where

- $\varphi_{\mu,\lambda}$ is the function defined by

$$\forall(r,x)\in\mathbb{R}\times\mathbb{R}^n,\ \varphi_{\mu,\lambda}(r,x)=\mathscr{R}\left(\cos(\mu.)e^{-i\langle\lambda|.\rangle}\right)(r,x).$$

- ν_n is the measure defined on $[0,+\infty[\times\mathbb{R}^n$, by

$$d\nu_n(r,x)=\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})(2\pi)^{\frac{n}{2}}}\ r^n\ dr\otimes dx.$$

- Γ is the set given by

$$\Gamma=\mathbb{R}\times\mathbb{R}^n\cup\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n,\ |\mu|\leqslant|\lambda|\}.$$

They have constructed the harmonic analysis related to the Fourier transform $\mathscr{F}$ (Inversion formula, Schwartz theorem, Paley-Wiener theorem, Plancherel theorem).

There are many ways to define Besov Spaces [4, 5, 13, 16, 20, 23]. It is well known that Besov spaces can be defined for instance in terms of convolutions $f*\phi_t$ with different kinds of smooth functions ϕ and that can be also described by means of differences $\triangle_x f$ [10, 11, 22].

In this work, we define and study a class of homogeneous Besov spaces connected with the spherical mean operator $\mathscr{R}$. More precisely, let ϕ be a smooth function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable. For all $p, q \in [1, +\infty]$ and $\gamma \in \mathbb{R}$, we define the Besov space $\mathscr{B}^{\gamma,\phi}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ to be the space of tempered distributions f on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable such that

$$f = \int_0^{+\infty} f * \phi_t * \phi_t \frac{dt}{t},$$

where $*$ is the convolution product associated with the spherical mean operator and $\phi_t;\ t > 0$ is the dilated function of ϕ defined by

$$\forall (r,x) \in [0,+\infty[\times\mathbb{R}^n, \quad \phi_t(r,x) = \frac{1}{t^{2n+1}}\ \phi(\frac{r}{t}, \frac{x}{t})$$

(see Definition 10 below).

The space $\mathscr{B}^{\gamma,\phi}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ is equipped firstly with the norm

$$M^{\gamma,\phi}_{p,q}(f) = \begin{cases} \left(\displaystyle\int_0^{+\infty} \Big(\frac{\|f*\phi_t\|_{p,\nu_n}}{t^\gamma}\Big)^q \frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } 1 \leqslant q < +\infty; \\ \operatorname*{esssup}_{t>0} \dfrac{\|f*\phi_t\|_{p,\nu_n}}{t^\gamma}, & \text{if } q = +\infty. \end{cases}$$

with

$$\|f*\phi_t\|_{p,\nu_n} = \begin{cases} \left(\displaystyle\int_{\mathbb{R}^n}\int_0^{+\infty} |f*\phi_t(r,x)|^p\ d\nu_n(r,x)\right)^{\frac{1}{p}}, & \text{if } p \in [1,+\infty[; \\ \operatorname*{esssup}_{(r,x)\in[0,+\infty[\times\mathbb{R}^n} |f*\phi_t(r,x)|, & \text{if } p = +\infty. \end{cases}$$

Then we have established the coming results

- The Besov space $\mathscr{B}^{\gamma,\phi}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ is independent of the choice of the function ϕ and will be denoted by $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$. This means that for all smooth functions ϕ and ψ, there exists a positive constant $C_{\phi,\psi}$ such that

$$\forall f\in\mathscr{B}^{\gamma,\phi}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big),\ M^{\gamma,\phi}_{p,q}(f)\leqslant C_{\phi,\psi}\ M^{\gamma,\psi}_{p,q}(f).$$

- The space $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ is homogeneous with degree equal to $(2n+1)/p-\gamma-2n-1$, that is for all $f\in\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ and $t>0$, the distribution f_t belongs to the space $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$and we have

$$M^{\gamma,\phi}_{p,q}(f_t)=t^{\frac{2n+1}{p}-\gamma-2n-1}\ M^{\gamma,\phi}_{p,q}(f).$$

- The Besov space is a Banach one when $\gamma<(2n+1)/p$.

We have also proved some continuous embeddings and density of subspaces.

Next, we define the following discrete norm on the space $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ by setting

$$N^{\gamma,\phi}_{p,q}(f)=\begin{cases}\left(\sum_{k\in\mathbb{Z}}\left(\frac{\|f*\phi_{2^k}\|_{p,\nu_n}}{2^{k\gamma}}\right)^q\right)^{\frac{1}{q}}, & \text{if } 1\leqslant q<+\infty;\\ \operatorname*{esssup}_{k\in\mathbb{Z}}\frac{\|f*\phi_{2^k}\|_{p,\nu_n}}{2^{k\gamma}}, & \text{if } q=+\infty.\end{cases}$$

We show that this norm defines the same topology as the norm $M^{\gamma,\phi}_{p,q}$. We prove that this space is homogeneous in a weaker sense when equipped with the norm $N^{\gamma,\phi}_{p,q}$, that is there exist two positive constants C_1 and C_2 such that for all $f\in\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ and $t>0$

$$C_1\ t^{\frac{2n+1}{p}-2n-1-\gamma}\ N^{\gamma,\phi}_{p,q}(f)\leqslant N^{\gamma,\phi}_{p,q}(f_t)\leqslant C_2\ t^{\frac{2n+1}{p}-2n-1-\gamma}\ N^{\gamma,\phi}_{p,q}(f).$$

Finally, we establish some new continuous embedding.

2 Fourier transform associated with the spherical mean operator

In this section, we recall some harmonic analysis results related to the Fourier transform associated with the spherical mean operator.
Let $\varphi_{\mu,\lambda},\ (\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$, be the function defined by

$$\forall (r,x)\in\mathbb{R}\times\mathbb{R}^n,\ \varphi_{\mu,\lambda}(r,x)=\mathscr{R}\left(\cos(\mu.)e^{-i\langle\lambda|.\rangle}\right)(r,x).$$

It's well known ([18, 21]) that

i. The function $\varphi_{\mu,\lambda}$ is given by

$$\forall (r,x)\in\mathbb{R}\times\mathbb{R}^n,\quad \varphi_{\mu,\lambda}(r,x)e^{-i\langle\lambda|x\rangle}j_{(n-1)/2}(r\sqrt{\mu^2+\lambda_1^2+\ldots+\lambda_n^2}),$$

where $j_{(n-1)/2}$ is the modified Bessel function defined by

$$j_{\frac{n-1}{2}}(s)=2^{\frac{n-1}{2}}\Gamma\Big(\frac{n+1}{2}\Big)\frac{J_{\frac{n-1}{2}}(s)}{s^{\frac{n-1}{2}}}=\Gamma\Big(\frac{n+1}{2}\Big)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\alpha+k+1)}\Big(\frac{s}{2}\Big)^{2k},$$

and $J_{(n-1)/2}$ is the Bessel function of first kind and index $(n-1)/2$ [7, 8, 17, 27].

ii. For all $(\mu,\lambda)\in\mathbb{C}\times\mathbb{C}^n$, $\varphi_{\mu,\lambda}$ is the unique infinitely differentiable function on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable, satisfying

$$\begin{cases} D_j u(r,x_1,...,x_n)=-i\lambda_j u(r,x_1,...,x_n), & 1\leqslant j\leqslant n,\\ \Xi u(r,x_1,...,x_n)=-\mu^2 u(r,x_1,...,x_n), & \\ u(0,...,0)=1,\ \frac{\partial u}{\partial r}(0,x_1,...,x_n)=0, & \forall (x_1,...,x_n)\in\mathbb{R}^n.\end{cases}$$

where

$$D_j=\frac{\partial}{\partial x_j};\quad 1\leqslant j\leqslant n,\quad \text{and}\quad \Xi=\frac{\partial^2}{\partial r^2}+\frac{n}{r}\frac{\partial}{\partial r}-\sum_{j=1}^{n}D_j^2. \tag{2.1}$$

iii. The function $\varphi_{\mu,\lambda}$ is bounded on $\mathbb{R}\times\mathbb{R}^n$ if, and only if (μ,λ) belongs to the set Γ given by

$$\Gamma=\mathbb{R}\times\mathbb{R}^n\cup\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n,\ |\mu|\leqslant|\lambda|\}. \tag{2.2}$$

In this case, we have

$$\sup_{(r,x)\in\mathbb{R}\times\mathbb{R}^n}|\varphi_{\mu,\lambda}(r,x)|=1.$$

We denote by

- $L^p(d\nu_n)$, $p\in[1,+\infty]$, the space of measurable functions f on $[0,+\infty[\times\mathbb{R}^n$, such that

$$\|f\|_{p,\nu_n}=\begin{cases}\Big(\int_{\mathbb{R}^n}\int_0^{+\infty}|f(r,x)|^p\,d\nu_n(r,x)\Big)^{\frac{1}{p}}<+\infty, & \text{if } p\in[1,+\infty[;\\ \operatorname{esssup}_{(r,x)\in[0,+\infty[\times\mathbb{R}^n}|f(r,x)|<+\infty, & \text{if } p=+\infty,\end{cases}$$

where ν_n is the measure defined in the introduction.

- Γ_+ the subset of Γ given by

$$\Gamma_+=[0,+\infty[\times\mathbb{R}^n\cup\{(i\mu,\lambda);\ (\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n,\ 0\leqslant\mu\leqslant|\lambda|\}.$$

- $\mathcal{B}_{\Gamma_+}$ the σ-algebra on Γ_+ defined by

$$\mathcal{B}_{\Gamma_+} = \theta^{-1}(\mathcal{B}_{[0,+\infty[\times\mathbb{R}^n}),$$

where θ is the bijective function defined on Γ_+ by

$$\theta(\mu,\lambda) = (\sqrt{\mu^2+\lambda^2},\lambda). \tag{2.3}$$

- γ_n the measure defined on Γ_+ by

$$\gamma_n(A) = \nu_n(\theta(A));\quad A\in\mathcal{B}_{\Gamma_+}.$$

- $L^p(d\gamma_n),\ p\in[1,+\infty]$, the space of measurable functions on Γ_+ satisfying

$$\|f\|_{p,\gamma_n} < +\infty.$$

Then we have the coming properties

Proposition 1. i) For all non negative measurable function f on Γ_+ (respectively integrable on Γ_+ with respect to the measure $d\gamma_n$), we have

$$\int\int_{\Gamma_+} f(\mu,\lambda)d\gamma_n(\mu,\lambda)\frac{1}{2^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})(2\pi)^{\frac{n}{2}}}\Big\{\int_{\mathbb{R}^n}\int_0^{+\infty} f(\mu,\lambda)\left(\mu^2+|\lambda|^2\right)^{\frac{n-1}{2}}\mu\,d\mu\,d\lambda$$
$$+\int_{\mathbb{R}^n}\int_0^{|\lambda|} f(i\mu,\lambda)\left(|\lambda|^2-\mu^2\right)^{\frac{n-1}{2}}\mu\,d\mu\,d\lambda\Big\}.$$

ii) For all non negative measurable function g on $[0,+\infty[\times\mathbb{R}^n$ (respectively integrable on $[0,+\infty[\times\mathbb{R}^n$ with respect to the measure $d\nu_n$), the function $g\circ\theta$ is measurable positive on Γ_+ (respectively integrable on Γ_+ with respect to the measure $d\gamma_n$) and we have

$$\int_{\mathbb{R}^n}\int_0^{\infty} g(r,x)d\nu_n(r,x) = \int\int_{\Gamma_+} g\circ\theta(\mu,\lambda)d\gamma_n(\mu,\lambda).$$

In the following, we shall define the translation operator and the convolution product associated with the spherical mean operator. For this, we use the product formula for the function $\varphi_{\mu,\lambda}$, for all $(r,x),(s,y)\in\mathbb{R}\times\mathbb{R}^n$, we have

$$\varphi_{\mu,\lambda}(r,x)\varphi_{\mu,\lambda}(s,y)\frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}\int_0^{\pi}\varphi_{\mu,\lambda}\left(\sqrt{r^2+s^2+2rs\cos\theta},x+y\right)\sin^{n-1}(\theta)d\theta \tag{2.4}$$

Definition 2. i) For all $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, the translation operator $\tau_{(r,x)}$ associated with the spherical mean operator is defined on $L^p(d\nu_n),\ p\in[1,+\infty]$, by

$$\tau_{(r,x)}(f)(s,y) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}\right)}\int_0^{\pi} f\left(\sqrt{r^2+s^2+2rs\cos\theta},x+y\right)\sin^{n-1}(\theta)d\theta.$$

ii) The convolution product of $f, g \in L^1(d\nu_n)$ is defined by

$$\forall (r,x) \in [0,+\infty[\times \mathbb{R}^n, \quad f * g(r,x) = \int_{\mathbb{R}^n}\int_0^{+\infty} f(s,y)\tau_{(r,-x)}(\check{g})(s,y)d\nu_n(s,y),$$

where

$$\check{g}(s,y) = g(s,-y).$$

We have the following properties

- For all $(r,x),(s,y) \in [0,+\infty[\times \mathbb{R}^n$, the relation (2.4) can be written

$$\tau_{(r,x)}(\varphi_{\mu,\lambda})(s,y)\varphi_{\mu,\lambda}(r,x)\ \varphi_{\mu,\lambda}(s,y). \tag{2.5}$$

- If $f \in L^p(d\nu_n)$, $1 \leqslant p \leqslant +\infty$, then for all $(s,y) \in [0,+\infty[\times \mathbb{R}^n$, the function $\tau_{(s,y)}(f)$ belongs to $L^p(d\nu_n)$ and we have

$$\left\|\tau_{(s,y)}(f)\right\|_{p,\nu_n} \leqslant \|f\|_{p,\nu_n}. \tag{2.6}$$

- Let $p,\ q,\ r \in [1,+\infty]$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1$. Then for all $f \in L^p(d\nu_n)$ and $g \in L^q(d\nu_n)$, the function $f * g$ belongs to $L^r(d\nu_n)$ and we have

$$\|f * g\|_{r,\nu_n} \leqslant \|f\|_{p,\nu_n}\ \|g\|_{q,\nu_n}. \tag{2.7}$$

Now, we will define the Fourier transform $\mathscr{F}$ connected with the spherical mean operator and we recall some properties that we need in the next section.

Definition 3. The Fourier transform associated with the spherical mean operator is defined on $L^1(d\nu_n)$ by

$$\forall (\mu,\lambda) \in \Gamma, \quad \mathscr{F}(f)(\mu,\lambda) = \int_{\mathbb{R}^n}\int_0^{+\infty} f(r,x)\ \varphi_{\mu,\lambda}(r,x)\, d\nu_n(r,x),$$

where Γ is the set defined by the relation (2.2).

The Fourier transform $\mathscr{F}$ satisfies the properties

- For every f in $L^1(d\nu_n)$ and $(r,x) \in [0,+\infty[\times \mathbb{R}^n$, we have

$$\forall (\mu,\lambda) \in \Gamma, \quad \mathscr{F}\left(\tau_{(r,-x)}(f)\right)(\mu,\lambda)\varphi_{\mu,\lambda}(r,x)\mathscr{F}(f)(\mu,\lambda). \tag{2.8}$$

- For all $f, g \in L^1(d\nu_n)$, we have

$$\forall (\mu,\lambda) \in \Gamma, \quad \mathscr{F}(f * g)(\mu,\lambda) = \mathscr{F}(f)(\mu,\lambda)\mathscr{F}(g)(\mu,\lambda). \tag{2.9}$$

- For all $f \in L^1(d\nu_n)$, we have

$$\forall (\mu,\lambda) \in \Gamma, \quad \mathscr{F}(f)(\mu,\lambda)\widetilde{\mathscr{F}}(f) \circ \theta(\mu,\lambda), \tag{2.10}$$

where

$$\forall(\mu,\lambda)\in\mathbb{R}\times\mathbb{R}^n,\quad \widetilde{\mathscr{F}}(f)(\mu,\lambda)=\int_{\mathbb{R}^n}\int_0^{+\infty} f(r,x)j_{\frac{n-1}{2}}(r\mu)e^{-i\langle\lambda|x\rangle}\,d\nu_n(r,x) \tag{2.11}$$

and θ is the function defined by the relation (2.3).

Theorem 4. (Inversion formula for $\mathscr{F}$) Let $f\in L^1(d\nu_n)$ such that the function $\mathscr{F}(f)$ belongs to $L^1(d\gamma_n)$, then for almost every $(r,x)\in[0,+\infty[\times\mathbb{R}^n$, we have

$$f(r,x)=\int\int_{\Gamma_+}\mathscr{F}(f)(\mu,\lambda)\,\overline{\varphi_{\mu,\lambda}(r,x)}\,d\gamma_n(\mu,\lambda).$$

We denote by

• $\mathscr{E}_*(\mathbb{R}\times\mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable.

• $S_*(\mathbb{R}\times\mathbb{R}^n)$ the subspace of $\mathscr{E}_*(\mathbb{R}\times\mathbb{R}^n)$ consisting of functions rapidly decreasing together with all their derivatives.

• $S_*(\Gamma)$ the space of functions $f:\Gamma\longrightarrow\mathbb{C}$ infinitely differentiable, even with respect to the first variable and rapidly decreasing together with all their derivatives, i.e

$$\forall k_1,k_2\in\mathbb{N},\ \forall\alpha\in\mathbb{N}^n,\quad \sup_{(\mu,\lambda)\in\Gamma}\ \left(1+\mu^2+2|\lambda|^2\right)^{k_1}\left|\left(\frac{\partial}{\partial\mu}\right)^{k_2}D_\lambda^\alpha f(\mu,\lambda)\right|<+\infty,$$

where

$$\frac{\partial f}{\partial\mu}(\mu,\lambda)\begin{cases}\dfrac{\partial}{\partial r}\left(f(r,\lambda)\right), & \text{if } \mu=r\in\mathbb{R}\\[2ex] \dfrac{1}{i}\dfrac{\partial}{\partial t}\left(f(it,\lambda)\right), & \text{if } \mu=it, |t|\leqslant|\lambda|\end{cases}$$

and

$$D_\lambda^\alpha=\left(\frac{\partial}{\partial\lambda_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial\lambda_n}\right)^{\alpha_n}.$$

• $S'_*(\mathbb{R}\times\mathbb{R}^n)$ and $S'_*(\Gamma)$ are respectively the topological dual spaces of $S_*(\mathbb{R}\times\mathbb{R}^n)$ and $S_*(\Gamma)$.

Each of these spaces is equipped with its usual topology.

Theorem 5. (Schwartz theorem)[2, 18] i) The Fourier transform $\mathscr{F}$ is a topological isomorphism from $S_*(\mathbb{R}\times\mathbb{R}^n)$ onto $S_*(\Gamma)$. The inverse mapping is given by

$$\forall(r,x)\in\mathbb{R}\times\mathbb{R}^n,\ \mathscr{F}^{-1}(f)(r,x)\ =\ \int\int_{\Gamma_+}f(\mu,\lambda)\,\overline{\varphi_{\mu,\lambda}(r,x)}\,d\gamma_n(\mu,\lambda). \tag{2.12}$$

ii) (Plancherel formula) For all $f,g\in S_*(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^n}f(r,x)\,\overline{g(r,x)}\,d\nu_n(r,x)\ =\ \int\int_{\Gamma_+}\mathscr{F}(f)(\mu,\lambda)\,\overline{\mathscr{F}(g)(\mu,\lambda)}\,d\gamma_n(\mu,\lambda).$$

In particular

$$\|\mathscr{F}(f)\|_{2,\gamma_n}\ \|f\|_{2,\nu_n}\ .$$

Theorem 6. (Plancherel theorem) The Fourier transform $\mathscr{F}$ can be extended to an isometric isomorphism from $L^2(d\nu_n)$ onto $L^2(d\gamma_n)$.

For $T \in S'_*(\mathbb{R}\times\mathbb{R}^n)$, we put

$$\langle \mathscr{F}(T), \varphi\rangle \quad = \quad \langle T, \mathscr{F}^{-1}(\varphi)\rangle; \quad \varphi \in S_*(\Gamma). \tag{2.13}$$

Then from Theorem 5, we get the following result

Corollary 7. The transform $\mathscr{F}$ defined by the relation (2.13) is a topological isomorphism from $S'_*(\mathbb{R}\times\mathbb{R}^n)$ onto $S'_*(\Gamma)$.

Proposition 8. i) Let $f \in \mathscr{E}_*(\mathbb{R}\times\mathbb{R}^n)$, f slowly increasing and let $g \in S_*(\mathbb{R}\times\mathbb{R}^n)$. Then the function $f*g$ belongs to the space $\mathscr{E}_*(\mathbb{R}\times\mathbb{R}^n)$.

ii) For all $f \in S_*(\mathbb{R}\times\mathbb{R}^n)$ and $T \in S'_*(\mathbb{R}\times\mathbb{R}^n)$. The function $T*f$ defined by

$$\forall (r,x) \in \mathbb{R}\times\mathbb{R}^n, \quad T*f(r,x) = \langle T, \tau_{(r,-x)}(\check{f})\rangle$$

belongs to the space $\mathscr{E}_*(\mathbb{R}\times\mathbb{R}^n)$ and is slowly increasing. Moreover, we have

$$\mathscr{F}\left(T_{T*f}\right) = \mathscr{F}(\check{f})\mathscr{F}(T).$$

3 Besov spaces

This section contains the main result of this paper. Indeed, we define and study a class of Besov spaces $\mathscr{B}^{\gamma,\phi}_{p,q}\left([0,+\infty[\times\mathbb{R}^n\right)$, where ϕ is a smooth function. We show that this space is independant of the choice of ϕ and is a Banach space for $\gamma < (2n+1)/p$. Next, we prove that $\mathscr{B}^{\gamma,\phi}_{p,q}\left([0,+\infty[\times\mathbb{R}^n\right)$ is an homogeneous space with degree equal to $(2n+1)/p-\gamma-2n-1$.

Lemma 9. Let $a,\ b,\ a_1,\ b_1$ be real numbers such that $0 < a_1 < a < b < b_1$. Then there exists a function $\psi \in S_*(\mathbb{R}\times\mathbb{R}^n)$ satisfying the following assumptions

i) $\forall (\mu,\lambda) \in \Gamma, \quad \mathscr{F}(\psi)(\mu,\lambda) \geqslant 0.$

ii) $\forall (\mu,\lambda) \in \Gamma; \quad a^2 \leqslant \mu^2 + 2|\lambda|^2 \leqslant b^2, \quad \mathscr{F}(\psi)(\mu,\lambda) = C$
where C is a positive constante.

iii) $\mathscr{F}(\psi)(\mu,\lambda) = 0$ if $\mu^2 + 2|\lambda|^2 > b_1^2$ or $\mu^2 + 2|\lambda|^2 < a_1^2$.

iv) For all $(\mu,\lambda)\in\Gamma\setminus\{(0,0)\}$,

$$\int_0^{+\infty}\big(\mathscr{F}(\psi)(t\mu,t\lambda)\big)^2\frac{dt}{t}=1.$$

Proof. From Uryshon's lemma, there exists an infinitely differentiable function ω on $\mathbb{R}$ such that

- $\forall t\in\mathbb{R};\quad 0\leqslant\omega(t)\leqslant 1.$
- $\forall t\in[a,b];\quad \omega(t)=1.$
- $\mathrm{supp}(\omega)\subset]a_1,b_1[.$

Let g be the function defined on $\mathbb{R}\times\mathbb{R}^n$ by

$$g(r,x)=\frac{\omega\big(\sqrt{r^2+|x|^2}\big)}{\Big(\int_0^{+\infty}(\omega(t))^2\frac{dt}{t}\Big)^{\frac{1}{2}}},$$

then the function g belongs to the space $S_*(\mathbb{R}\times\mathbb{R}^n)$. Since, the transform $\widetilde{\mathscr{F}}$ defined by the relation (2.11) is a topological isomorphism from the space $S_*(\mathbb{R}\times\mathbb{R}^n)$ onto itself [24, 25], then there exists $\psi\in S_*(\mathbb{R}\times\mathbb{R}^n)$ such that $\widetilde{\mathscr{F}}(\psi)=g$. Thus, by the relation (2.10), we deduce that the function ψ satisfies the hypothesis of the lemma. □

We denote by

- $\mathscr{D}_*(\Gamma)$ the space of real infinitely differentiable functions g on Γ, even with respect to the first variable such that, there exist two positive real numbers
$0<a<b$ verifying

$$g(\mu,\lambda)=0\quad\text{if}\quad \mu^2+2|\lambda|^2<a^2\quad\text{or}\quad \mu^2+2|\lambda|^2>b^2.$$

- $S_{*,0}(\mathbb{R}\times\mathbb{R}^n)$ the subspace of $S_*(\mathbb{R}\times\mathbb{R}^n)$ consisting of functions f such that $\mathscr{F}(f)$ belongs to the space $\mathscr{D}_*(\Gamma)$.
- $S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$ the subspace of $S_{*,0}(\mathbb{R}\times\mathbb{R}^n)$ formed by the functions f such that

$$\forall(\mu,\lambda)\in\Gamma\setminus\{(0,0)\},\quad \int_0^{+\infty}\big(\mathscr{F}(f)(t\mu,t\lambda)\big)^2\frac{dt}{t}\ =\ 1. \tag{3.1}$$

These functions are known as wavelets on $[0,+\infty[\times\mathbb{R}^n$ [19, 26].

- $L^p(\frac{dt}{t})$; $p \in [1,+\infty]$, the space of measurable functions on $]0,+\infty[$ such that

$$\|f\|_{L^p(\frac{dt}{t})} = \begin{cases} \left(\int_0^{+\infty} |f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}} < +\infty, & 1 \leqslant p < +\infty; \\ \operatorname{esssup}_{t>0} |f(t)| < +\infty, & p = +\infty. \end{cases}$$

- $\star$ the convolution product defined on the group $\big(]0,+\infty[,.\big)$ by

$$f \star g(s) \quad = \quad \int_0^{+\infty} f(t)\ g(\frac{s}{t})\frac{dt}{t}. \tag{3.2}$$

- For all measurable function ϕ on $[0,+\infty[\times \mathbb{R}^n$, the dilated ϕ_t; $t > 0$ of ϕ is defined by

$$\forall (r,x) \in [0,+\infty[\times \mathbb{R}^n, \quad \phi_t(r,x) = \frac{1}{t^{2n+1}}\ \phi(\frac{r}{t},\frac{x}{t}).$$

Then we have the following properties

- Let $p,\ q,\ r \in [1,+\infty]$ such that $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$. Then for all $f \in L^p(\frac{dt}{t})$ and $g \in L^q(\frac{dt}{t})$, the function $f \star g$ belongs to $L^r(\frac{dt}{t})$ and we have

$$\|f \star g\|_{L^r(\frac{dt}{t})} \quad \leqslant \quad \|f\|_{L^p(\frac{dt}{t})}\|g\|_{L^q(\frac{dt}{t})}. \tag{3.3}$$

- For every $\phi \in L^p(d\nu_n)$; $p \in [1,+\infty]$, the function ϕ_t belongs to $L^p(d\nu_n)$ and we have

$$\|\phi_t\|_{p,\nu_n} \quad = \quad t^{-\frac{2n+1}{p'}}\ \|\phi\|_{p,\nu_n}, \tag{3.4}$$

where $p' = p/(p-1)$.

- For all $\phi \in L^1(d\nu_n)$ and for every $(\mu,\lambda) \in \Gamma$,

$$\mathscr{F}(\phi_t)(\mu,\lambda) \quad = \quad \mathscr{F}(\phi)(t\mu,t\lambda). \tag{3.5}$$

Definition 10. Let $p,\ q \in [1,+\infty]$, $\gamma \in \mathbb{R}$ and $\phi \in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$. We define the Besov space $\mathscr{B}^{\gamma,\phi}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ to be the space of tempered distributions f on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable and satisfying

- For all $t > 0$, the function $f * \phi_t$ belongs to the space $L^p(d\nu_n)$.
- The function

$$t \longmapsto \frac{\|f * \phi_t\|_{p,\nu_n}}{t^\gamma}$$

belongs to the space $L^q(\frac{dt}{t})$.

- The integral

$$(r,x) \longmapsto \int_0^{+\infty} f * \phi_t * \phi_t(r,x)\frac{dt}{t}$$

is convergent in $S'_*(\mathbb{R}\times\mathbb{R}^n)$ and

$$f = \int_0^{+\infty} f*\phi_t*\phi_t\frac{dt}{t}. \tag{3.6}$$

The space $\mathscr{B}^{\gamma,\phi}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ is equipped with the norm

$$M^{\gamma,\phi}_{p,q}(f)=\begin{cases}\left(\int_0^{+\infty}\Big(\frac{\|f*\phi_t\|_{p,\nu_n}}{t^\gamma}\Big)^q\frac{dt}{t}\right)^{\frac{1}{q}}, & \text{if } 1\leqslant q<+\infty;\\[2mm] \operatorname*{esssup}_{t>0}\frac{\|f*\phi_t\|_{p,\nu_n}}{t^\gamma}, & \text{if } q=+\infty.\end{cases}$$

Lemma 11. let $\psi\in S_*(\mathbb{R}\times\mathbb{R}^n)$ and let $\phi\in S_{*,0}(\mathbb{R}\times\mathbb{R}^n)$. Then for all $k\in\mathbb{N}$, there exists $\phi_k\in S_{*,0}(\mathbb{R}\times\mathbb{R}^n)$ such that

$$\psi*\phi_t=t^{2k}\,\big(\Delta^k\psi\big)*(\phi_k)_t,$$

where Δ is the differential operator defined by

$$\Delta=-\Big(\frac{\partial^2}{\partial r^2}+\frac{n}{r}\,\frac{\partial}{\partial r}+\sum_{j=1}^{n}\big(\frac{\partial}{\partial x_j}\big)^2\Big).$$

Moreover, for all $p\in[1,+\infty]$

$$\big\|\psi*\phi_t\big\|_{p,\nu_n} \leqslant t^{2k}\,\big\|\Delta^k\psi\big\|_{p,\nu_n}\big\|\phi_k\big\|_{1,\nu_n} \tag{3.7}$$

and

$$\big\|\psi*\phi_t\big\|_{p,\nu_n} \leqslant t^{-\frac{2n+1}{p'}}\,\big\|\psi\big\|_{1,\nu_n}\big\|\phi\big\|_{p,\nu_n}. \tag{3.8}$$

Proof. The operator Δ is continuous from $S_*(\mathbb{R}\times\mathbb{R}^n)$ into itself and for all $f\in S_*(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\mathscr{F}\big(\Delta f\big)(\mu,\lambda) = \big(\mu^2+2|\lambda|^2\big)\,\mathscr{F}(f)(\mu,\lambda). \tag{3.9}$$

Let $\psi\in S_*(\mathbb{R}\times\mathbb{R}^n)$ and let $\phi\in S_{*,0}(\mathbb{R}\times\mathbb{R}^n)$. From the relations (2.9) and (3.5), we get

$$\begin{aligned}\mathscr{F}\big(\psi*\phi_t\big)(\mu,\lambda) &= \mathscr{F}(\psi)(\mu,\lambda)\,\mathscr{F}(\phi)(t\mu,t\lambda)\\ &= t^2\big(\mu^2+2|\lambda|^2\big)\,\mathscr{F}(\psi)(\mu,\lambda)\,\frac{\mathscr{F}(\phi)(t\mu,t\lambda)}{t^2\big(\mu^2+2|\lambda|^2\big)},\end{aligned}$$

and from the equality (3.9), we obtain

$$\mathscr{F}\big(\psi*\phi_t\big)(\mu,\lambda) = t^2\mathscr{F}(\Delta\psi)(\mu,\lambda)\,\frac{\mathscr{F}(\phi)(t\mu,t\lambda)}{t^2\big(\mu^2+2|\lambda|^2\big)}. \tag{3.10}$$

Since, the function ϕ belongs to the space $S_{*,0}(\mathbb{R}\times\mathbb{R}^n)$ then the function

$$(\mu,\lambda)\longmapsto \frac{\mathscr{F}(\phi)(\mu,\lambda)}{\mu^2+2|\lambda|^2}$$

belongs to the space $S_*(\Gamma)$ and from Theorem 5, there exists $\phi_1\in S_*(\mathbb{R}\times\mathbb{R}^n)$ such that

$$\mathscr{F}(\phi_1)(\mu,\lambda) = \frac{\mathscr{F}(\phi)(\mu,\lambda)}{\mu^2+2|\lambda|^2}.$$

In particular, ϕ_1 lies in $S_{*,0}(\mathbb{R}\times\mathbb{R}^n)$ and the relation (3.10) leads to

$$\mathscr{F}\big(\psi*\phi_t\big)(\mu,\lambda) = t^2\mathscr{F}(\Delta\psi)(\mu,\lambda)\ \mathscr{F}\big((\phi_1)_t\big)(\mu,\lambda),$$

which implies that

$$\psi*\phi_t = t^2\ (\Delta\psi)*(\phi_1)_t.$$

By induction, for all $k\in\mathbb{N}^*$, there exists $\phi_k\in S_{*,0}(\mathbb{R}\times\mathbb{R}^n)$ verifying

$$\psi*\phi_t = t^{2k}\ (\Delta^k\psi)*(\phi_k)_t. \tag{3.11}$$

On the other hand, for every $t>0$ and by the relation (3.4), we get

$$\begin{aligned}\big\|\psi*\phi_t\big\|_{p,\nu_n} &\leqslant \|\psi\|_{1,\nu_n}\ \|\phi_t\|_{p,\nu_n}\\ &= t^{-\frac{2n+1}{p'}}\ \|\psi\|_{1,\nu_n}\|\phi\|_{p,\nu_n}\end{aligned}$$

as the same way and using the relation (3.11), it follows that

$$\big\|\psi*\phi_t\big\|_{p,\nu_n} \leqslant t^{2k}\ \|\Delta^k\psi\|_{p,\nu_n}\|\phi_k\|_{1,\nu_n}.$$

□

Proposition 12. Let $\phi\in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$.

i) For all $f\in L^2(d\nu_n)$ we have

$$f=\int_0^{+\infty} f*\phi_t*\phi_t\frac{dt}{t};\quad \text{in } L^2(d\nu_n).$$

ii) Let $\gamma\in\mathbb{R}$; $\gamma<(2n+1)/p$ and $f\in S'_*(\mathbb{R}\times\mathbb{R}^n)$ such that for all $t>0$, the function $f*\phi_t$ belongs to $L^p(d\nu_n)$ and the function

$$t\longmapsto\frac{\|f*\phi_t\|_{p,\nu_n}}{t^\gamma}$$

belongs to the space $L^q(\frac{dt}{t})$. Then the integral

$$\int_0^{+\infty} f*\phi_t*\phi_t\frac{dt}{t}$$

converges in $S'_*(\mathbb{R}\times\mathbb{R}^n)$.

Proof. i) Let $f \in L^2(d\nu_n)$ and let $F_{a,b}(f)$ be the function defined by

$$\forall (r,x) \in [0,+\infty[\times \mathbb{R}^n, \quad F_{a,b}(f)(r,x) = \int_a^b f * \phi_t * \phi_t(r,x)\,\frac{dt}{t}; \quad 0 < a < b.$$

The function $F_{a,b}(f)$ is well defined and by the relation (3.4) we have

$$\begin{aligned}
\left|F_{a,b}(f)(r,x)\right| &\leqslant \int_a^b \|f\|_{2,\nu_n}\ \|\phi_t * \phi_t\|_{2,\nu_n}\ \frac{dt}{t}\\
&\leqslant \|f\|_{2,\nu_n} \int_a^b \|\phi_t\|_{1,\nu_n}\ \|\phi_t\|_{2,\nu_n}\ \frac{dt}{t}\\
&\leqslant \|f\|_{2,\nu_n}\ \|\phi\|_{1,\nu_n}\ \|\phi\|_{2,\nu_n} \int_a^b t^{-\frac{2n+1}{2}-1}\,dt\\
&< +\infty.
\end{aligned}$$

Moreover, the function $F_{a,b}(f)$ belongs to $L^2(d\nu_n)$. Indeed by Minkowski's inequality [12] and the relation (3.4) we get

$$\begin{aligned}
\left\|F_{a,b}(f)\right\|_{2,\nu_n} &\leqslant \int_a^b \left\|f * \phi_t * \phi_t\right\|_{2,\nu_n} \frac{dt}{t}\\
&\leqslant \int_a^b \|f\|_{2,\nu_n}\ \|\phi_t\|^2_{1,\nu_n}\ \frac{dt}{t}\\
&= \|f\|_{2,\nu_n}\ \|\phi\|^2_{1,\nu_n}\ \log\left(\frac{b}{a}\right)\\
&< +\infty.
\end{aligned}$$

On the other hand, by Fubini's theorem and the relation (3.5), we have

$$\mathscr{F}\big(F_{a,b}(f)\big)(\mu,\lambda) = \mathscr{F}(f)(\mu,\lambda) \int_a^b \Big(\mathscr{F}(\phi)(t\mu,t\lambda)\Big)^2 \frac{dt}{t}.$$

Thus, by the Plancherel theorem

$$\begin{aligned}
\left\|f - F_{a,b}(f)\right\|^2_{2,\nu_n} &= \left\|\mathscr{F}(f) - \mathscr{F}(F_{a,b}(f))\right\|^2_{2,\gamma_n}\\
&= \int\!\!\int_{\Gamma_+} \left|\mathscr{F}(f)(\mu,\lambda)\right|^2 \left|1 - \int_a^b \Big(\mathscr{F}(\phi)(t\mu,t\lambda)\Big)^2 \frac{dt}{t}\right| d\gamma_n(\mu,\lambda).
\end{aligned}$$

Using the fact that

$$\int_0^{+\infty} \Big(\mathscr{F}(\phi)(t\mu,t\lambda)\Big)^2 \frac{dt}{t} = 1,$$

we have

$$\forall (\mu,\lambda) \in \Gamma\backslash\{(0,0)\}, \quad \left|1 - \int_a^b \Big(\mathscr{F}(\phi)(t\mu,t\lambda)\Big)^2 \frac{dt}{t}\right| \leqslant 1$$

and applying the dominated convergence theorem, we deduce that

$$\lim_{\substack{a\to 0^+\\ b\to+\infty}} \left\|f - F_{a,b}(f)\right\|_{2,\nu_n} = 0.$$

ii) Let f be in $S'_*(\mathbb{R}\times\mathbb{R}^n)$ satisfying the hypothesis, then the function $F_{a,b}(f)$ defined above is bounded on $\mathbb{R}\times\mathbb{R}^n$. In fact

$$\begin{aligned}
\left|F_{a,b}(f)(r,x)\right| &\leqslant \int_a^b \left\|f*\phi_t\right\|_{p,\nu_n}\left\|\phi_t\right\|_{p',\nu_n}\frac{dt}{t}\\
&= \left\|\phi\right\|_{p',\nu_n}\int_a^b \frac{\left\|f*\phi_t\right\|_{p,\nu_n}}{t^{\gamma}}\; t^{-\frac{2n+1}{p}+\gamma}\;\frac{dt}{t}\\
&\leqslant \left\|\phi\right\|_{p',\nu_n}\Big[\int_a^b\Big(\frac{\left\|f*\phi_t\right\|_{p,\nu_n}}{t^{\gamma}}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}}\Big[\int_a^b t^{(-\frac{2n+1}{p}+\gamma)\;q'}\frac{dt}{t}\Big]^{\frac{1}{q'}}\\
&< +\infty,
\end{aligned}$$

where q' is the conjugate exponent of q.
Thus for all $a,\ b\in\mathbb{R};\ b>a>0$, the function $F_{a,b}(f)$ defines an element of $S'_*(\mathbb{R}\times\mathbb{R}^n)$.
Let $\psi\in S_*(\mathbb{R}\times\mathbb{R}^n)$, by Fubini's theorem, we have

$$\begin{aligned}
&\langle F_{a,b}(f),\psi\rangle = \int_0^{+\infty}\int_{\mathbb{R}^n}\Big\{\int_a^b f*\phi_t*\phi_t(r,x)\;\psi(r,x)\;\frac{dt}{t}\Big\}d\nu_n(r,x)\\
&= \int_a^b\Big\{\int_0^{+\infty}\int_{\mathbb{R}^n} f*\phi_t*\phi_t(r,x)\;\psi(r,x)\;d\nu_n(r,x)\Big\}\frac{dt}{t}\\
&= \int_a^b\Big\{\int_0^{+\infty}\int_{\mathbb{R}^n}\psi(r,x)\Big[\int_0^{+\infty}\int_{\mathbb{R}^n} f*\phi_t(s,y)\;\tau_{(r,-x)}(\check{\phi}_t)(s,y)\;d\nu_n(s,y)\Big]\;d\nu_n(r,x)\Big\}\frac{dt}{t}\\
&= \int_a^b\Big\{\int_0^{+\infty}\int_{\mathbb{R}^n} f*\phi_t(s,y)\Big[\int_0^{+\infty}\int_{\mathbb{R}^n}\psi(r,x)\;\tau_{(s,-y)}(\phi_t)(r,x)\;d\nu_n(r,x)\Big]\;d\nu_n(s,y)\Big\}\frac{dt}{t}\\
&= \int_a^b\Big[\int_0^{+\infty}\int_{\mathbb{R}^n} f*\phi_t(s,y)\;\check{\phi}_t*\psi(s,y)\;d\nu_n(s,y)\Big]\;\frac{dt}{t}.
\end{aligned}$$

However,

$$\begin{aligned}
&\int_0^{+\infty}\Big[\int_0^{+\infty}\int_{\mathbb{R}^n}\left|f*\phi_t(s,y)\right|\;\left|\check{\phi}_t*\psi(s,y)\right|\;d\nu_n(s,y)\Big]\;\frac{dt}{t}\\
&\qquad\leqslant \int_0^{+\infty}\left\|f*\phi_t\right\|_{p,\nu_n}\;\left\|\check{\phi}_t*\psi\right\|_{p',\nu_n}\;\frac{dt}{t}\\
&\qquad\leqslant \int_0^1\left\|f*\phi_t\right\|_{p,\nu_n}\;\left\|\check{\phi}_t*\psi\right\|_{p',\nu_n}\;\frac{dt}{t}+\int_1^{+\infty}\left\|f*\phi_t\right\|_{p,\nu_n}\;\left\|\check{\phi}_t*\psi\right\|_{p',\nu_n}\;\frac{dt}{t}.
\end{aligned}$$

Using the relations (3.7) and (3.8), we get

$$\begin{aligned}
&\int_0^{+\infty}\left\{\int_0^{+\infty}\int_{\mathbb{R}^n}\left|f*\phi_t(s,y)\right|\,\left|\check{\phi}_t*\psi(s,y)\right|\,d\nu_n(s,y)\right\}\frac{dt}{t}\\
&\qquad\leqslant \left\|\Delta^k\psi\right\|_{p',\nu_n}\left\|\phi_k\right\|_{1,\nu_n}\int_0^1 t^{2k}\left\|f*\phi_t\right\|_{p,\nu_n}\frac{dt}{t}\\
&\qquad+\left\|\psi\right\|_{1,\nu_n}\left\|\phi\right\|_{p',\nu_n}\int_1^{+\infty}t^{-\frac{2n+1}{p}}\left\|f*\phi_t\right\|_{p,\nu_n}\frac{dt}{t}\\
&\qquad\left\|\Delta^k\psi\right\|_{p',\nu_n}\left\|\phi_k\right\|_{1,\nu_n}\int_0^{+\infty}t^{2k+\gamma}\,\mathbf{1}_{[0,1]}(t)\,\frac{\left\|f*\phi_t\right\|_{p,\nu_n}}{t^\gamma}\frac{dt}{t}\\
&\qquad+\left\|\psi\right\|_{1,\nu_n}\left\|\phi\right\|_{p',\nu_n}\int_0^{+\infty}t^{-\frac{2n+1}{p}+\gamma}\,\mathbf{1}_{[1,+\infty[}(t)\,\frac{\left\|f*\phi_t\right\|_{p,\nu_n}}{t^\gamma}\frac{dt}{t}.
\end{aligned}$$

Let k be sufficiently large. Using the hypothesis $\gamma<(2n+1)/p$ and applying Hölder's inequality, we obtain

$$\begin{aligned}
&\int_0^{+\infty}\left\{\int_0^{+\infty}\int_{\mathbb{R}^n}\left|f*\phi_t(s,y)\right|\,\left|\check{\phi}_t*\psi(s,y)\right|\,d\nu_n(s,y)\right\}\frac{dt}{t}\\
&\leqslant \left\|\Delta^k\psi\right\|_{p',\nu_n}\left\|\phi_k\right\|_{1,\nu_n}\left\|t^{2k+\gamma}\,\mathbf{1}_{[0,1]}\right\|_{L^{q'}(\frac{dt}{t})}\left\|\frac{\left\|f*\phi_t\right\|_{p,\nu_n}}{t^\gamma}\right\|_{L^q(\frac{dt}{t})}\\
&+\left\|\psi\right\|_{1,\nu_n}\left\|\phi\right\|_{p',\nu_n}\left\|t^{-\frac{2n+1}{p}+\gamma}\,\mathbf{1}_{[1,+\infty[}\right\|_{L^{q'}(\frac{dt}{t})}\left\|\frac{\left\|f*\phi_t\right\|_{p,\nu_n}}{t^\gamma}\right\|_{L^q(\frac{dt}{t})}\\
&<+\infty.
\end{aligned}$$

This shows that for all $\psi\in S_*(\mathbb{R}\times\mathbb{R}^n)$, $\lim\limits_{\substack{a\to 0^+\\ b\to+\infty}}\langle F_{a,b}(f),\psi\rangle$ exists and

$$\lim_{\substack{a\to 0^+\\ b\to+\infty}}\langle F_{a,b}(f),\psi\rangle = \int_0^{+\infty}\int_0^{+\infty}\int_{\mathbb{R}^n} f*\phi_t(s,y)\ \check{\phi}_t*\psi(s,y)\ d\nu_n(s,y)\ \frac{dt}{t}.$$

This means that the integral

$$\int_0^{+\infty} f*\phi_t*\phi_t\ \frac{dt}{t}$$

converges in $S'_*(\mathbb{R}\times\mathbb{R}^n)$. □

Lemma 13. 1) Let $f\in\mathscr{B}^{\gamma,\phi}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$. Then

i) For all $\psi\in S_*(\mathbb{R}\times\mathbb{R}^n)$, we have

$$f*\psi=\int_0^{\infty} f*\phi_t*\phi_t*\psi\ \frac{dt}{t}.$$

ii) For all $\psi\in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$,

$$f=\int_0^{+\infty} f*\psi_\rho*\psi_\rho\ \frac{d\rho}{\rho}.$$

2) For all $g \in S_*(\mathbb{R} \times \mathbb{R}^n)$ and for all $\psi \in S^1_{*,0}(\mathbb{R} \times \mathbb{R}^n)$, we have

$$\int_0^{+\infty} g * \psi_\rho * \psi_\rho \, \frac{d\rho}{\rho} = g.$$

Proof. 1) Let $f \in \mathscr{B}^{\gamma,\phi}_{p,q}\left([0,+\infty[\times \mathbb{R}^n\right)$.

i) For every $\psi \in S_*(\mathbb{R} \times \mathbb{R}^n)$, we have

$$\begin{aligned} f * \psi(r,x) &= \langle f, \tau_{(r,-x)}\check{\psi}\rangle \\ &= \lim_{\substack{a\to 0^+\\ b\to+\infty}} \langle \int_a^b f * \phi_t * \phi_t \, \frac{dt}{t}, \tau_{(r,-x)}\check{\psi}\rangle \\ &= \lim_{\substack{a\to 0^+\\ b\to+\infty}} \int_0^{+\infty}\int_{\mathbb{R}^n}\left(\int_a^b f * \phi_t * \phi_t(s,y)\, \frac{dt}{t}\right)\tau_{(r,-x)}\check{\psi}(s,y)\, d\nu_n(s,y), \end{aligned}$$

and by Fubini's theorem, we obtain

$$\begin{aligned} f * \psi(r,x) &= \lim_{\substack{a\to 0^+\\ b\to+\infty}} \int_a^b \left(\int_0^{+\infty}\int_{\mathbb{R}^n} f * \phi_t * \phi_t(s,y)\; \tau_{(r,-x)}\check{\psi}(s,y)\, d\nu_n(s,y)\right)\frac{dt}{t} \\ &= \lim_{\substack{a\to 0^+\\ b\to+\infty}} \int_a^b f * \phi_t * \phi_t * \psi(r,x)\, \frac{dt}{t} \\ &= \int_0^{+\infty} f * \phi_t * \phi_t * \psi(r,x)\, \frac{dt}{t}. \end{aligned}$$

ii) Let $\psi \in S^1_{*,0}(\mathbb{R} \times \mathbb{R}^n)$. For all positive real number ρ, we have

$$\psi_\rho * \psi_\rho \;=\; (\psi * \psi)_\rho. \tag{3.12}$$

Applying i) we get

$$f * \psi_\rho * \psi_\rho = \int_0^{+\infty} f * \phi_t * \phi_t * \psi_\rho * \psi_\rho \, \frac{dt}{t}.$$

Now, let $a_1,\ a_2,\ b_1,\ b_2$ be positive real numbers such that

$$\mathscr{F}(\phi)(\mu,\lambda) = 0 \quad \text{if} \quad \mu^2 + 2|\lambda|^2 < a_1^2 \quad \text{or} \quad \mu^2 + 2|\lambda|^2 > b_1^2$$

and

$$\mathscr{F}(\psi)(\mu,\lambda) = 0 \quad \text{if} \quad \mu^2 + 2|\lambda|^2 < a_2^2 \quad \text{or} \quad \mu^2 + 2|\lambda|^2 > b_2^2$$

then

$$\mathscr{F}(\phi_t)(\mu,\lambda)\mathscr{F}(\psi_\rho)(\mu,\lambda) = 0 \quad \text{if} \quad \frac{t}{\rho} \notin \left[\frac{a_1}{b_2},\ \frac{b_1}{a_2}\right] = [\alpha,\ \beta]\,,$$

and consequently, by the relation (2.9) and Theorem 4

$$\phi_t * \psi_\rho = 0 \quad \text{if} \quad \frac{t}{\rho} \notin [\alpha,\ \beta]\,. \tag{3.13}$$

Thus,

$$f * \psi_\rho * \psi_\rho = \int_{\rho\alpha}^{\rho\beta} f * \phi_t * \phi_t * \psi_\rho * \psi_\rho \, \frac{dt}{t}.$$

So for all $a,\ b \in \mathbb{R};\ 0 < a < b$,

$$\int_a^b f * \psi_\rho * \psi_\rho \, \frac{d\rho}{\rho} = \int_a^b \Big(\int_{\rho\alpha}^{\rho\beta} f * \phi_t * \phi_t * \psi_\rho * \psi_\rho \, \frac{dt}{t} \Big) \frac{d\rho}{\rho}.$$

By Fubini's theorem, we get

$$\int_a^b f * \psi_\rho * \psi_\rho \, \frac{d\rho}{\rho} \quad = \quad \int_{a\alpha}^{b\beta} \Big(\int_{\frac{t}{\beta}}^{\frac{t}{\alpha}} f * \phi_t * \phi_t * \psi_\rho * \psi_\rho \, \frac{d\rho}{\rho} \Big) \frac{dt}{t}. \tag{3.14}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\frac{t}{\beta}}^{\frac{t}{\alpha}} f * \phi_t * \phi_t * \psi_\rho * \psi_\rho(r,x) \, \frac{d\rho}{\rho} \\
&= \int_{\frac{t}{\beta}}^{\frac{t}{\alpha}} \Big(\int_0^{+\infty} \int_{\mathbb{R}^n} \phi_t * \psi_\rho * \psi_\rho(s,y) \ \tau_{(r,-x)}(f * \check{\phi}_t)(s,y) d\nu_n(s,y) \Big) \frac{d\rho}{\rho} \\
&= \int_0^{+\infty} \int_{\mathbb{R}^n} \tau_{(r,-x)}(f * \check{\phi}_t)(s,y) \Big(\int_{\frac{t}{\beta}}^{\frac{t}{\alpha}} \phi_t * \psi_\rho * \psi_\rho(s,y) \, \frac{d\rho}{\rho} \Big) d\nu_n(s,y) \\
&= \int_0^{+\infty} \int_{\mathbb{R}^n} \tau_{(r,-x)}(f * \check{\phi}_t)(s,y) \Big(\int_0^{+\infty} \phi_t * \psi_\rho * \psi_\rho(s,y) \, \frac{d\rho}{\rho} \Big) d\nu_n(s,y).
\end{aligned}$$

However by i) of Proposition 12, it follows that

$$\begin{aligned}
\int_{\frac{t}{\beta}}^{\frac{t}{\alpha}} f * \phi_t * \ \phi_t * \psi_\rho * \psi_\rho(r,x) \, \frac{d\rho}{\rho} &= \int_0^{+\infty} \int_{\mathbb{R}^n} \phi_t(s,y) \ \tau_{(r,-x)}(f * \check{\phi}_t)(s,y) \ d\nu_n(s,y) \\
&= f * \phi_t * \phi_t(r,x).
\end{aligned}$$

Replacing in the equality (3.14), we obtain

$$\int_a^b f * \psi_\rho * \psi_\rho \, \frac{d\rho}{\rho} = \int_{a\alpha}^{b\beta} f * \phi_t * \phi_t \, \frac{dt}{t}.$$

2) We know that for all $g \in S_*(\mathbb{R} \times \mathbb{R}^n)$ and $\psi \in S^1_{*,0}(\mathbb{R} \times \mathbb{R}^n)$, the function $g * \psi_\rho * \psi_\rho$ belongs to the space $S_*(\mathbb{R} \times \mathbb{R}^n)$. By Theorem 4 and the relation (3.5), we have

$$g * \psi_\rho * \psi_\rho(r,x) = \int\int_{\Gamma_+} \mathscr{F}(g)(\mu,\lambda) \ \big(\mathscr{F}(\psi)(\rho\mu,\rho\lambda)\big)^2 \overline{\varphi_{\mu,\lambda}(r,x)} \ d\gamma_n(\mu,\lambda),$$

then

$$\int_0^{+\infty} g * \psi_\rho * \psi_\rho(r,x) \, \frac{d\rho}{\rho} \int\int_{\Gamma_+} \mathscr{F}(g)(\mu,\lambda) \overline{\varphi_{\mu,\lambda}(r,x)} \Big[\int_0^{+\infty} \big(\mathscr{F}(\psi)(\rho\mu,\rho\lambda)\big)^2 \, \frac{d\rho}{\rho} \Big] \ d\gamma_n(\mu,\lambda),$$

and by the relation (3.1) and Theorem 4, we get

$$\begin{aligned}\int_0^{+\infty} g * \psi_\rho * \psi_\rho(r,x)\, \frac{d\rho}{\rho} &= \iint_{\Gamma_+} \mathscr{F}(g)(\mu,\lambda)\, \overline{\varphi_{\mu,\lambda}(r,x)}\, d\gamma_n(\mu,\lambda) \\ &= g(r,x).\end{aligned}$$

□

Theorem 14. Let $p,\ q \in [1,+\infty]$ and $\gamma \in \mathbb{R}$, the space $\mathscr{B}^{\gamma,\phi}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ is independent of the choice of the function ϕ in $S^1_{*,0}\big(\mathbb{R}\times\mathbb{R}^n\big)$ and will be denoted by $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$.

Proof. Let $f \in \mathscr{B}^{\gamma,\phi}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ and let $\psi \in S^1_{*,0}\big(\mathbb{R}\times\mathbb{R}^n\big)$. From Lemma 13 and the relation (3.13), we have

$$\begin{aligned} f * \psi_\rho &= \int_0^{+\infty} f * \phi_t * \phi_t * \psi_\rho\, \frac{dt}{t} \\ &= \int_{\rho\alpha}^{\rho\beta} f * \phi_t * \phi_t * \psi_\rho\, \frac{dt}{t} \\ &= \int_\alpha^\beta f * \phi_{\rho s} * \phi_{\rho s} * \psi_\rho\, \frac{ds}{s}.\end{aligned}$$

Thus, from Minkowski's inequality and the relations (2.7) and (3.4), we get

$$\begin{aligned} \|f*\psi_\rho\|_{p,\nu_n} &\leqslant \int_\alpha^\beta \|f*\phi_{\rho s}*\psi_\rho*\phi_{\rho s}\|_{p,\nu_n}\, \frac{ds}{s} \\ &\leqslant \int_\alpha^\beta \|f*\phi_{\rho s}\|_{p,\nu_n}\|\psi_\rho*\phi_{\rho s}\|_{1,\nu_n}\, \frac{ds}{s} \\ &\leqslant \|\psi\|_{1,\nu_n}\|\phi\|_{1,\nu_n}\int_\alpha^\beta \|f*\phi_{\rho s}\|_{p,\nu_n}\, \frac{ds}{s}, \end{aligned} \tag{3.15}$$

and by Hölder's inequality, it follows that

$$\begin{aligned} \|f*\psi_\rho\|_{p,\nu_n} &\leqslant \|\psi\|_{1,\nu_n}\|\phi\|_{1,\nu_n}\left(\int_\alpha^\beta \Big(\frac{\|f*\phi_{\rho s}\|_{p,\nu_n}}{(\rho s)^\gamma}\Big)^q \frac{ds}{s}\right)^{\frac{1}{q}} \left(\int_\alpha^\beta \Big(\rho s\Big)^{\gamma\, q'}\, \frac{ds}{s}\right)^{\frac{1}{q'}} \\ &\leqslant \|\psi\|_{1,\nu_n}\|\phi\|_{1,\nu_n}\ M^{\gamma,\phi}_{p,q}(f)\rho^\gamma\left(\int_\alpha^\beta s^{\gamma\, q'}\, \frac{ds}{s}\right)^{\frac{1}{q'}} \\ &< +\infty, \end{aligned}$$

where q' is the conjugate exponent of q. Now, by the relation (3.15), we have

$$\begin{aligned} \frac{\|f*\psi_\rho\|_{p,\nu_n}}{\rho^\gamma} &\leqslant \|\psi\|_{1,\nu_n}\|\phi\|_{1,\nu_n}\int_\alpha^\beta s^\gamma \frac{\|f*\phi_{\rho s}\|_{p,\nu_n}}{(\rho s)^\gamma}\, \frac{ds}{s} \\ &\leqslant \|\psi\|_{1,\nu_n}\|\phi\|_{1,\nu_n}\int_{\frac{1}{\beta}}^{\frac{1}{\alpha}} t^{-\gamma} \frac{\|f*\phi_{\frac{\rho}{t}}\|_{p,\nu_n}}{(\frac{\rho}{t})^\gamma}\, \frac{dt}{t} \\ &= \|\psi\|_{1,\nu_n}\|\phi\|_{1,\nu_n}\left[t^{-\gamma}\mathbf{1}_{[\frac{1}{\beta},\frac{1}{\alpha}]} \star \Big(\frac{\|f*\phi_t\|_{p,\nu_n}}{t^\gamma}\Big)\right](\rho), \end{aligned}$$

where $\star$ is the convolution product defined on $]0,+\infty[$ by the relation (3.2). By the relation (3.3), we obtain

$$\begin{aligned} M_{p,q}^{\gamma,\psi}(f) &\leqslant \|\psi\|_{1,\nu_n}\|\phi\|_{1,\nu_n}\left\|t^{-\gamma}\mathbf{1}_{[\frac{1}{\beta},\frac{1}{\alpha}]}\right\|_{L^1(\frac{dt}{t})}M_{p,q}^{\gamma,\phi}(f)\\ &< +\infty, \end{aligned}$$

and the proof is complete if we take into account Lemma 13. □

Proposition 15. Let $p,\ q\in[1,+\infty]$ and $\gamma\in\mathbb{R}$. The Besov space $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ is homogeneous of degree equal to $(2n+1)/p-\gamma-2n-1$, that is for every $f\in\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ and $t>0$, the distribution f_t belongs to the space $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$and we have

$$M_{p,q}^{\gamma,\phi}(f_t)=t^{\frac{2n+1}{p}-\gamma-2n-1}\ M_{p,q}^{\gamma,\phi}(f),$$

where

$$\langle f_t,\varphi\rangle=\langle f,\frac{1}{t^{2n+1}}\varphi_{\frac{1}{t}}\rangle;\quad \varphi\in S_*(\mathbb{R}\times\mathbb{R}^n).$$

Proof. Let $\phi\in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\begin{aligned} f_t*\phi_\rho(r,x) &= \langle f_t,\tau_{(r,-x)}(\check{\phi}_\rho)\rangle\\ &= \langle f,\frac{1}{t^{2n+1}}\big(\tau_{(r,-x)}(\check{\phi}_\rho)\big)_{\frac{1}{t}}\rangle. \end{aligned}$$

However,

$$\begin{aligned} \frac{1}{t^{2n+1}}\big(\tau_{(r,-x)}(\check{\phi}_\rho)\big)_{\frac{1}{t}}(s,y) &= \tau_{(r,-x)}(\check{\phi}_\rho)(ts,ty)\\ &= \frac{1}{t^{2n+1}}\ \tau_{(\frac{r}{t},-\frac{x}{t})}(\check{\phi}_{\frac{\rho}{t}})(s,y) \end{aligned}$$

consequently,

$$\begin{aligned} f_t*\phi_\rho(r,x) &= \langle f,\frac{1}{t^{2n+1}}\ \tau_{(\frac{r}{t},-\frac{x}{t})}(\check{\phi}_{\frac{\rho}{t}})\rangle\\ &= \big(f*\phi_{\frac{\rho}{t}}\big)_t(r,x). \end{aligned} \tag{3.16}$$

Hence, from the relation (3.4), we get

$$\|f_t*\phi_\rho\|_{p,\nu_n}=t^{-\frac{2n+1}{p'}}\|f*\phi_{\frac{\rho}{t}}\|_{p,\nu_n},$$

this shows that for all $\rho>0$, the function $f_t*\phi_\rho$ belongs to $L^p(d\nu_n)$ and we have

$$\begin{aligned} \left\|\frac{\|f_t*\phi_\rho\|_{p,\nu_n}}{\rho^\gamma}\right\|^q_{L^q(\frac{d\rho}{\rho})} &= t^{-\frac{2n+1}{p'}q}\int_0^{+\infty}\Big(\frac{\|f*\phi_{\frac{\rho}{t}}\|_{p,\nu_n}}{\rho^\gamma}\Big)^q\ \frac{d\rho}{\rho}\\ &= t^{-\frac{2n+1}{p'}q}\ t^{-\gamma\ q}\int_0^{+\infty}\Big(\frac{\|f*\phi_s\|_{p,\nu_n}}{s^\gamma}\Big)^q\ \frac{ds}{s}\\ &= t^{-q\left(\frac{2n+1}{p'}+\gamma\right)}\Big[M_{p,q}^{\gamma,\phi}(f)\Big]^q, \end{aligned}$$

which proves that the function

$$\rho \longmapsto \frac{\|f_t * \phi_\rho\|_{p,\nu_n}}{\rho^\gamma}$$

belongs to the space $L^q(\frac{d\rho}{\rho})$ and that

$$M^{\gamma,\phi}_{p,q}(f_t) = t^{\frac{2n+1}{p}-\gamma-2n-1}\; M^{\gamma,\phi}_{p,q}(f).$$

On the other hand, from the relations (3.12) and (3.16), we have

$$\begin{aligned}\int_0^{+\infty} f_t * \phi_\rho * \phi_\rho(r,x)\,\frac{d\rho}{\rho} & \qquad \frac{1}{t^{2n+1}}\int_0^{+\infty} f * \big(\phi*\phi\big)_s(\frac{r}{t},\frac{x}{t})\,\frac{ds}{s} \\ &= \frac{1}{t^{2n+1}}\int_0^{+\infty} f * \phi_s * \phi_s(\frac{r}{t},\frac{x}{t})\,\frac{ds}{s},\end{aligned}$$

and from the relation (3.6), it follows

$$\begin{aligned}\int_0^{+\infty} f_t * \phi_\rho * \phi_\rho(r,x)\,\frac{d\rho}{\rho} & \qquad \frac{1}{t^{2n+1}} f(\frac{r}{t},\frac{x}{t}) \\ &= f_t(r,x).\end{aligned}$$

This completes the proof. □

Proposition 16. Let $p,\ q \in [1,+\infty]$ and $\gamma \in \mathbb{R}$.
The space $\mathscr{B}^\gamma_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big) \cap \mathscr{E}_*\big(\mathbb{R}\times\mathbb{R}^n\big)$ is dense in $\mathscr{B}^\gamma_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$.

Proof. Let $f \in \mathscr{B}^\gamma_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ and $\phi \in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$. For all $t>0$, the function

$$(r,x) \longmapsto f*\phi_t(r,x) = \langle f, \tau_{(r,-x)}(\check{\phi_t})\rangle$$

belongs to the space $\mathscr{E}_*\big(\mathbb{R}\times\mathbb{R}^n\big)$ and is slowly increasing. From i) of Proposition 8, we deduce that the function $f*\phi_t*\phi_t$ belongs to the space $\mathscr{E}_*\big(\mathbb{R}\times\mathbb{R}^n\big)$. Thus, from derivative's theorem it follows that for all $k\in\mathbb{N}^*$; the function

$$f_k(r,x) = \int_{\frac{1}{k}}^{k} f*\phi_t*\phi_t(r,x)\,\frac{dt}{t}$$

is infinitely differentiable on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable. On the other hand, let $\psi \in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$, by Fubini's theorem, we have

$$f_k * \psi_\rho \;=\; \int_{\frac{1}{k}}^{k} f*\phi_t*\phi_t*\psi_\rho\,\frac{dt}{t}.$$

And by the same way as the proof of Theorem 14, we deduce that for all $\rho>0$, the function $f_k*\psi_\rho$ belongs to $L^p(d\nu_n)$ and that the function

$$\rho \longmapsto \frac{\|f_k * \psi_\rho\|_{p,\nu_n}}{\rho^\gamma}$$

belongs to $L^q(\frac{d\rho}{\rho})$. Again, by Fubini's theorem, for all $\psi \in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$,

$$\begin{aligned}\int_0^{+\infty} f_k * \psi_\rho * \psi_\rho \,\frac{d\rho}{\rho} &= \int_0^{+\infty}\Big(\int_{\frac{1}{k}}^{k} f*\phi_t*\phi_t*\psi_\rho*\psi_\rho\frac{dt}{t}\Big)\,\frac{d\rho}{\rho}\\ &= \int_{\frac{1}{k}}^{k}\Big(\int_0^{+\infty} f*\psi_\rho*\psi_\rho*\phi_t*\phi_t\frac{d\rho}{\rho}\Big)\,\frac{dt}{t},\end{aligned}$$

and by Lemma 13 and Theorem 14, we obtain

$$\begin{aligned}\int_0^{+\infty} f_k*\psi_\rho*\psi_\rho\,\frac{d\rho}{\rho} &= \int_{\frac{1}{k}}^{k} f*\phi_t*\phi_t\,\frac{dt}{t}\\ &= f_k.\end{aligned}$$

This shows that for all $k \in \mathbb{N}^*$, the function f_k belongs to the space $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)\cap\mathscr{E}_*\big(\mathbb{R}\times\mathbb{R}^n\big)$. Moreover, for every $\varphi \in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$, we have

$$f_k*\varphi_\rho = \int_{\frac{1}{k}}^{k} f*\phi_t*\phi_t*\varphi_\rho\,\frac{dt}{t},$$

and by i) of Lemma 13, we get

$$f*\varphi_\rho = \int_0^{+\infty} f*\phi_t*\phi_t*\varphi_\rho\,\frac{dt}{t},$$

Thus,

$$\begin{aligned}\big(f-f_k\big)*\varphi_\rho &= \int_0^{\frac{1}{k}} f*\phi_t*\phi_t*\varphi_\rho\,\frac{dt}{t}+\int_k^{+\infty} f*\phi_t*\phi_t*\varphi_\rho\,\frac{dt}{t}\\ &= \int_{[0,\frac{1}{k}]\cup[k,+\infty[} f*\phi_t*\phi_t*\varphi_\rho\,\frac{dt}{t}.\end{aligned}$$

Now using the relation (3.13), we obtain

$$\begin{aligned}\big(f-f_k\big)*\varphi_\rho &= \int_{\big([0,\frac{1}{k\rho}]\cup[\frac{k}{\rho},+\infty[\big)\cap[\alpha,\beta]} f*\phi_{\rho s}*\phi_{\rho s}*\varphi_\rho\,\frac{ds}{s}\\ &= \int_0^{+\infty}\mathbf{1}_{\big([0,\frac{1}{k\rho}]\cup[\frac{k}{\rho},+\infty[\big)\cap[\alpha,\beta]}(s)\; f*\phi_{\rho s}*\phi_{\rho s}*\varphi_\rho\,\frac{ds}{s}.\end{aligned}$$

Now Minkowski's inequality leads to

$$\begin{aligned}\big\|\big(f-f_k\big)*\varphi_\rho\big\|_{p,\nu_n} &\leqslant \int_\alpha^\beta \mathbf{1}_{\big([0,\frac{1}{k\rho}]\cup[\frac{k}{\rho},+\infty[\big)}(s)\big\|f*\phi_{\rho s}*\phi_{\rho s}*\varphi_\rho\big\|_{p,\nu_n}\,\frac{ds}{s}\\ &\leqslant \int_\alpha^\beta \mathbf{1}_{\big([0,\frac{1}{k\rho}]\cup[\frac{k}{\rho},+\infty[\big)}(s)\;\big\|f*\phi_{\rho s}\big\|_{p,\nu_n}\big\|\phi_{\rho s}*\varphi_\rho\big\|_{1,\nu_n}\,\frac{ds}{s}\\ &\leqslant \|\phi\|_{1,\nu_n}\;\|\varphi\|_{1,\nu_n}\int_\alpha^\beta \mathbf{1}_{\big([0,\frac{1}{k\rho}]\cup[\frac{k}{\rho},+\infty[\big)}(s)\;\big\|f*\phi_{\rho s}\big\|_{p,\nu_n}\,\frac{ds}{s}.\end{aligned}$$

Consequently;

$$\begin{aligned}\frac{\big\|(f-f_k)*\varphi_\rho\big\|_{p,\nu_n}}{\rho^\gamma} &\leqslant \|\phi\|_{1,\nu_n}\ \|\varphi\|_{1,\nu_n}\int_\alpha^\beta \mathbf{1}_{\left([0,\frac{1}{k}]\cup[k,+\infty[\right)}(\rho s)\ \frac{\big\|f*\phi_{\rho s}\big\|_{p,\nu_n}}{\rho^\gamma}\ \frac{ds}{s}\\ &\leqslant \|\phi\|_{1,\nu_n}\ \|\varphi\|_{1,\nu_n}\int_{\frac{1}{\beta}}^{\frac{1}{\alpha}} \mathbf{1}_{\left([0,\frac{1}{k}]\cup[k,+\infty[\right)}(\frac{\rho}{t})\ t^{-\gamma}\ \frac{\big\|f*\phi_{\frac{\rho}{t}}\big\|_{p,\nu_n}}{(\frac{\rho}{t})^\gamma}\ \frac{dt}{t}\\ &= \|\phi\|_{1,\nu_n}\ \|\varphi\|_{1,\nu_n}\Big(t^{-\gamma}\ \mathbf{1}_{[\frac{1}{\beta},\frac{1}{\alpha}]}\star\frac{\big\|f*\phi_t\big\|_{p,\nu_n}}{t^\gamma}\ \mathbf{1}_{\left([0,\frac{1}{k}]\cup[k,+\infty[\right)}\Big)(\rho).\end{aligned}$$

Thus, by the relation (3.3), we obtain

$$\begin{aligned}M_{p,q}^{\gamma,\varphi}(f_k-f) \leqslant \|\phi\|_{1,\nu_n}\ \|\varphi\|_{1,\nu_n}\big\|t^{-\gamma}\ \mathbf{1}_{[\frac{1}{\beta},\frac{1}{\alpha}]}\big\|_{L^1(\frac{dt}{t})}\\ \times\Big[\int_0^{\frac{1}{k}}\Big(\frac{\big\|f*\phi_t\big\|_{p,\nu_n}}{t^\gamma}\Big)^q\ \frac{dt}{t}+\int_k^{+\infty}\Big(\frac{\big\|f*\phi_t\big\|_{p,\nu_n}}{t^\gamma}\Big)^q\ \frac{dt}{t}\Big]^{\frac{1}{q}}.\end{aligned}$$

So,

$$\lim_{k\to+\infty} M_{p,q}^{\gamma,\varphi}(f_k-f)=0$$

because

$$\int_0^{+\infty}\Big(\frac{\big\|f*\phi_t\big\|_{p,\nu_n}}{t^\gamma}\Big)^q\ \frac{dt}{t}<+\infty$$

and the proof is complete. □

We denote by $L^q\big(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\big)$ the space of measurable functions g on $]0,+\infty[\times[0,+\infty[\times\mathbb{R}^n$ such that for all $t>0$, the function $g(t,(.,.))$ belongs to the space $L^p(d\nu_n)$ and the function

$$t\longmapsto\big\|g(t,(.,.))\big\|_{p,\nu_n}$$

belongs to $L^q(\frac{dt}{t})$. This space is equipped with the norm

$$\|g\|_{L^q\left(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\right)} = \Big(\int_0^{+\infty}\|g(t,(.,.))\|_{p,\nu_n}^q\ \frac{dt}{t}\Big)^{\frac{1}{q}}.$$

Then we have

Lemma 17. Let $p,q\in[1,+\infty]$ and let $\gamma<(2n+1)/p$. For all $\phi\in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$, the mapping F defined by

$$F(g)(r,x) = \int_0^{+\infty} t^\gamma\ g\big(t,(.,.)\big)*\phi_t(r,x)\ \frac{dt}{t}$$

is continuous from $L^q\big(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\big)$ into $\mathscr{B}^\gamma_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$.

Proof. Let $\phi \in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$ and $g \in L^q\big(]0,+\infty[,\ L^p(d\nu_n),\ \dfrac{dt}{t}\big)$.

- Let $a,\ b$ be real numbers such that $b > a > 0$ and

$$\mathscr{F}(\phi)(\mu,\lambda) = 0 \quad \text{if} \quad \mu^2+2|\lambda|^2 < a^2 \quad \text{or} \quad \mu^2+2|\lambda|^2 > b^2.$$

Let $\psi \in S_*(\mathbb{R}\times\mathbb{R}^n)$ such that

$$\mathscr{F}(\psi)(\mu,\lambda) = 1 \quad \text{if} \quad a^2 \leqslant \mu^2+2|\lambda|^2 \leqslant b^2,$$

then from the relation (2.9), we deduce that for every $t > 0$

$$\psi_t * \phi_t \quad = \quad \phi_t. \tag{3.17}$$

For every $k \in \mathbb{N}^*$, the function $F(g)_k$ defined by

$$F(g)_k(r,x) \quad = \quad \int_{\frac{1}{k}}^{k} t^{\gamma}\ g(t,(.,.)) * \phi_t(r,x)\ \frac{dt}{t}.$$

is bounded on $\mathbb{R}\times\mathbb{R}^n$. In fact, from the relations (2.7) and (3.4), we deduce that for all $(r,x) \in \mathbb{R}\times\mathbb{R}^n$,

$$\begin{aligned}
\big|F(g)_k(r,x)\big| &\leqslant \int_{\frac{1}{k}}^{k} t^{\gamma}\ \big\|g(t,(.,.))\big\|_{p,\nu_n}\ \big\|\phi_t\big\|_{p',\nu_n}\ \frac{dt}{t}\\
&\leqslant \big\|\phi\big\|_{p',\nu_n} \int_{\frac{1}{k}}^{k} t^{\gamma-\frac{2n+1}{p}}\ \big\|g\big(t,(.,.)\big)\big\|_{p,\nu_n}\ \frac{dt}{t}\\
&\leqslant \big\|\phi\big\|_{p',\nu_n}\ \big\|g\big\|_{L^q\left(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\right)} \Big[\int_{\frac{1}{k}}^{k} t^{(\gamma-\frac{2n+1}{p})\ q'}\ \frac{dt}{t}\Big]^{\frac{1}{q'}}\\
&< +\infty.
\end{aligned}$$

Thus, for all $k \in \mathbb{N}^*$ the function $F(g)_k$ defines a tempered distribution on $\mathbb{R}\times\mathbb{R}^n$, even with respect to the first variable. Moreover, for all $h \in S_*(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\begin{aligned}
\langle F(g)_k, h\rangle &= \int_{\frac{1}{k}}^{k} t^{\gamma}\ \Big[\int_0^{+\infty}\int_{\mathbb{R}^n} h(r,x)\ g(t,(.,.)) * \phi_t(r,x)\ d\nu_n(r,x)\Big]\ \frac{dt}{t}\\
&= \int_{\frac{1}{k}}^{k} t^{\gamma}\ \langle g(t,(.,.)) * \phi_t, h\rangle\ \frac{dt}{t},
\end{aligned}$$

and by the relation (3.17), it follows that

$$\begin{aligned}
\langle F(g)_k, h\rangle &= \int_{\frac{1}{k}}^{k} t^{\gamma}\ \langle g(t,(.,.)) * \phi_t * \psi_t, h\rangle\ \frac{dt}{t}\\
&= \int_{\frac{1}{k}}^{k} t^{\gamma}\ \langle g(t,(.,.)) * \phi_t, h * \check{\psi}_t\rangle\ \frac{dt}{t}.
\end{aligned} \tag{3.18}$$

However,

$$\int_0^{+\infty} t^{\gamma} \left|\langle g(t,(.,.)) * \phi_t, h * \check{\psi}_t\rangle\right| \frac{dt}{t}$$
$$\leqslant \int_0^{+\infty} t^{\gamma} \left\|g(t,(.,.)) * \phi_t\right\|_{\infty,\nu_n} \left\|h * \check{\psi}_t\right\|_{1,\nu_n} \frac{dt}{t}$$
$$\leqslant \int_0^{+\infty} t^{\gamma} \left\|g(t,(.,.))\right\|_{p,\nu_n} \left\|\phi_t\right\|_{p',\nu_n} \left\|h * \check{\psi}_t\right\|_{1,\nu_n} \frac{dt}{t}$$
$$= \left\|\phi\right\|_{p',\nu_n} \int_0^{+\infty} t^{\gamma - \frac{2n+1}{p}} \left\|g(t,(.,.))\right\|_{p,\nu_n} \left\|h * \check{\psi}_t\right\|_{1,\nu_n} \frac{dt}{t}$$
$$= \left\|\phi\right\|_{p',\nu_n} \left\{ \int_0^1 t^{\gamma - \frac{2n+1}{p}} \left\|g(t,(.,.))\right\|_{p,\nu_n} \left\|h * \check{\psi}_t\right\|_{1,\nu_n} \frac{dt}{t} \right.$$
$$\left. + \int_1^{+\infty} t^{\gamma - \frac{2n+1}{p}} \left\|g(t,(.,.))\right\|_{p,\nu_n} \left\|h * \check{\psi}_t\right\|_{1,\nu_n} \frac{dt}{t} \right\}$$

Applying the relations (3.7) and (3.8), we get

$$\int_0^{+\infty} t^{\gamma} \left|\langle g(t,(.,.)) * \phi_t, h * \check{\psi}_t\rangle\right| \frac{dt}{t}$$
$$\leqslant \left\|\phi\right\|_{p',\nu_n} \|\Delta^k h\|_{1,\nu_n} \|\check{\psi_k}\|_{1,\nu_n} \int_0^1 t^{2k+\gamma - \frac{2n+1}{p}} \left\|g(t,(.,.))\right\|_{p,\nu_n} \frac{dt}{t}$$
$$+ \left\|\phi\right\|_{p',\nu_n} \|h\|_{1,\nu_n} \|\check{\psi}\|_{1,\nu_n} \int_1^{+\infty} t^{\gamma - \frac{2n+1}{p}} \left\|g(t,(.,.))\right\|_{p,\nu_n} \frac{dt}{t};$$

and by Hölder's inequality, we have

$$\int_0^{+\infty} t^{\gamma} \left|\langle g(t,(.,.)) * \phi_t, h * \check{\psi}_t\rangle\right| \frac{dt}{t}$$
$$\leqslant \left\|\phi\right\|_{p',\nu_n} \left\|g\right\|_{L^q\left(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\right)} \left\{ \|\Delta^k h\|_{1,\nu_n} \|\check{\psi_k}\|_{1,\nu_n} \left(\int_0^1 t^{(2k+\gamma - \frac{2n+1}{p})\, q'} \frac{dt}{t}\right)^{\frac{1}{q'}} \right.$$
$$\left. + \|h\|_{1,\nu_n} \|\check{\psi}\|_{1,\nu_n} \left(\int_1^{+\infty} t^{(\gamma - \frac{2n+1}{p})\, q'} \frac{dt}{t}\right)^{\frac{1}{q'}} \right\} < +\infty.$$

The last inequality together with the relation (3.18) show that for all $h \in S_*(\mathbb{R} \times \mathbb{R}^n)$, $\lim\limits_{k\to+\infty} \langle F(g)_k, h\rangle$ exists and

$$\lim_{k\to+\infty} \langle F(g)_k, h\rangle \quad = \quad \int_0^{+\infty} t^{\gamma} \langle g(t,(.,.)) * \phi_t, h\rangle \frac{dt}{t}.$$

Consequently, the function

$$F(g)(r,x) \quad = \quad \int_0^{+\infty} t^{\gamma}\, g\big(t,(.,.)\big) * \phi_t(r,x) \frac{dt}{t}$$

defines an element of $S_*^{'}(\mathbb{R} \times \mathbb{R}^n)$.

• Let $\varphi \in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$, we have

$$\begin{aligned}
F(g)*\varphi_\rho(r,x) &= \langle F(g), \tau_{(r,-x)}\check{\varphi}_\rho\rangle \\
&= \lim_{k\to+\infty}\langle F(g)_k, \tau_{(r,-x)}\check{\varphi}_\rho\rangle \\
&= \lim_{k\to+\infty} F(g)_k * \varphi_\rho(r,x) \\
&= \lim_{k\to+\infty}\int_{\frac{1}{k}}^{k} t^\gamma\ g(t,(.,.)) * \phi_t * \varphi_\rho(r,x)\ \frac{dt}{t}.
\end{aligned}$$

However, the relation (3.13) implies

$$\begin{aligned}
&\int_0^{+\infty} t^\gamma \big|g(t,(.,.)) * \phi_t * \varphi_\rho(r,x)\big|\ \frac{dt}{t}\int_{\rho\alpha}^{\rho\beta} t^\gamma\ \big|g(t,(.,.)) * \phi_t * \varphi_\rho(r,x)\big|\ \frac{dt}{t} \\
&\leqslant \int_{\rho\alpha}^{\rho\beta} t^\gamma\ \big\|g(t,(.,.))\big\|_{p,\nu_n}\big\|\phi_t * \varphi_\rho\big\|_{p',\nu_n}\ \frac{dt}{t} \\
&\leqslant \|\phi\|_{p',\nu_n}\|\varphi\|_{1,\nu_n}\int_{\rho\alpha}^{\rho\beta} t^{\gamma\ -\frac{2n+1}{p}}\ \big\|g(t,(.,.))\big\|_{p,\nu_n}\ \frac{dt}{t}, \\
&\leqslant \|\phi\|_{p',\nu_n}\ \|\varphi\|_{1,\nu_n}\ \Big(\int_{\rho\alpha}^{\rho\beta} t^{(\gamma\ -\frac{2n+1}{p})\ q'}\frac{dt}{t}\Big)^{\frac{1}{q'}}\ \|g\|_{L^q\left(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\right)} \\
&< +\infty.
\end{aligned}$$

Thus,

$$\begin{aligned}
F(g)*\varphi_\rho(r,x) &= \int_0^{+\infty} t^\gamma\ g(t,(.,.)) * \phi_t * \varphi_\rho(r,x)\ \frac{dt}{t} \\
&= \int_\alpha^\beta (\rho s)^\gamma\ g(\rho s,(.,.)) * \phi_{\rho s} * \varphi_\rho(r,x)\ \frac{ds}{s}. \qquad (3.19)
\end{aligned}$$

By Minkowski's inequality, we obtain

$$\begin{aligned}
\big\|F(g)*\varphi_\rho\big\|_{p,\nu_n} &\leqslant \int_\alpha^\beta(\rho s)^\gamma\ \big\|g(\rho s,(.,.)) * \phi_{\rho s} * \varphi_\rho\big\|_{p,\nu_n}\ \frac{ds}{s} \\
&\leqslant \big\|\phi\big\|_{1,\nu_n}\ \big\|\varphi\big\|_{1,\nu_n}\ \int_\alpha^\beta(\rho s)^\gamma\ \big\|g(\rho s,(.,.))\big\|_{p,\nu_n}\ \frac{ds}{s} \\
&< +\infty
\end{aligned}$$

and

$$\begin{aligned}
\frac{\big\|F(g)*\varphi_\rho\big\|_{p,\nu_n}}{\rho^\gamma} &\leqslant \big\|\phi\big\|_{1,\nu_n}\ \big\|\varphi\big\|_{1,\nu_n}\ \int_\alpha^\beta s^\gamma\ \big\|g(\rho s,(.,.))\big\|_{p,\nu_n}\ \frac{ds}{s} \\
&= \big\|\phi\big\|_{1,\nu_n}\ \big\|\varphi\big\|_{1,\nu_n}\ \int_{\frac{1}{\beta}}^{\frac{1}{\alpha}} t^{-\gamma}\ \big\|g(\frac{\rho}{t},(.,.))\big\|_{p,\nu_n}\ \frac{dt}{t} \\
&= \big\|\phi\big\|_{1,\nu_n}\ \big\|\varphi\big\|_{1,\nu_n}\Big(t^{-\gamma}\mathbf{1}_{[\frac{1}{\beta},\frac{1}{\alpha}]}\star\big\|g(t,(.,.))\big\|_{p,\nu_n}\Big)(\rho),
\end{aligned}$$

and by the relation (3.3) it follows that

$$\left\| \frac{\|F(g) * \varphi_\rho\|_{p,\nu_n}}{\rho^\gamma} \right\|_{L^q(\frac{d\rho}{\rho})} \leqslant \|\phi\|_{1,\nu_n} \; \|\varphi\|_{1,\nu_n} \left\| t^{-\gamma} \mathbf{1}_{[\frac{1}{\beta}, \frac{1}{\alpha}]} \right\|_{L^1(\frac{dt}{t})} \|g\|_{L^q\left(]0,+\infty[, L^p(d\nu_n), \frac{dt}{t}\right)} < +\infty. \tag{3.20}$$

- Let $\varphi \in S^1_{*,0}(\mathbb{R} \times \mathbb{R}^n)$, from the relation (3.19), we have

$$F(g) * \varphi_\rho(r,x) = \int_\alpha^\beta (\rho s)^\gamma \; g(\rho s, (.,.)) * \phi_{\rho s} * \varphi_\rho(r,x) \; \frac{ds}{s},$$

and by Fubini's theorem, we get

$$\begin{aligned} F(g) * \varphi_\rho * \varphi_\rho(r,x) &= \int_\alpha^\beta (\rho s)^\gamma \; g(\rho s, (.,.)) * \phi_{\rho s} * \varphi_\rho * \varphi_\rho(r,x) \; \frac{ds}{s} \\ &= \int_{\rho\alpha}^{\rho\beta} t^\gamma \; g(t, (.,.)) * \phi_t * \varphi_\rho * \varphi_\rho(r,x) \; \frac{dt}{t}. \end{aligned}$$

Thus,

$$\begin{aligned} \int_{\frac{1}{k}}^k F(g) * \varphi_\rho * \varphi_\rho(r,x) \frac{d\rho}{\rho} & \int_{\frac{1}{k}}^k \left[\int_{\rho\alpha}^{\rho\beta} t^\gamma \; g(t, (.,.)) * \phi_t * \varphi_\rho * \varphi_\rho(r,x) \; \frac{dt}{t} \right] \frac{d\rho}{\rho} \\ &= \int_{\frac{\alpha}{k}}^{\beta k} t^\gamma \left[\int_{\frac{t}{\beta}}^{\frac{t}{\alpha}} \; g(t, (.,.)) * \phi_t * \varphi_\rho * \varphi_\rho(r,x) \; \frac{d\rho}{\rho} \right] \frac{dt}{t}. \end{aligned} \tag{3.21}$$

However,

$$\begin{aligned} & \int_{\frac{t}{\beta}}^{\frac{t}{\alpha}} \; g(t, (.,.)) * \phi_t * \varphi_\rho * \varphi_\rho(r,x) \; \frac{d\rho}{\rho} \\ &= \int_{\frac{t}{\beta}}^{\frac{t}{\alpha}} \; \left[\int_0^{+\infty} \int_{\mathbb{R}^n} \tau_{(r,-x)} \check{g}(t, (.,.))(s,y) \; \phi_t * \varphi_\rho * \varphi_\rho(s,y) \; d\nu_n(s,y) \right] \; \frac{d\rho}{\rho}. \end{aligned}$$

Again, by Fubini's theorem, we have

$$\begin{aligned} & \int_{\frac{t}{\beta}}^{\frac{t}{\alpha}} \; g(t, (.,.)) * \phi_t * \varphi_\rho * \varphi_\rho(r,x) \; \frac{d\rho}{\rho} \\ &= \int_0^{+\infty} \int_{\mathbb{R}^n} \tau_{(r,-x)} \check{g}(t, (.,.))(s,y) \; \left[\int_0^{+\infty} \; \phi_t * \varphi_\rho * \varphi_\rho(s,y) \; \frac{d\rho}{\rho} \right] \; d\nu_n(s,y). \end{aligned}$$

applying 2) of Lemma 13, we obtain

$$\begin{aligned} \int_{\frac{t}{\beta}}^{\frac{t}{\alpha}} \; g(t, (.,.)) * \phi_t * \varphi_\rho * \varphi_\rho(r,x) \; \frac{d\rho}{\rho} &= \int_0^{+\infty} \int_{\mathbb{R}^n} \tau_{(r,-x)} \check{g}(t, (.,.))(s,y) \; \phi_t(s,y) \; d\nu_n(s,y) \\ &= g(t, (.,.)) * \phi_t(r,x). \end{aligned}$$

Replacing in the equality (3.21), it follows that

$$\int_{\frac{1}{k}}^{k} F(g)*\varphi_\rho*\varphi_\rho(r,x)\frac{d\rho}{\rho} = \int_{\frac{\alpha}{k}}^{\beta k} t^\gamma\ g(t,(.,.))*\phi_t(r,x)\ \frac{dt}{t}.$$

Hence,

$$\int_0^{+\infty} F(g)*\varphi_\rho*\varphi_\rho\ \frac{d\rho}{\rho} = F(g).$$

This shows that the function $F(g)$ belongs to the space $\mathscr{B}^\gamma_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ and from the inequality (3.20), we have

$$M^{\gamma,\varphi}_{p,q}(F(g)) \leqslant \|\phi\|_{1,\nu_n}\ \|\varphi\|_{1,\nu_n}\Big\|t^{-\gamma}\mathbf{1}_{[\frac{1}{\beta},\frac{1}{\alpha}]}\Big\|_{L^1(\frac{dt}{t})}\|g\|_{L^q\left(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\right)}$$

which means that the mapping F is continuous from $L^q\big(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\big)$ into $\mathscr{B}^\gamma_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$. □

Theorem 18. Let $p,q\in[1,+\infty]$ and let $\gamma\in\mathbb{R}$, $\gamma<(2n+1)/p$. Then the Besov space $\mathscr{B}^\gamma_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ is a Banach one.

Proof. Let $\phi\in S^1_{*,0}(\mathbb{R}\times\mathbb{R}^n)$. We define the mapping G on the space $\mathscr{B}^\gamma_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ by setting

$$G(f)(t,(r,x)) = \frac{f*\phi_t(r,x)}{t^\gamma}.$$

The mapping G is continuous from $\mathscr{B}^\gamma_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ into $L^q\big(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\big)$ and we have

$$\|G(f)\|_{L^q\left(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\right)} = M^{\gamma,\phi}_{p,q}(f). \tag{3.22}$$

Moreover, for all $f\in\mathscr{B}^\gamma_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$, we have

$$\begin{aligned} F\circ G(f)(r,x) &= \int_0^{+\infty} t^\gamma\ G(f)(t,(.,.))*\phi_t(r,x)\ \frac{dt}{t}\\ &= \int_0^{+\infty} t^\gamma\ \frac{f*\phi_t*\phi_t(r,x)}{t^\gamma}\ \frac{dt}{t}\\ &= \int_0^{+\infty} f*\phi_t*\phi_t(r,x)\ \frac{dt}{t}\end{aligned}$$

and by ii) of Lemma 13, we get

$$F\circ G(f) = f.$$

This equality shows that

$$G\big(\mathscr{B}^\gamma_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)\big)\ \ker\big(G\circ F - Id_{(L^q\left(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\right))}\big).$$

In particular, $G\big(\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)\big)$ is a closed subspace of $L^q\big(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\big)$.

Let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$. From the relation (3.22), the sequence $\big(G(f_k)\big)_k$ is a Cauchy's one in $L^q\big(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\big)$.
Since $G\big(\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)\big)$ is a closed subspace of $L^q\big(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\big)$, then there exists a function f in $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ such that

$$\lim_{k\to+\infty} G(f_k) = G(f) \quad \text{in} \quad L^q\Big(]0,+\infty[,\ L^p(d\nu_n),\ \frac{dt}{t}\Big).$$

Again by the relation (3.22),

$$\lim_{k\to+\infty} f_k = f \quad \text{in} \quad \mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big).$$

□

Proposition 19. i) Let $q\in[1,+\infty]$, $p_1,\ p_2\in[1,+\infty]$; $p_1<p_2$ and let γ_1, $\gamma_2\in\mathbb{R}$ such that

$$\frac{2n+1}{p_1}-\gamma_1 = \frac{2n+1}{p_2}-\gamma_2. \tag{3.23}$$

Then

$$\mathscr{B}^{\gamma_1}_{p_1,q}\big([0,+\infty[\times\mathbb{R}^n\big)\hookrightarrow\mathscr{B}^{\gamma_2}_{p_2,q}\big([0,+\infty[\times\mathbb{R}^n\big).$$

ii) For all $p\in[1,+\infty]$,

$$\mathscr{B}^{0}_{p,1}\big([0,+\infty[\times\mathbb{R}^n\big)\hookrightarrow L^p(d\nu_n).$$

Proof. i) Let $p_1,\ p_2,\ \gamma_1,\ \gamma_2,\ q$ be real numbers satisfying the hypothesis. Let p_3 be an exponent such that

$$\frac{1}{p_1}+\frac{1}{p_3} = 1+\frac{1}{p_2}. \tag{3.24}$$

Finally, let $f\in\mathscr{B}^{\gamma_1}_{p_1,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ and $\phi\in S^1_{*,0}\big(\mathbb{R}\times\mathbb{R}^n\big)$ such that

$$\mathscr{F}(\phi)(\mu,\lambda)=0 \quad \text{if} \quad \mu^2+2|\lambda|^2>b^2 \quad \text{or} \quad \mu^2+2|\lambda|^2<a^2.$$

Let us take $\psi\in S_*\big(\mathbb{R}\times\mathbb{R}^n\big)$ satisfying

$$\forall(\mu,\lambda)\in\Gamma;\quad a^2\leqslant\mu^2+2|\lambda|^2\leqslant b^2,\quad \mathscr{F}(\psi)(\mu,\lambda)=1.$$

Then for all $t>0$, we have

$$\phi_t*\psi_t=\phi_t$$

and

$$\begin{aligned} M^{\gamma_2,\phi}_{p_2,q}(f) &= \Big(\int_0^{+\infty}\Big(\frac{\|f*\phi_t\|_{p_2,\nu_n}}{t^{\gamma_2}}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}} \\ &= \Big(\int_0^{+\infty}\Big(\frac{\|f*\phi_t*\psi_t\|_{p_2,\nu_n}}{t^{\gamma_2}}\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}. \end{aligned}$$

By the relations (2.7), (3.4), (3.23) and (3.24) we get

$$\begin{aligned} M_{p_2,q}^{\gamma_2,\varphi}(f) &\leqslant \|\psi\|_{p_3,\nu_n}\Big[\int_0^{+\infty}\Big(\frac{\|f*\varphi_t\|_{p_1,\nu_n}}{t^{\gamma_1}}\Big)^q\frac{dt}{t}\Big]^{\frac{1}{q}} \\ &\leqslant \|\psi\|_{p_3,\nu_n}\, M_{p_1,q}^{\gamma_1,\varphi}(f). \end{aligned}$$

This shows that the space $\mathscr{B}_{p_1,q}^{\gamma_1}\big([0,+\infty[\times\mathbb{R}^n\big)$ is contained in $\mathscr{B}_{p_2,q}^{\gamma_2}\big([0,+\infty[\times\mathbb{R}^n\big)$ and that the canonical injection is continuous from $\mathscr{B}_{p_1,q}^{\gamma_1}\big([0,+\infty[\times\mathbb{R}^n\big)$ into the space $\mathscr{B}_{p_2,q}^{\gamma_2}\big([0,+\infty[\times\mathbb{R}^n\big)$.

ii) Let $f\in\mathscr{B}_{p,1}^{0}\big([0,+\infty[\times\mathbb{R}^n\big);\ p\in[1,+\infty]$. From ii) of Lemma 13, we have

$$f=\int_0^{+\infty} f*\varphi_t*\varphi_t\,\frac{dt}{t};\quad \varphi\in S_{*,0}^1\big(\mathbb{R}\times\mathbb{R}^n\big)$$

thus,

$$\begin{aligned} \|f\|_{p,\nu_n} &\leqslant \int_0^{+\infty}\big\|f*\varphi_t*\varphi_t\big\|_{p,\nu_n}\,\frac{dt}{t} \\ &\leqslant \|\varphi\|_{1,\nu_n}\, M_{p,1}^{0,\varphi}(f). \end{aligned}$$

This completes the proof. □

In the following, we shall define a discrete norm on the Besov space $\mathscr{B}_{p,q}^{\gamma}\big([0,+\infty[\times\mathbb{R}^n\big)$ and we will prove that it is equivalent to the norm $M_{p,q}^{\gamma,\varphi};\ \varphi\in S_{*,0}^1\big(\mathbb{R}\times\mathbb{R}^n\big)$. More precisely, we have

Theorem 20. Let $p,\ q\in[1,+\infty],\ \gamma\in\mathbb{R}$. Let $a,\ b$ be real numbers such that $0<a<b$ and $\varphi\in S_{*,0}\big(\mathbb{R}\times\mathbb{R}^n\big)$ verifying

$$\mathscr{F}(\varphi)(\mu,\lambda)=1 \quad \text{if} \quad a^2\leqslant\mu^2+2|\lambda|^2\leqslant b^2.$$

Then the mapping $N_{p,q}^{\gamma,\varphi}$ defined by

$$N_{p,q}^{\gamma,\varphi}(f)=\begin{cases}\left(\displaystyle\sum_{k\in\mathbb{Z}}\Big(\frac{\|f*\varphi_{2^k}\|_{p,\nu_n}}{2^{k\gamma}}\Big)^q\right)^{\frac{1}{q}}, & \text{if } 1\leqslant q<+\infty;\\[2ex] \displaystyle\operatorname*{esssup}_{k\in\mathbb{Z}}\frac{\|f*\varphi_{2^k}\|_{p,\nu_n}}{2^{k\gamma}}, & \text{if } q=+\infty\end{cases}$$

is a norm on the Besov space $\mathscr{B}_{p,q}^{\gamma}\big([0,+\infty[\times\mathbb{R}^n\big)$ which defines the same topology as the norm $M_{p,q}^{\gamma,\psi};\ \psi\in S_{*,0}^1\big(\mathbb{R}\times\mathbb{R}^n\big)$.

Proof. • From Lemma 9, there exists $\psi\in S_{*,0}^1\big(\mathbb{R}\times\mathbb{R}^n\big)$ such that

$$\mathscr{F}(\psi)(\mu,\lambda)=0 \quad \text{if} \quad \mu^2+2|\lambda|^2<a^2 \quad \text{or} \quad \mu^2+2|\lambda|^2>b^2.$$

Then for all $s \in [1,2]$ and $k \in \mathbb{Z}$, we have

$$\mathscr{F}(\psi)(2^k s\mu, 2^k s\lambda) = \mathscr{F}(\psi)(2^k s\mu, 2^k s\lambda)\ \mathscr{F}(\phi)(2^k \mu, 2^k \lambda)$$

which leads to

$$\psi_{2^k s} = \psi_{2^k s} * \phi_{2^k}$$

and therefore, for all $f \in \mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times \mathbb{R}^n\big)$

$$f * \psi_{2^k s} \quad = \quad f * \phi_{2^k} * \psi_{2^k s}. \tag{3.25}$$

Then for all $q \in [1,+\infty[$

$$\begin{aligned} M^{\gamma,\psi}_{p,q}(f) &= \left(\int_0^{+\infty} \Big(\frac{\|f*\psi_t\|_{p,\nu_n}}{t^{\gamma}}\Big)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &= \left(\sum_{k\in\mathbb{Z}} \int_{2^k}^{2^{k+1}} \Big(\frac{\|f*\psi_t\|_{p,\nu_n}}{t^{\gamma}}\Big)^q \frac{dt}{t}\right)^{\frac{1}{q}} \\ &= \left(\sum_{k\in\mathbb{Z}} \int_1^{2} \Big(\frac{\|f*\psi_{2^k s}\|_{p,\nu_n}}{(2^k s)^{\gamma}}\Big)^q \frac{ds}{s}\right)^{\frac{1}{q}}. \end{aligned}$$

Using the relations (2.7), (3.4) and (3.25), we obtain

$$\begin{aligned} M^{\gamma,\psi}_{p,q}(f) &\leqslant \|\psi\|_{1,\nu_n} \left[\sum_{k\in\mathbb{Z}} \Big(\frac{\|f*\phi_{2^k}\|_{p,\nu_n}}{2^{k\gamma}}\Big)^q \int_1^2 \frac{ds}{s^{\gamma q+1}}\right]^{\frac{1}{q}} \\ &= \|\psi\|_{1,\nu_n} \Big(\frac{1-2^{-q\gamma}}{q\gamma}\Big)^{\frac{1}{q}} N^{\gamma,\phi}_{p,q}(f). \end{aligned}$$

On the other hand, for $q = +\infty$ and again by the relation (3.25), we deduce that for all $k \in \mathbb{Z}$ and $s \in [1,2]$

$$\frac{\|f*\psi_{2^k s}\|_{p,\nu_n}}{(2^k s)^{\gamma}} \leqslant (1+2^{-\gamma})\ \|\psi\|_{1,\nu_n}\ \frac{\|f*\phi_{2^k}\|_{p,\nu_n}}{2^{k\gamma}}.$$

Consequently, for all $k \in \mathbb{Z}$ and $t \in [2^k, 2^{k+1}]$

$$\frac{\|f*\psi_t\|_{p,\nu_n}}{t^{\gamma}} \leqslant (1+2^{-\gamma})\ \|\psi\|_{1,\nu_n}\ N^{\gamma,\phi}_{p,\infty}(f),$$

which shows that

$$M^{\gamma,\psi}_{p,\infty}(f) \leqslant (1+2^{-\gamma})\ \|\psi\|_{1,\nu_n}\ N^{\gamma,\phi}_{p,\infty}(f).$$

- Let $a_1,\ b_1$ be two real numbers; $0 < a_1 < a < b < b_1$ such that

$$\mathscr{F}(\phi)(\mu,\lambda) = 0 \quad \text{if} \quad \mu^2 + 2|\lambda|^2 < a_1^2 \quad \text{or} \quad \mu^2 + 2|\lambda|^2 > b_1^2.$$

From Lemma 9, there exists $\psi \in S^1_{*,0}\big(\mathbb{R}\times\mathbb{R}^n\big)$ such that

$$\mathscr{F}(\psi)(\mu,\lambda) = C, \quad \text{for all} \quad (\mu,\lambda)\in\Gamma; \quad a_1^2 \leqslant \mu^2 + 2|\lambda|^2 \leqslant 4b_1^2$$

where C is a positive constant. Then for all $k \in \mathbb{Z}$ and $s \in [1,2]$,

$$C\mathscr{F}(\phi)(2^k\mu, 2^k\lambda) = \mathscr{F}(\phi)(2^k\mu, 2^k\lambda)\ \mathscr{F}(\psi)(2^k\mu s, 2^k\lambda s)$$

so,

$$C.\phi_{2^k} = \phi_{2^k} * \psi_{2^k s}.$$

Hence, for all $f \in \mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$

$$C\ f * \phi_{2^k} \quad = \quad f * \psi_{2^k s} * \phi_{2^k} \tag{3.26}$$

and

$$C\ \frac{\|f * \phi_{2^k}\|_{p,\nu_n}}{2^{k\gamma}} \leqslant (1+2^{\gamma})\ \|\phi\|_{1,\nu_n}\ \frac{\|f * \psi_{2^k s}\|_{p,\nu_n}}{(2^k s)^{\gamma}}.$$

Integrating over $[1,2]$ with respect to the measure $\frac{ds}{s}$, we get for all $q \in [1,+\infty[$,

$$\Big(\frac{\|f * \phi_{2^k}\|_{p,\nu_n}}{2^{k\gamma}}\Big)^q \leqslant \frac{\big((1+2^{\gamma})\ \|\phi\|_{1,\nu_n}\big)^q}{C^q \log 2} \int_{2^k}^{2^{k+1}} \Big(\frac{\|f * \psi_t\|_{p,\nu_n}}{t^{\gamma}}\Big)^q \frac{dt}{t}$$

which leads to

$$N^{\gamma,\phi}_{p,q}(f) \leqslant \frac{1}{C}(\log 2)^{-\frac{1}{q}}\ (1+2^{\gamma})\ \|\phi\|_{1,\nu_n}\ M^{\gamma,\psi}_{p,q}(f).$$

On the other hand, for $q = +\infty$ and using the relation (3.26), we deduce that for all $k \in \mathbb{Z}$

$$\frac{\|f * \phi_{2^k}\|_{p,\nu_n}}{2^{k\gamma}} \leqslant \frac{(1+2^{\gamma})}{C}\ \|\phi\|_{1,\nu_n}\ M^{\gamma,\psi}_{p,\infty}(f),$$

which implies that

$$N^{\gamma,\phi}_{p,\infty}(f) \leqslant \frac{(1+2^{\gamma})}{C}\ \|\phi\|_{1,\nu_n}\ M^{\gamma,\psi}_{p,\infty}(f).$$

This completes the proof of theorem. □

Remark 21. 1) From Theorem 14 and Theorem 20, we deduce that the Besov space $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ is independent of the choice of the function $\phi \in S_{*,0}\big(\mathbb{R}\times\mathbb{R}^n\big)$, when it is endowed with the norm $N^{\gamma,\phi}_{p,q}$.

From Proposition 15 and Theorem 20, we deduce the following proposition

Proposition 22. The Besov space $\mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ is homogeneous in a weaker sense when equipped with the norm $N^{\gamma,\phi}_{p,q}$, that is there exist $C_1,\ C_2 > 0$ such that for all $f \in \mathscr{B}^{\gamma}_{p,q}\big([0,+\infty[\times\mathbb{R}^n\big)$ and $t > 0$

$$C_1\ t^{\frac{2n+1}{p}-2n-1-\gamma}\ N^{\gamma,\phi}_{p,q}(f) \leqslant N^{\gamma,\phi}_{p,q}(f_t) \leqslant C_2\ t^{\frac{2n+1}{p}-2n-1-\gamma}\ N^{\gamma,\phi}_{p,q}(f).$$

Proposition 23. Let $p \in [1,+\infty]$ and $\gamma \in \mathbb{R}$. Then for all $q_1,\ q_2 \in [1,+\infty]$;
$q_1 \leqslant q_2$, we have the continuous embedding

$$\mathscr{B}^{\gamma}_{p,q_1}\big([0,+\infty[\times\mathbb{R}^n\big) \hookrightarrow \mathscr{B}^{\gamma}_{p,q_2}\big([0,+\infty[\times\mathbb{R}^n\big).$$

Proof. Let $f \in \mathscr{B}^{\gamma}_{p,q_1}\big([0,+\infty[\times\mathbb{R}^n\big)$ and $\phi \in S_{*,0}\big(\mathbb{R}\times\mathbb{R}^n\big)$. Since

$$\sum_{k\in\mathbb{Z}}\Big(\frac{\|f*\phi_{2^k}\|_{p,\nu_n}}{2^{k\gamma}}\Big)^{q_1} < +\infty$$

then,

$$N^{\gamma,\phi}_{p,\infty}(f) < +\infty$$

and we have

$$N^{\gamma,\phi}_{p,q_2}(f) \leqslant \big(N^{\gamma,\phi}_{p,\infty}(f)\big)^{1-\frac{q_1}{q_2}}\ \big(N^{\gamma,\phi}_{p,q_1}(f)\big)^{\frac{q_1}{q_2}}.$$

However,

$$N^{\gamma,\phi}_{p,\infty}(f) \leqslant N^{\gamma,\phi}_{p,q_1}(f)$$

and consequently,

$$N^{\gamma,\phi}_{p,q_2}(f) \leqslant N^{\gamma,\phi}_{p,q_1}(f).$$

□

Received: October 2009. Revised: December 2009.

References

[1] L. E. Andersson, *On the determination of a function from spherical averages*, SIAM J. Math. Anal 19 (1988), 214–234.

[2] C. Baccar, N. B. Hamadi, and L. T. Rachdi, *Inversion formulas for the Riemann-Liouville transform and its dual associated with singular partial differential operators.* Int. J. Math. Math. Sci. 2006, Article ID 86238, (2006), 1–26.

[3] C. Baccar, S. Omri, and L. T. Rachdi, *Fock spaces connected with spherical mean operator and associated operators.* Mediterr. J. Math. 6 (2009), 1–25.

[4] J. J. Betancor and L. Rodriguez-Mesa, *On the Besov-Hankel spaces*, J. Math. Soc. Japan 50, (3), (1998), 781-788.

[5] O. V. Besov, *On a family of function spaces in connection with embeddings and extensions,* Tr. Mat. Inst. Steklova 60, (1961), 42–81.

[6] M. Dziri, M. Jelassi and L. T. Rachdi, *Spaces of* D_{L^p} *type and a convolution product associated with the spherical mean operator.* Int. J. Math. Math. Sci.,3,(2005), 357–381.

[7] A. Erdely, *Higher Transcendental Functions, vol. II*, Mc Graw-Hill, New York, (1953).

[8] A. Erdely, *Tables of Integral Transforms, vol. I.* Mc GRAW-Hill Book Compagny, INC, (1954).

[9] J. A. Fawcett, *Inversion of N-dimensional spherical means.* SIAM J. Appl. Math. 45, (1985), 336–341.

[10] T. M. Flett, *Lipschitz spaces of functions on the circle and the disc*, J. Math. Anal. Appl. 39 (1972), 125–158.

[11] T. M. Flett, *Temperatures, Bessel potentials and Lipschizt spaces*, Proc. London Math. Soc. 20, (1970), 749–768.

[12] G. B. Folland, *Real analysis: Modern Techniques and Their Applications.* New York, (1984).

[13] D. V. Giang and F. Móriez, *A New characterization of Besov spaces on the real line*, J. Math. Anal. Appl., 189,(1995), 533–551.

[14] H. Helesten and L. E. Anderson, *An inverse method for the processing of synthetic aperture radar data.* Inv. Prob. 3 (1987), 111–124.

[15] M. Herberthson, *A numerical implementation of an inverse formula for CARABAS Raw Data.* Internal Report D 30430-3.2, National Defense Research Institute, Linköping, Sweden, (1986).

[16] L. Kamoun, *Besov-type spaces for the Dunkl operator on the real line.* J. Comput. Appl. Math. 199, no. 1, (2007), 56–67.

[17] N. N. Lebedev, *Special Functions and Their Applications.* Dover publications, Inc. New-York, (1972).

[18] M. M. Nessibi, L. T. Rachdi, and K. Trimèche, *Ranges and inversion formulas for spherical mean operator and its dual.* J. Math. Anal. Appl. 196, (1995), 861–884.

[19] M. M. Nessibi, and K. Trimèche, *Inversion of the Radon transform on the Laguerre hypergroup by using generalized wavelets.* J. Math. Anal. Appl. 208 (1997), 337–363.

[20] J. Peetre, *New Thoughts on Besov Spaces*, Duke Univ. Math. Series, Durham, NC, (1976).

[21] L. T. Rachdi, and K. Trimèche, *Weyl transforms associated with the spherical mean operator.* Anal. Appl. 1, no 2, (2003), 141–164.

[22] M. Taibleson, *On the theory of lipschizt spaces of distributions on euclidean n-space.* I, II, III, J. Math. Mech. 13 (1964), 407–480; 65, 821–840; 15 (1966), 973–981.

[23] H. Triebel, *Theory of Function Spaces*, Monographs in Math., vol. 78, Birkhäuser, Verlag, Basel, (1983).

[24] K. Trimèche, *Transformation intégrale de Weyl et théorème de Paley-Wiener associés a un opérateur différentiel singulier sur* $(0, +\infty)$, J. Math. Pures Appl., 60 (1981), 51–98.

[25] K. Trimèche, *Inversion of the Lions translation operator using generalized wavelets*, Appl. Comput. Harmon. Anal. 4 (1997), 97–112.

[26] K. Trimèche, Generalized Harmonic Analysis and Wavelet Packets: An Elementary Treatment of Theory & Applications, Gordon and Breach Science Publishers, (2001).

[27] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed. Cambridge Univ. Press. London/New-York, (1966).

CUBO A Mathematical Journal
Vol.13, Nº02, (37–57). June 2011

Remarks on the Generation of Semigroups of Nonlinear Operators on p-Fréchet Spaces, $0 < p < 1$

Sorin G. Gal
University of Oradea,
Department of Mathematics,
Str. Universitatii 1, 410087 Oradea, Romania
email: galso@uoradea.ro

ABSTRACT

In this paper we study the convergence properties of the Crandall-Liggett sequence $J^n_{t/n}(A)(x) = \left(I - \frac{t}{n}A\right)^{-n}(x)$, $n \in \mathbb{N}$, for A a nonlinear operator on some important non-locally convex F-spaces (called p-Fréchet spaces with $0 < p < 1$) and the generation of the corresponding strongly continuous one-parameter nonlinear semigroups.

RESUMEN

En este trabajo se estudian las propiedades de convergencia de la secuencia de Crandall-Liggett $J^n_{t/n}(A)(x) = \left(I - \frac{t}{n}A\right)^{-n}(x)$, $n \in \mathbb{N}$ para A un operador lineal en algunos importantes F-espacios no-localmente convexos (llamado p-Fréchet espacios con $0 < p < 1$) y la generación de los correspondientes semigrupos fuertemente continuos no lineales con un parámetro.

Keywords and phrases:: p-Fréchet space, $0 < p < 1$, Cauchy problem, affine semigroup, nonlinear semigroup, Crandall-Liggett type theorem.

Mathematics Subject Classification: 47H06, 47H20.

1. Introduction

It is well known that an F-space $(X, +, \cdot, \|\cdot\|)$ is a linear space (over the field $K = \mathbb{R}$ or $K = \mathbb{C}$) such that $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$, $\|x\| = 0$ if and only if $x = 0$, $\|\lambda x\| \leq \|x\|$, for all scalars λ with $|\lambda| \leq 1, x \in X$, and with respect to the metric $d(x, y) = \|x - y\|$, X is a complete metric space (see e.g. [4, p. 52] or [7]).

In addition, if there exists $0 < p < 1$ with $\|\lambda x\| = |\lambda|^p \|x\|$, for all $\lambda \in K, x \in X$, then $\|\cdot\|$ will be called a p-norm and X will be called p-Fréchet space. (This is only a slight abuse of terminology. Note that in e.g. [1] these spaces are called p-Banach spaces).

It is known that the F-spaces are not necessarily locally convex spaces. Three classical examples of p-Fréchet spaces, non-locally convex, are the Hardy space H^p with $0 < p < 1$ that consists in the class of all analytic functions $f : D \to \mathbb{C}$, $D = \{z \in \mathbb{C}; |z| < 1\}$ with the property

$$\|f\| = \frac{1}{2\pi} \sup\{\int_0^{2\pi} |f(re^{it})|^p dt; r \in [0, 1)\} < +\infty,$$

the sequence space

$$l^p = \{x = (x_n)_n; \|x\| = \sum_{n=1}^{\infty} |x_n|^p < \infty\}$$

for $0 < p < 1$, and the $L^p[0, 1]$ space, $0 < p < 1$, given by

$$L^p = L^p[0, 1] = \{f : [0, 1] \to \mathbb{R}; \|f\| = \int_0^1 |f(t)|^p dt < \infty.\}$$

Some important characteristics of the F-spaces are given by the following remarks.

Remarks. 1) Three of the basic results in Functional Analysis hold in F-spaces too : the Principle of Uniform Boundedness (see e.g. [4, p. 52]), the Open Mapping Theorem and the Closed Graph Theorem (see e.g. [7, p. 9-10]).

2) The Hahn-Banach Theorem fails in non-locally convex F-spaces. More exactly, if in an F-space the Hahn-Banach theorem holds, then that space is a necessarily locally convex space (see e.g. [6, Chapter 4]).

The beginning of a theory of semigroups of linear operators on p-Fréchet spaces, $0 < p < 1$, was developed in the very recent paper [5]. One of the main result in [5] is the Chernoff-type formula $e^{tA}(x) = \text{lím}_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n}(x)$, for A a bounded linear operator on a p-Fréchet space with $0 < p < 1$.

The aim of the present paper is to look for similar results, that is for convergence properties of the sequence $J^n_{t/n}(A)(x) = \left(I - \frac{t}{n}A\right)^{-n}(x), n \in \mathbb{N}$, in the case when A is a nonlinear operator on a p-Fréchet space with $0 < p < 1$. A very careful examination of the proofs in [3] shows us that because of the property $\|\lambda x\| = |\lambda|^p \|x\|$ with $0 < p < 1$, the estimate for $\|J^n_{t/n}(A)(x) - J^m_{t/m}(A)(x)\|$ does not converges to zero as $m, n \to \infty$ and in fact the sequence $J^n_{t/n}(A)(x), n \in \mathbb{N}$, is not, in general, a convergent one.

However, by using techniques in Functional Analysis, we will be able to prove that the sequence $(J^n_{t/n}(A)(x))_{n\in\mathbb{N}}$ contains some convergent subsequences in the spaces l^p and H^p with $0<p<1$, while this kind of result seems to fail in the space $L^p[0,1]$, $0<p<1$. Moreover, in the simplest nonlinear case when A is an affine operator, we prove that the sequence $(J^n_{t/n}(A)(x))_{n\in\mathbb{N}}$ is still convergent and some results in the case of Banach spaces in [6] will be extended to p-Fréchet spaces $(0<p<1)$ too.

The plan of the paper goes as follows. In Section 2 we study the case when A is an affine operator on an arbitrary p-Fréchet space, $0<p<1$, Section 3 deals with the case when A is a nonlinear Lipschitz operator on l^p, $0<p<1$, while the Sections 4 and 5 deal with the similar problem in the spaces H^p and $L^p[0,1]$, respectively, with $0<p<1$.

2. Affine Semigroups

As we will see, the affine case is closely connected to the linear case.

First we need a result in operator theory on p-Fréchet spaces (well-known in the case of classical Banach spaces).

Lemma 2.1 *Let $A,B:X\to X$ be bounded linear operators on the p-Fréchet space $(X,\|\cdot\|)$, $0<p<1$. If A is bijection and $|||A^{-1}B|||<1$ then $A+B$ is bounded linear bijection on X.*

Proof. Since A is a bijection, as a consequence of the Open Mapping Theorem it follows that A^{-1} is a bounded linear operator (see e.g. [1, Theorem 14, p. 20 and Corollary 2, p. 23]).

Next we reason as in the case of Banach spaces. Let $y\in X$ be arbitrary fixed and define $T_y(x)=A^{-1}(y)-(A^{-1}B)(x)$. Then the equation $(A+B)(x)=y$ is equivalent to the equation $T_y(x)=x$. But $\|T_y(x_1)-T_y(x_2)\|\le|||A^{-1}B|||\cdot\|x_1-x_2\|$, which shows that T_y is a contraction in the complete metric space X (with respect to the metric $d(x_1,x_2)=\|x_1-x_2\|$). Therefore it has a unique fixed point x, which shows that $A+B$ is bijective and the lemma is proved.

The first result on affine semigroups is the following.

Theorem 2.2 *Let $(X,\|\cdot\|)$ be a p-Fréchet space, $0<p<1$, $A(x)=B(x)+x_0$, where $x_0\in X$ is fixed and $B:X\to X$ is a bounded, linear and strictly dissipative operator, i.e $|||(I-\lambda B)^{-1}|||<1$, for all $\lambda>0$ sufficiently small. Then B is invertible and if we define*

$$J_\lambda(A)(x)=(I-\lambda A)^{-1}(x),$$

(here I defines the identity operator) then

$$T(t)(x)=\lim_{n\to+\infty}J^n_{t/n}(A)(x)=e^{tB}(x)+B^{-1}[e^{tB}(x_0)]-B^{-1}(x_0),$$

is a strongly continuous semigroup of nonlinear (affine) operators on X, (where according to [4], $e^{tB}(x)=\lim_{n\to+\infty}J^n_{t/n}(B)(x)$ is a strongly continuous semigroup of linear operators on X).

Proof. By easy calculation we can write

$$J_\lambda(A)(x) = J_\lambda(B)(x+\lambda x_0) = (I-\lambda B)^{-1}(x+\lambda x_0),$$

and in general

$$J_\lambda^n(A)(x) = (I-\lambda B)^{-n}(x) + \lambda\left[\sum_{k=1}^{n}(I-\lambda B)^{-k}(x_0)\right].$$

But it is easy to show that for any operator G we have the identity

$$(I-G)(I+G+G^2+...+G^{n-1}) = I-G^n.$$

Replacing G by $J_\lambda(B)$, by Lemma 2.1 it follows that $I-J_\lambda(B)$ is invertible and we immediately obtain

$$\left[\sum_{k=1}^{n}(I-\lambda B)^{-k}(x_0)\right] = \lambda J_\lambda(B)[I-J_\lambda(B)]^{-1}[I-J_\lambda^n(B)]^{-n}(x_0).$$

But

$$J_\lambda(B)[I-J_\lambda(B)]^{-1} = (I-\lambda B)^{-1}[I-J_\lambda(B)]^{-1} =$$
$$\{[I-(I-\lambda B)^{-1}][I-\lambda B]\}^{-1} =$$
$$\{-\lambda B\}^{-1} = -\frac{1}{\lambda}B^{-1},$$

which implies that

$$\left[\sum_{k=1}^{n}(I-\lambda B)^{-k}(x_0)\right] = -B^{-1}[(I-\lambda B)^{-n}](x_0).$$

Taking $\lambda = \frac{t}{n}$, passing to limit with $n\to+\infty$ and taking into account the important Remark after the Theorem 2.11 in [4] which says that

$$e^{tB}(x) = \lim_{n\to+\infty}(I-\frac{t}{n}B)^{-n},$$

we arrive at

$$\lim_{n\to+\infty} J_{t/n}^n(A)(x) = e^{tB} + B^{-1}[e^{tB}(x_0)] - B^{-1}(x_0).$$

Also, simple calculations show that if we denote $T(t)(x) = e^{tB}(x) + B^{-1}[e^{tB}(x_0)] - B^{-1}(x_0)$, then $T(0) = I$, $\{T(t), t\geq 0\}$ has the semigroup property, $T(\cdot)(x)$ is continuous as function of t, $A(x) = \lim_{h\searrow 0}\frac{T(h)(x)-x}{h}$, for all $x\in X$ and $T'(t)(x) = B[T(t)(x)] + x_0$, which proves the theorem.

Remarks. 1) According to Theorem 2.2, $T(t)(u_0)$ is the unique solution of the Cauchy problem

$$u'(t) = B[u(t)] + x_0, u(0) = u_0.$$

(The uniqueness of the solution follows from Lemma 2.12 in [5] concerning the uniqueness of the solution for the inhomogeneous Cauchy problem in p-Fréchet spaces, $0<p<1$.)

2) Let us give a simple example satisfying Theorem 2.2. Let $(X, \|\cdot\|)$ be a p-Fréchet space, $0 < p < 1$, and define $A : X \to X$ by $A(x) = B(x) + x_0$, where $B(x) = -x$ for all $x \in X$ and $x_0 \in X$ is fixed. B obviously is strictly dissipative and A obviously is nonlinear, strictly dissipative, with

$$\|(I - \lambda A)^{-1}\|_{\mathrm{Lip}} = \frac{1}{1+\lambda} < 1,$$

for all $\lambda > 0$.

We see that $B^{-1} = B$, $e^{tB}(x) = xe^{-t}$ and $T(t)(x) = (x - x_0)e^{-t} + x_0$ and in this case $T(t)(u_0)$ is the unique solution to the nonlinear Cauchy problem

$$\frac{du}{dt} = -u(t) + x_0, u(0) = u_0.$$

3) From the proof of Theorem 2.2, it easily follows the following.

Corollary 2.3 *In the case when $(X, \|\cdot\|)$ is a Banach space (i.e. a p-Fréchet space with $p = 1$), the statement of Theorem 2.2 still remains true.*

In what follows, let us consider some concepts introduced in [6] for Banach spaces. They remain unchanged for the case of p-Fréchet spaces too.

Definition 2.4 By an affine semigroup $(S(t) : t \geq 0)$ on a p-Fréchet space X, $0 < p < 1$, we mean a family of continuous affine transformations on X with the properties :

(i) S(0)=I, S(t+s)=S(t)[S(s)], for all $t, s \geq 0$;

(ii) For each $x \in X$, $t \to S(t)(x)$ is a continuous function from $[0, +\infty)$ into X.

(iii) Any family $(S(t) : t \geq 0)$ of affine transformations on X can be written in the form $S(t)(x) = T(t)(x) + z(t)$, for all $t \geq 0$, $x \in X$, where $T(t)(x) = S(t)(x) - S(t)(0)$ is its linear part and $z(t) = S(t)(0)$ is its translation part ($z : [0, +\infty) \to X$).

(iv) Let us denote by $\overline{X} = X \times \mathbb{R}$. It is a p-Fréchet space, endowed with the p-norm $\|(x, r)\| = \text{máx}\{\|x\|, |r|^p\}$. If $(S(t) : t \geq 0)$ is a family of affine transformations on X of the form $S(t)(x) = T(t)(x) + z(t)$, for all $t \geq 0$, $x \in X$, where $T(t)(x)$ is its linear part and $z(t)$ is its translation part, the augmented family associated with $(S(t) : t \geq 0)$, is a family $(\overline{T}(t); t \geq 0)$ of linear transformations on $\overline{X}$, defined by

$$\overline{T}(t)[x, r] = [T(t)(x) + rz(t), r].$$

Having introduced these concepts, Propositions 1.1 and 1.2 proved in [6] for Banach spaces, hold (with the same proofs) for p-Fréchet spaces too, summarized as follows.

Theorem 2.5 *(i) Let $(S(t) : t \geq 0)$ be a family of affine transformations on the p-Fréchet space X, $0 < p < 1$, with its linear part $(T(t) : t \geq 0)$ and its translation part $z(t); t \geq 0)$. Then $(S(t) : t \geq 0)$ is an affine semigroup on X if and only if $(T(t) : t \geq 0)$ is a linear semigroup on X and $z(\cdot)$ is a continuous map from $[0, +\infty)$ into X satisfying*

$$z(t + s) = T(t)[z(s)] + z(t), s, t \geq 0.$$

(ii) Let $(S(t) : t \geq 0)$ *be a family of affine transformations on the* p*-Fréchet space* X*,* $0 < p < 1$*, and let* $(\overline{T}(t) : t \geq 0)$ *be the augmented family on* $\overline{X}$*, associated with* $S(\cdot)$*. Then* $(S(t) : t \geq 0)$ *is an affine semigroup on* X*, if and only if* $(\overline{T} : t \geq 0)$ *is a linear semigroup on* $\overline{X}$*.*

Remark. While Proposition 2.1 in [6] remains valid in the case of p-Fréchet spaces too, $0 < p < 1$, the other results in [6, Section 2] (i.e. Corollary 2.2, Proposition 2.3, Proposition 2.4 and Corollary 2.5) seem to be not valid. The reason is that they use the Fundamental Theorem of Calculus in Banach spaces, which, as it was pointed out in [5], does not hold in p-Fréchet spaces, $0 < p < 1$.

It would be of interest to see what other results for affine semigroups on Banach spaces in [6], would remain valid for p-Fréchet spaces too, $0 < p < 1$.

3. Nonlinear Semigroups on l^p, $0 < p < 1$

Before to starting the study in the concrete l^p-case, $0 < p < 1$, let us briefly recall the problem and make a useful remark, valid in any p-Fréchet space, $0 < p < 1$.

For $(X, \|\cdot\|_X)$ a p-Fréchet space, $0 < p \leq 1$ (the case $p = 1$ means that X is a Banach space), let $A : X \to X$ be a nonlinear operator and let us consider the abstract Cauchy problem

$$\frac{d}{dt}u(t) = A[u(t)], t \geq 0,$$

$$u(0) = x,$$

where the solution is $u : \mathbb{R}_+ \to X$ and $x \in X$ is fixed. The nonlinear operator A is considered a Lipschitz mapping, that is

$$\|A(x) - A(y)\|_X \leq |||A|||_{Lip}\|x - y\|_X, \text{ for every } x, y \in X,$$

where $|||A|||_{Lip} = \sup\{\|A(x)-A(y)\|_X/\|x-y\|_X; x, y \in X, x \neq y\} < +\infty$. If we replace this differential equation by the difference equation

$$\frac{1}{\varepsilon}[u_\varepsilon(t) - u_\varepsilon(t - \varepsilon)] = A[u_\varepsilon(t)], t \geq 0,$$

with initial condition $u_\varepsilon(s) = x, -\varepsilon \leq s \leq 0$, then we easily get by recurrence that $u_\varepsilon(t) = \left(I - \frac{t}{n}A\right)^{-n}(x)$, for $\varepsilon = \frac{t}{n}$.

Remark. Without loss of generality, we may suppose $A(0) = 0$. Indeed, if we suppose that $A(0) \neq 0$, then denoting $B(u) = A(u) - A(0)$ we get $B(0) = 0$ and if $v(t)$ is solution of the abstract Cauchy problem

$$\frac{d}{dt}v(t) = B[v(t)], v(0) = u_0,$$

then $u(t) = v(t)+tA(0)$ is a solution of the above (in A) mentioned problem. Moreover, if for a fixed $\omega \in \mathbb{R}$, the operator $A - \omega I$ is dissipative, then $B - \omega I$ also is dissipative. Indeed, from $B - \omega I =$

$(A-\omega I)-A(0)$, since $A-\omega I$ is injective and surjective, it easily follows that $B-\omega I$ is injection and surjection and , in addition, from the relationship $(B-\omega I)^{-1}(y)=(A-\omega I)^{-1}(y+A(0))$, we get $\|(B-\omega I)^{-1}\|_{Lip}=\|(A-\omega I)^{-1}\|_{Lip}\le 1$.

In what follows, we denote the p-norm in l^p by $\|\cdot\|_p$. The first main result of this section is the following.

Theorem 3.1 *Let* $A: l^p\to l^p$, $0<p<1$, *be nonlinear and Lipschitz, such that* $A(0)=0$ *and there exists* $\omega\in\mathbb{R}$ *with* $A-\omega I$ *dissipative. Then the sequence in* l^p *defined by* $J^n_{t/n}(x)=(I-\frac{t}{n}A)^{-n}(x), n\in\mathbb{N}, t\ge 0, x\in l^p$, *contains a subsequence* $J^{n_k}_{t/n_k}(x), k\in\mathbb{N}$ *(the same subsequence for all* $x\in l^p$ *and all* $t\in\mathcal{R}_+$ *= the set of all rational numbers* ≥ 0*), convergent to an element of* l^p *in the weak topology of* l^p.

Proof. By $A(0)=0$ we get $(I-\frac{t}{n}A)^{-n}(0)=0$, for all $n\in\mathbb{N}$. The dissipative property implies $\|[I-t(A-\omega I)]^{-1}\|_{Lip}\le 1$, for all $t\ge 0$, which is equivalent to $\|(I-\frac{t}{1+t\omega}A)^{-1}\|_{Lip}\le|1+t\omega|^p$, for all $t\ge 0$ with $1+t\omega\ne 0$. For $\lambda=\frac{t}{1+t\omega}$ we get $\|(I-\lambda A)^{-1}\|_{Lip}\le(1-\lambda\omega)^{-p}$, in particular for all $\lambda>0$ with $\lambda\omega<1$. (Note that for n sufficiently great, depending on t and ω, we have $\frac{t}{n}\omega<1$.) Therefore,

$$\|(I-\frac{t}{n}A)^{-1}\|_{Lip}\le(1-\frac{t}{n}\omega)^{-p}$$

and by mathematical induction

$$\|(I-\frac{t}{n}A)^{-n}\|_{Lip}\le(1-\frac{t}{n}\omega)^{-np}.$$

But it is known that the sequence $(1+\frac{s}{n})^n$ converges (for $n\to+\infty$) to e^s, and for any $s\in\mathbb{R}$ it is monotonically increasing, for all $n\ge[|s|]+1$ (see e.g. [10, p. 263]), which implies that $(1-\frac{t}{n}\omega)^{-np}$ converges to $e^{t\omega p}$, monotonically decreasing, for all $n\ge[|tw|]+1$. Therefore, the greatest value of $(1-\frac{t}{n}\omega)^{-np}$ is for $n=[|t\omega|]+1$, which means that there exists $M=M(t,p,\omega)>0$ (depending only on t, p and ω) such that

$$\|(I-\frac{t}{n}A)^{-n}\|_{Lip}\le M,$$

for all $n\in\mathbb{N}$.

We obtain

$$\|J^n_{t/n}(x)-J^n_{t/n}(y)\|_p\le(1-\frac{t}{n}\omega)^{-np}\|x-y\|_p\le M\|x-y\|_p, \tag{3.1}$$

for all $x,y\in l^p$. Taking $y=0$ we have $J^n_{t/n}(0)=0$ and denoting $J^n_{t/n}(x)=(g_{n,r}(t)(x))_r\in l^p$, we obtain $\|J^n_{t/n}(x)\|_p\le M\|x\|_p$, i.e.

$$\sum_{r=1}^{+\infty}|g_{n,r}(t)(x)|^p\le M\|x\|_p<+\infty, \tag{3.2}$$

for all $n\in\mathbb{N}$.

Now, since l^p, $0 < p < 1$, has a Schauder basis (see e.g. [7, p. 20]), it follows that it is separable, denote by Y a countable dense subset of l^p. Also, denote by $\mathcal{R}_+$, the set of all positive nonnegative rational numbers and define $G_n : \mathbb{N} \times \mathcal{R}_+ \times Y \to K$ (where $K = \mathbb{R}$ or $\mathbb{C}$), by $G_n(r, t, y) = g_{n,r}(t)(y)$.

Since $E := \mathbb{N} \times \mathcal{R}_+ \times Y$ is countable and by (2) the sequence $(G_n)_n$ is pointwise bounded on E, by the Cantor's diagonal process (see e.g. [11, p. 156-157]), there exists a subsequence $G_{n_k}, k \in \mathbb{N}$, pointwise convergent on E. Denote $g_r(t)(y) = \text{lím}_{k\to+\infty} g_{n_k,r}(t)(y)$, for all $(r, t, y) \in E$.

We will show that in fact there exists the limit (in $\mathbb{R}$), $\text{lím}_{k\to\infty} g_{n_k,r}(t)(x)$, for all $r \in \mathbb{N}$, $t \in \mathcal{R}_+$ and $x \in l^p$. For this purpose, we will show that $(g_{n_k,r}(t)(x))_k$ is a Cauchy sequence in $\mathbb{R}$ (i.e. it is convergent).

For this purpose, let $x \in l^p$ and $y \in Y$. We have

$$|g_{n_k,r}(t)(x) - g_{n_s,r}(t)(x)| \le |g_{n_k,r}(t)(x) - g_{n_k,r}(t)(y)| +$$

$$|g_{n_k,r}(t)(y) - g_{n_s,r}(t)(y)| + |g_{n_s,r}(t)(y) - g_{n_s,r}(t)(x)|.$$

Taking into account (1) too, we immediately obtain

$$|g_{n_k,r}(t)(x) - g_{n_s,r}(t)(x)| \le 2M^{1/p} \cdot \|x - y\|_p^{1/p} + |g_{n_k,r}(t)(y) - g_{n_s,r}(t)(y)|.$$

Now, since Y is dense in l^p, for $x \in l^p$ and $\varepsilon > 0$, let $y \in Y$ such that $2M^{1/p}\|x - y\|_p^{1/p} < \frac{\varepsilon}{2}$, which implies

$$|g_{n_k,r}(t)(x) - g_{n_s,r}(t)(x)| < \frac{\varepsilon}{2} + |g_{n_k,r}(t)(y) - g_{n_s,r}(t)(y)|.$$

But for this y, the sequence $(g_{n_k,r}(t)(y))_k$ is convergent, i.e. it is a Cauchy sequence, which implies that there exists l_0 such that for all $k, s \ge l_0$ we have

$$|g_{n_k,r}(t)(y) - g_{n_s,r}(t)(y)| < \frac{\varepsilon}{2}.$$

This leads to

$$|g_{n_k,r}(t)(x) - g_{n_s,r}(t)(x)| < \varepsilon,$$

for all $k, s \ge l_0$, i.e. $(g_{n_k,r}(t)(x))_k$ is a Cauchy sequence in $\mathbb{R}$. Therefore, we can write

$$g_r(t)(x) = \lim_{k\to\infty} g_{n_k,r}(t)(x),$$

for all $t \in \mathcal{R}_+$ and $x \in l^p$.

By (2) it follows

$$\sum_{r=1}^{m} |g_{n_k,r}(t)(x)|^p \le M\|x\|_p < +\infty,$$

for all $k, m \in \mathbb{N}$. Passing here to limit with $k \to \infty$, we get

$$\sum_{r=1}^{m} |g_r(t)(x)|^p \le M\|x\|_p < +\infty,$$

for all $m \in \mathbb{N}$, which obviously implies

$$\sum_{r=1}^{\infty} |g_r(t)(x)|^p \leq M\|x\|_p < +\infty,$$

i.e. $g(t)(x) := (g_r(t)(x))_r$ belongs to l^p.

Now, we will show that for any $x^* \in (l^1)^*$, i.e. of the form (see e.g. [8, pp. 36-37]) $x^*(z) = \sum_{j=1}^{\infty} u_j z_j$, for all $z \in l^1$, where $(u_j)_j \in m$, with m denoting the space of all bounded sequences, we have $x^*(J^{n_k}_{t/n_k}(x)) \to x^*(g(t)(x))$, when $k \to \infty$, for any fixed $t \in \mathcal{R}_+, x \in l^p$. Note that $x^*(g(t)(x))$ has sense for $g(t)(x) \in l^p$, because $l^p \subset l^1$.

It is obvious that each functional of the form $x_i^*(x) = x_i$, for all $x = (x_i)_i \in l^1$, is linear and continuous on l^1, since $|x_i^*(x)| = |x_i| \leq \sum_{j=1}^{\infty} |x_j| = \|x\|_{l^1}$ and for $k \to \infty$, $x_i^*[J^{n_k}_{t/n_k}(x)] = g_{n_k,i}(t)(x) \to g_i(t)(x) = x_i^*(g(t)(x))$, for all $i \in \mathbb{N}$.

Then, obviously that for any $y^* \in span\{x_1^*, ..., x_i^*, ...,\} =: Y^*$ we also have $y^*[J^{n_k}_{t/n_k}(x)] \to y^*[g(t)(x)]$, for $k \to \infty$.

We show that Y^* is dense in $(l^1)^*$ in the weak topology on $(l^1)^*$. Indeed, let $x^* \in (l^1)^*$ be arbitrary, $x^*(u) = \sum_{i=1}^{\infty} \alpha_i u_i$, for all $u = (u_j)_j \in l^1$, where $\alpha = (\alpha_j)_j \in m$. Since $z_n^*(u) = \sum_{j=1}^{n} \alpha_j u_j = \sum_{j=1}^{n} \alpha_j x_j^*(u)$, it follows $z_n^* \in Y^*$ and we get

$$|x^*(u) - z_n^*(u)| \leq \sum_{j=n+1}^{+\infty} |\alpha_j u_j| \leq \|\alpha\|_m \sum_{i=n+1}^{+\infty} |u_i| \leq M_0 \sum_{i=n+1}^{+\infty} |u_i| \to 0,$$

for $n \to \infty$.

This implies that $z_n^* \to x^*$ in the weak topology (i.e. the density of Y^* in $(l^1)^*$ in the weak topology) and that for any $\varepsilon > 0$ and any $u_1, u_2 \in l^1$, $x^* \in (l^1)^*$, there exists $y^* \in Y^*$, such that $|x^*(u_j) - y^*(u_j)| < \varepsilon$, $j = 1, 2$.

For $u_1 = J^{n_k}_{t/n_k}(x)$ and $u_2 = g(t)(x)$, we get

$$|x^*[J^{n_k}_{t/n_k}(x)] - x^*[g(t)(x)]| \leq |x^*[J^{n_k}_{t/n_k}(x)] - y^*[J^{n_k}_{t/n_k}(x)]| +$$

$$|y^*[J^{n_k}_{t/n_k}(x)] - y^*[g(t)(x)]| + |y^*[g(t)(x)] - x^*[g(t)(x)]| <$$

$$2\varepsilon + |y^*[J^{n_k}_{t/n_k}(x)] - y^*[g(t)(x)]| < 3\varepsilon,$$

for all $k > k_0$, with k_0 depending on ε, t and x.

This shows that for any $x^* \in (l^1)^*$, if $k \to \infty$ then we have $x^*[J^{n_k}_{t/n_k}(x)] \to x^*[g(t)(x)]$, for any fixed $t \in \mathcal{R}_+$ and $x \in l^p$.

Finally, since according to [7], p. 27, l^1 is the so-called Banach envelope of l^p and $(l^1)^* = (l^p)^*$ (with the same dual norms too), the theorem is proved.

Remarks. 1) We may repeat the reasonings in the proof of Theorem 3.1 for the sequence $(J^n_{t/n}(x), n \in \mathbb{N}, n \neq n_k)$, where n_k is the subsequence in Theorem 3.1, so that by mathematically induction we easily obtain that the sequence $(J^n_{t/n}(x))_{n\in\mathbb{N}}$ has at most a countable set of limit points in the weak topology of l^p, denote that set by $T^*(t)(x)$, where $t \in \mathcal{R}_+, x \in l^p$. For any fixed $t \in \mathcal{R}_+$, an element $a \in T^*(t)$ is in fact a mapping $a : l^p \to l^p$.

2) From the proof of Theorem 3.1, we easily can derive that in addition, the functions $g_{n,r}(t) : l^p \to \mathbb{R}$ are Lipschitz functions, i.e. $|g_{n,r}(t)(x) - g_{n,r}(t)(y)| \le M^{1/p}\|x-y\|_p^{1/p}$, which implies that the family $(g_{n,r}(t))_{n,r\in\mathbb{N}}$ is equicontinuous. Also, for any $x \in l^p$, $t \in \mathcal{R}_+$, the sequence $(g_{n,r}(t)(x))_{n,r\in\mathbb{N}}$ is bounded . Unfortunately we cannot apply the classical Arzela-Ascoli theorem in l^p, because l^p is not locally compact.

However, we may impose some additional properties to the nonlinear operator A, which could imply better convergence results in Theorem 3.1, as follows.

Consider on l^p the so called lexicographic order, i.e. for $x = (x_j)_j, y = (y_j)_j \in l^p$, we write $x \le y$ if and only if $x_j \le y_j$, for all $j \in \mathbb{N}$ and $x < y$ if and only if $x \le y$ and there is a j with $x_j < y_j$.

The following simple result holds.

Lemma 3.2 *Suppose that* $A : l^p \to l^p$ *is a dissipative nonlinear operator, A(0)=0,* A *is convex and non-increasing with respect to the above order, i.e.*

$$A[\alpha x + (1-\alpha)y] \le \alpha A(x) + (1-\alpha)A(y),$$

for all $x, y \in l^p, \alpha \in [0,1]$ *and* $x < y$ *implies* $A(x) \ge A(y)$. *We have :*

(i) $I - \lambda A$ *is concave and non-decreasing, for any* $\lambda > 0$ *;*

(ii) $B := (I - \lambda A)^{-1}$ *is convex and non-decreasing, for any* $\lambda > 0$ *;*

(iii) B^n *is convex and non-decreasing, for any* $\lambda > 0$.

The proof is an easy exercise and it is left to the reader.

Remark. Lemma 3.2 says that if A is convex and non-increasing, then so is $J^n_{t/n}(x)$, which obviously implies that the functions $g_{n,r}(t) : l^p \to \mathbb{R}$ in the proof of Theorem 3.1 are convex and non-decreasing.

Corollary 3.3 *Denote by* $T(t)(x) \in l^p$ *the weak limit in* l^p *of the sequence* $(J^{n_k}_{t/n_k}(x))_k$, *for all* $t \in \mathcal{R}_+$ *and* $x \in l^p$, *where* $(n_k)_k$ *is the subsequence in Theorem 3.1. We have*

(i) $T(0) = I$ *;*

(ii) $\|T(t)(x) - T(t)(y)\|_{l^1} \le e^{t\omega}\|x-y\|_p^{1/p}$, *for all* $t \in \mathcal{R}_+$, $x, y \in l^p \subset l^1$ *;*

(iii) For any $t, s \in R_+$ *and* $a \in T^*(t+s)$, *there exist* $b \in T^*(t)$ *and* $c \in T^*(s)$ *such that* $a(x) = b[c(x)]$, *for all* $x \in l^p$.

Proof. (i) It is obvious by the definition of $J^{n_k}_{t/n_k}(x)$;

(ii) First, passing to limit with $k \to +\infty$ in the following inequality in the proof of Theorem

3.1

$$\|J^{n_k}_{t/n_k}(x) - J^{n_k}_{t/n_k}(y)\|_p \le (1 - \frac{t}{n_k}\omega)^{-n_k p}\|x - y\|_p,$$

we easily get

$$\lim_{k\to\infty}\|J^{n_k}_{t/n_k}(x) - J^{n_k}_{t/n_k}(y)\|_p \le \lim_{k\to\infty}(1 - \frac{t}{n_k}\omega)^{-n_k p}\|x - y\|_p = e^{t\omega p}\|x - y\|_p.$$

Let $x^* \in (l^p)^*$ be with $|||x^*|||_{(l^p)^*} \le 1$. According to [7, p. 27], it can be extended to a $x^* \in (l^1)^*$, preserving its norm, i.e. $|||x^*|||_{(l^1)^*} \le 1$. We have

$$|x^*[T(t)(x)] - x^*[T(t)(y)]| = |x^*[T(t)(x) - T(t)(y)]| \le$$

$$|x^*[T(t)(x)] - x^*[J^{n_k}_{t/n_k}(x)]| + |x^*[J^{n_k}_{t/n_k}(x)] - x^*[J^{n_k}_{t/n_k}(y)]| +$$

$$|x^*[J^{n_k}_{t/n_k}(y)] - x^*[T(t)(y)]| :=$$

$$a_k + |x^*[J^{n_k}_{t/n_k}(x)] - x^*[J^{n_k}_{t/n_k}(y)]| + b_k,$$

where $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$ by the definitions of $T(t)(x)$ and $T(t)(y)$.

On the other hand,

$$|x^*[J^{n_k}_{t/n_k}(x)] - x^*[J^{n_k}_{t/n_k}(y)]| \le |||x^*|||_{(l^1)^*}\|J^{n_k}_{t/n_k}(x) - J^{n_k}_{t/n_k}(y)\|_{l^1} \le$$

$$\|J^{n_k}_{t/n_k}(x) - J^{n_k}_{t/n_k}(y)\|_{l^1} \le \|J^{n_k}_{t/n_k}(x) - J^{n_k}_{t/n_k}(y)\|_p^{1/p}.$$

Passing in the above two inequalities to limit with $k \to \infty$, we get

$$|x^*[T(t)(x)] - x^*[T(t)(y)]| \le e^{t\omega}\|x - y\|_p^{1/p},$$

for all $x^* \in (l^1)^*$, with $|||x^*|||_{(l^1)^*} \le 1$. Passing here to supremum with such x^*, by a classical result in functional analysis for normed spaces, it follows

$$\sup_{|||x^*|||_{(l^1)^*}\le 1}|x^*[T(t)(x)] - x^*[T(t)(y)]| = \|T(t)(x) - T(t)(y)\|_{l^1} \le e^{t\omega}\|x - y\|_p^{1/p}.$$

(iii) Let $q \in \mathbb{N}$, $t \in \mathcal{R}_+, x \in l^p$ be fixed. For any $x^* \in (l^p)^*$, we have

$$\lim_{k\to+\infty} x^*[J^{n_k}_{qt/n_k}(x)] = x^*([T(qt)](x)),$$

which immediately implies

$$\lim_{k\to+\infty} x^*(J^{qn_k}_{qt/qn_k}(x)) = \lim_{k\to+\infty} x^*(J^{qn_k}_{t/n_k}(x)) = \lim_{k\to+\infty} x^*([T(t)]^q(x)).$$

Applying the same reasonings as in the proof of Theorem 3.1, there exists a subsequence of $(qn_k)_k$, let us denote it by $(q_k)_k$, such that

$$\lim_{k\to+\infty} x^*[J^{q_k}_{qt/q_k}(x)] = x^*([T(t)]^q(x)),$$

which shows that for all $t \in \mathcal{R}_+$, if $a \in T^*(qt)$, then there exists $b \in [T^*(t)]^q$ with $a = b$.

Then, for $l, k, r, s \in \mathbb{N}$, we easily get that for any $a \in T^*(\frac{l}{k} + \frac{r}{s}) = T^*\left(\frac{ls+rk}{ks}\right)$, there exists $d \in [T^*(\frac{1}{ks})]^{ls+kr}$ with $a = d$. On the other hand, denoting $ks = t$, we have $J_{t/n}^{n(ls+kr)}(x) = (I - \frac{t}{n}A)^{-n(ls+kr)}(x) = (I - \frac{t}{n}A)^{-nls}[(I - \frac{t}{n}A)^{-nkr}(x)]$, so for the above d, there exists a subsequence $(n_j)_j$ with $d = \lim_{j \to +\infty} x^*(J_{t/n_j}^{n_j(ls+kr)}(x))$ and there exist $b \in [T^*(t)]^{ls}$ and $c \in [T^*(t)]^{kr}$ such that $d(x) = b[c(x)]$. The corollary is proved.

The next result shows that for some particular nonlinear operators, the whole sequence $(J_{t/n}^n(x))_n$ is convergent in the l^p space.

Theorem 3.4 *Let $A : l^p \to l^p$, $0 < p < 1$, be a nonlinear operator of the form $A(x) = (f_k(x_k))_{k\in\mathbb{N}}$, for all $x = (x_k)_{k\in l^p}$, where $f_k : \mathbb{R} \to \mathbb{R}$ are non-increasing continuous functions, $f_k(0) = 0$ and there exists $M > 0$ such that $|f_k(\alpha) - f_k(\beta)| \leq M|\alpha - \beta|$, for all $k \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$. Then, for any $t \geq 0$ and $x \in l^p$, the sequence $J_{t/n}^n(x) = (I - \frac{t}{n}A)^{-n}(x), n \in \mathbb{N}$ is strongly convergent to a limit in l^p.*

Proof. First by definition it easily follows that A is a Lipschitz operator with respect to the $\|\cdot\|_p$-norm in l^p. Then, we can write $J_{t/n}^n(x) = (g_{n,k}(t)(x_k))_k$, where $g_{n,k}(t) : \mathbb{R} \to \mathbb{R}$ are given by $g_{n,k}(t)(u) = (I - \frac{t}{n}f_k)^{-n}(u)$. By the hypothesis, it follows that each sequence $(g_{n,k}(t)(u))_k$ is convergent in the Banach space $\mathbb{R}$, denote $g_k(t)(u) = \lim_{n\to+\infty} g_{n,k}(t)(u)$.

We know that l^p has the basis $\{e_1, e_2, ..., e_n, ...,\}$, where $e_i = (\delta_{in})_{n\in\mathbb{N}}$. Due to the particular form of $J_{t/n}^n(x) = (g_{n,k}(t)(x_k))_k$, we have $g_{n,k}(t)(0) = 0$ for all $k \in \mathbb{N}$ and it is obvious that if $x \in \mathrm{span}\{e_1, ..., e_i, ...\} =: Y$, then $J_{t/n}^n(x)$ becomes a sequence with only a finite number of non-zero elements. This means that for such x, $J_{t/n}^n(x)$ is convergent in l^p. Also, obviously Y is dense in l^p.

Let $x \in l^p$ and $\varepsilon > 0$ be arbitrary. There exists $y \in Y$ such that $\|x - y\|_p < \varepsilon$. We get

$$\|J_{t/n}^n(x) - J_{t/m}^m(x)\|_p \leq \|J_{t/n}^n(x) - J_{t/n}^n(y)\|_p + \|J_{t/n}^n(y) - J_{t/m}^m(y)\|_p +$$

$$\|J_{t/m}^m(y) - J_{t/m}^m(x)\|_p \leq$$

$$2M\|x - y\|_p + \|J_{t/n}^n(y) - J_{t/m}^m(y)\|_p < 2M\varepsilon + \|J_{t/n}^n(y) - J_{t/m}^m(y)\|_p,$$

where $\|J_{t/n}^n\|_{Lip} \leq M = 1$ (see the proof of Theorem 3.1, where we take $\omega = 0$).

Since $(J_{t/n}^n(y))_n$ is convergent in l^p, it is a Cauchy sequence and therefore given $\delta > 0$, there is a n_0 such that $\|J_{t/n}^n(y) - J_{t/m}^m(y)\|_p < \delta$, for all $m, n > n_0$. Together with the above inequality this implies that $(J_{t/n}^n(x))_n$ is a Cauchy sequence in the complete metric space l^p, i.e. it is convergent in l^p. The theorem is proved.

As an application of Theorem 3.1, we obtain the following

Corollary 3.5 *Let $A : l^p \to l^p$, $0 < p < 1$, be nonlinear, Lipschitz, such that $A(0) = 0$, there exists $\omega \in \mathbb{R}$ with $A - \omega I$ dissipative and A is weakly continuous (that is for any $x^* \in (l^p)^*$, if $\lim_{n\to\infty} x^*(a_n) = x^*(a)$, then $\lim_{n\to\infty} x^*[A(a_n)] = x^*[A(a)]$).*

For $x \in l^p$ *and* $t \in \mathcal{R}_+$*, let us consider as in the statement and proof of Theorem 3.1, the sequence in* l^p*,* $u_k(x)(t) = J^{n_k}_{t/n_k}(x) = (g_{k,r}(x)(t))_{r\in\mathbb{N}}$*, convergent (as* $k \to \infty$*) in the weak topology of* l^p*, to* $u(x)(t) = (g_r(x)(t))_{r\in\mathbb{N}}$*.*

Let us suppose that for all $r \in \mathbb{N}$*,* $x \in l^p$*, the real functions* $g_r(x)(t)$ *are left differentiable with respect to* $t \in \mathcal{R}_+$*, that is there exists (finite)*

$$[g_r(x)]'_-(t) = \lim_{h\to 0, h\in\mathcal{R}_+} \frac{g_r(x)(t) - g_r(x)(t-h)}{h}, t \in \mathcal{R}_+,$$

and that for all $k, r \in \mathbb{N}$*,* $x \in l^p$*, the real functions* $g_{k,r}(x)(t)$ *are differentiable (in the classical sense) with respect to* $t \in [0, \sigma)$*, satisfying in addition the relation*

$$\lim_{t_k \nearrow t}[g_{k,r}(x)]'(t_k) = [g_r(x)]'_-(t),$$

for all $t \in [0, \sigma) \cap \mathcal{R}_+$ *and all* $t_k \in [0, \sigma)$ *with* $t_k \nearrow t$*. Here, for* $s < 0$ *we take by convention* $g_r(x)(s) = g_r(x)(0)$*,* $g_{k,r}(x)(s) = g_{k,r}(x)(0)$*, which gives sense to* $[g_r(x)]'_-(0) = 0$ *and* $[g_{k,r}(x)]'(s) = 0$*,* $s \le 0$*.*

Then, $v(t) = u(x)(t)$ *is a solution of the Cauchy problem*

$$v'_-(t) = A[v(t)], t \in [0, \sigma) \cap \mathcal{R}_+,$$

$$v(0) = x,$$

where $v'_-(t)$ *is defined componentwise as above and v(s)=v(0), for* $s < 0$*.*

Proof. Let $x^* \in (l^p)^*$ be arbitrary. According to [7, p. 27], it can be extended to a $x^* \in (l^1)^*$, preserving its norm. By the considerations from the beginning of this section, it follows that $u_k(x)(t)$ satisfies the difference equation

$$\frac{u_k(x)(t) - u_k(x)(t - t/n_k)}{\frac{t}{n_k}} = A[u_k(x)(t)], t \ge 0.$$

This obviously implies

$$x^* \left[\frac{u_k(x)(t) - u_k(x)(t - t/n_k)}{\frac{t}{n_k}} - A(u_k(x)(t))\right] = 0, t \ge 0.$$

But by Theorem 3.1 we have $\lim_{k\to\infty} x^*[u_k(x)(t)] = x^*[u(x)(t)]$, for all $t \in \mathcal{R}_+$, $x \in l^p$. Taking into account the weak continuity of A, first we obtain $\lim_{k\to\infty} x^*(A[u_k(x)(t)]) = x^*(A[u(x)(t)])$.

Next we will show that

$$\lim_{k\to\infty} x^* \left[\frac{u_k(x)(t) - u_k(x)(t - t/n_k)}{\frac{t}{n_k}}\right] = x^*([u(x)]'_-(t)) \qquad (3),$$

for all $t \in \mathcal{R}_+$, $x \in l^p$.

For this purpose, we reason as in the proof of Theorem 3.1, that is first we prove (3) for any $x_r^* \in (l^p)^*$, $r \in \mathbb{N}$ of the form $x_r^*(x) = x_r$, for all $x = (x_1, ..., x_r, ...) \in l^p$. This one reduces to

$$\lim_{k\to\infty} \frac{g_{k,r}(x)(t) - g_{k,r}(x)(t - t/n_k)}{\frac{t}{n_k}} = [g_r(x)]'_-(t),$$

for all $t \in \mathcal{R}_+$, $x \in l^p$, $r \in \mathbb{N}$.

By the mean value theorem, there exists $\xi_{t,k} \in (t-t/n_k, t)$ such that $\frac{g_{k,r}(x)(t)-g_{k,r}(x)(t-t/n_k)}{\frac{t}{n_k}} = [g_{k,r}(x)]'(\xi_{t,k})$, which by the hypothesis immediately implies that at the limit with $k \to \infty$ we obtain (3).

Also, it is clear that (3) holds for any $y^* \in span\{x_1^*, ..., x_r^*, ...\} = Y^*$. Reasoning now exactly as at the end of proof in Theorem 3.1 (since Y^* is dense in $(l^1)^*$ in the weak topology on $(l^1)^*$), we easily get that (3) is satisfied for all $x^* \in (l^1)^*$.

In conclusion, we get

$$x^*[(u(x))'_-(t) - A(u(x)(t))] = 0,$$

for all $t \in \mathcal{R}_+$ and all $x^* \in (l^1)^*$. Passing here to supremum with $|||x^*|||_{(l^1)^*} \leq 1$ and taking into account a classical result in functional analysis (since l^1 is a normed space), we obtain

$$||[u(x)]'_-(t) - A(u(x)(t))||_{l^1} = 0, t \in \mathcal{R}_+, x \in l^p,$$

which implies $[u(x)]'_-(t) = A(u(x)(t))$, for all $t \in \mathcal{R}_+$, $x \in l^p$. Also, obviously $u(x)(0) = x$, which proves the corollary.

A consequence of Theorem 3.4 is the following

Corollary 3.6 *For* $x = (x_1, ..., x_k, ...) \in l^p$ *and* $0 < p < 1$*, let us consider as in the statement and proof of Theorem 3.4, the operator* A*, the sequence* $u_n(t) := J^n_{t/n}(x) = (g_{n,k}(t)(x_k))_{k\in\mathbb{N}} \in l^p$*, where* $g_{n,k}(t) : \mathbb{R} \to \mathbb{R}$ *are given by* $g_{n,k}(t)(u) = (I - \frac{t}{n} f_k)^{-n}(u)$ *and* $u(t) = (g_k(t)(x_k))_{k\in\mathbb{N}} \in l^p$ *with* $\lim_{n\to\infty} ||u_n(t) - u(t)||_p = 0$*, for all* $t \geq 0$.

If, in addition, $u_n(t)$*,* $u(t)$ *are differentiable with respect to* $t \in [0, \sigma]$ *such that*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} ||g'_{n,k}(x_k) - g'_k(x_k)||^p = 0,$$

where $||g'_{n,k}(x_k) - g'_k(x_k)|| := \sup_{t\in[0,\sigma]} |g'_{n,k}(t)(x_k) - g'_k(t)(x_k)|$*, then* $v(t) = u(t)$ *represents the unique solution of the nonlinear Cauchy problem*

$$\frac{d}{dt} v(t) = A[v(t)], t \in [0, \sigma],$$

$$v(0) = x.$$

Proof. By the considerations from the beginning of this section, it follows that $u_n(t) = J^n_{t/n}(x)$ satisfies the difference equation

$$\frac{u_n(t) - u_n(t - t/n)}{\frac{t}{n}} = A[u_n(t)], t \geq 0.$$

Passing here to limit (with $n \to \infty$) in the $\|\cdot\|_p$-norm in l^p, since A is Lipschitz in l^p (see the proof of Theorem 3.4), it follows that $\lim_{n\to\infty} A(u_n(t)) = A(u(t))$, for all $t \in [0, \sigma]$.

For the left-hand side, we have

$$\left\| u'(t) - \frac{u_n(t) - u_n(t - t/n)}{\frac{t}{n}} \right\|_p \leq$$

$$\left\| u'(t) - \frac{u(t) - u(t - t/n)}{\frac{t}{n}} \right\|_p + \left\| \frac{u(t) - u(t - t/n)}{\frac{t}{n}} - \frac{u_n(t) - u_n(t - t/n)}{\frac{t}{n}} \right\|_p,$$

where $\lim_{n\to\infty} \|u'(t) - \frac{u(t)-u(t-t/n)}{\frac{t}{n}}\|_p = 0$ by the definition of derivative, while by the mean value theorem we obtain

$$\left\| \frac{u(t) - u(t - t/n)}{\frac{t}{n}} - \frac{u_n(t) - u_n(t - t/n)}{\frac{t}{n}} \right\|_p =$$

$$\sum_{k=1}^{\infty} \left| \frac{[g_k(t)(x_k) - g_{n,k}(t)(x_k)] - [g_k(t - t/n)(x_k) - g_{n,k}(t - t/n)(x_k)]}{t/n} \right|^p =$$

$$\sum_{k=1}^{\infty} |g'_k(\xi_{t,k,n})(x_k) - g'_{n,k}(\xi_{t,k,n})(x_k)|^p \leq \sum_{k=1}^{\infty} \|g'_k(x_k) - g'_{n,k}(x_k)\|^p,$$

which by the hypothesis implies

$$\lim_{n\to\infty} \left\| \frac{u(t) - u(t - t/n)}{\frac{t}{n}} - \frac{u_n(t) - u_n(t - t/n)}{\frac{t}{n}} \right\|_p = 0$$

and proves the corollary.

Example. A simple example satisfying the conditions (and the conclusions) in Corollary 3.6 is given as follows. Define the non-linear strictly decreasing continuous function $f : \mathbb{R} \to \mathbb{R}$, by $f(x) = -x$ if $x < 0$, $f(x) = -2x$ if $x \geq 0$ and $A : l^p \to l^p$ by $A(x) = (f(x_1), ..., f(x_k), ...)$, for all $x = (x_1, ..., x_k, ...) \in l^p$.

It is easy to check that $|f(\alpha) - f(\beta)| \leq 2|\alpha - \beta|$, for all $\alpha, \beta \in \mathbb{R}$, which implies that A is Lipschitz nonlinear operator. Also, it is easy to check that for all $\lambda > 0$, the operator $I - \lambda A$ is invertible, with $x = (x_1, ..., x_k, ...)$, $(I - \lambda A)^{-1}(x) = (g(x_1), ..., g(x_k), ...)$, $g(x_k) = \frac{x_k}{1+\lambda}$ if $x_k < 0$, $g(x_k) = \frac{x_k}{1+2\lambda}$ if $x_k \geq 0$, and

$$\|(I - \lambda A)^{-1}\|_{Lip} \leq \left(\frac{1}{1 + \lambda} \right)^p \leq 1,$$

which shows that A is dissipative.

Simple calculation shows that $u_n(t) = (g_n(t)(x_1), ..., g_n(t)(x_k), ...)$, where $g_n(t)(x_k) = \frac{x_k}{(1+(t/n))^n}$ if $x_k < 0$, $g_n(x_k) = \frac{x_k}{(1+2(t/n))^n}$ if $x_k \geq 0$, $u(t) = (g(t)(x_1), ..., g(t)(x_k), ...)$, where $g(t)(x_k) = x_k e^{-t}$ if $x_k < 0$, $g(t)(x_k) = x_k e^{-2t}$ if $x_k \geq 0$.

It is easy to prove that all the conditions in Corollary 3.6 are satisfied with $\sigma = 1$, which shows that $u(t)$ defined as above is the unique solution of the nonlinear Cauchy problem

$$\frac{d}{dt}v(t) = A[v(t)], t \in [0, 1],$$

$$v(0) = x.$$

4. Nonlinear Semigroups on H^p, $0 < p < 1$

In this section we consider the H^p space, $0 < p < 1$, where we denote its p-norm by $\|\cdot\|_p$. The main result is the following.

Theorem 4.1 *Let* $A : (H^p, \|\cdot\|_p) \to (H^p, \|\cdot\|_p)$, $0 < p < 1$, *be nonlinear and Lipschitz, such that* $A(0) = 0$ *and there exists* $\omega \in \mathbb{R}$ *with* $A - \omega I$ *dissipative. Then the sequence in* H^p *defined by* $J^n_{t/n}(x) = (I - \frac{t}{n}A)^{-n}(x), n \in \mathbb{N}, t \geq 0, x \in H^p$, *contains a subsequence* $J^{n_k}_{t/n_k}(x), k \in \mathbb{N}$ *(the same subsequence for all* $x \in H^p$ *and all* $t \in \mathcal{R}_+$ *= the set of all rational numbers* ≥ 0*), uniformly convergent on compacts in* $\mathbb{D}$.

Proof. By $A(0) = 0$ we get $(I - \frac{t}{n}A)^{-n}(0) = 0$, for all $n \in \mathbb{N}$. The dissipative property implies $\|I - t(A - \omega I)^{-1}\|_{Lip} \leq 1$, for all $t \geq 0$, which is equivalent to $\|(I - \frac{t}{1+t\omega}A)^{-1}\|_{Lip} \leq |1 + t\omega|^p$, for all $t \geq 0$ with $1 + t\omega \neq 0$. For $\lambda = \frac{t}{1+t\omega}$ we get $\|(I - \lambda A)^{-1}\|_{Lip} \leq (1 - \lambda\omega)^{-p}$, in particular for all $\lambda > 0$ with $\lambda\omega < 1$. (Note that for n sufficiently great, depending on t and ω, we have $\frac{t}{n}\omega < 1$.) Therefore,

$$\|(I - \frac{t}{n}A)^{-1}\|_{Lip} \leq (1 - \frac{t}{n}\omega)^{-p}$$

and by mathematical induction

$$\|(I - \frac{t}{n}A)^{-n}\|_{Lip} \leq (1 - \frac{t}{n}\omega)^{-np}.$$

But it is known that the sequence $(1 + \frac{s}{n})^n$ converges (for $n \to +\infty$) to e^s, and for any $s \in \mathbb{R}$ is monotonically increasing, for all $n \geq [|s|] + 1$ (see e.g. [10, p. 263]), which implies that $(1 - \frac{t}{n}\omega)^{-np}$ converges to $e^{t\omega p}$, monotonically decreasing, for all $n \geq [|tw|] + 1$. Therefore, the greatest value of $(1 - \frac{t}{n}\omega)^{-np}$ is for $n = [|t\omega|] + 1$, which means that there exists $M = M(t, p, \omega) > 0$ (depending only on t, p and ω) such that

$$\|(I - \frac{t}{n}A)^{-n}\|_{Lip} \leq M,$$

for all $n \in \mathbb{N}$.

We obtain

$$\|J^n_{t/n}(x) - J^n_{t/n}(y)\|_p \leq (1 - \frac{t}{n}\omega)^{-np}\|x - y\|_p \leq M\|x - y\|_p, \tag{4}$$

for all $x, y \in H^p$. Taking $y = 0$ we have $J^n_{t/n}(0) = 0$ and we obtain

$$\|J^n_{t/n}(x)\|_p \leq M(t, p, \omega)\|x\|_p < +\infty, \tag{5}$$

for all $n \in \mathbb{N}$.

Since H^p, $0 < p < 1$, has a Schauder basis (see e.g. [9]), it follows that it is separable, denote by Y a countable dense subset of H^p. Also, denote by $\mathcal{R}_+$, the set of all nonnegative rational numbers and define $E = \mathcal{R}_+ \times Y$. Obviously E is a countable set, let us denote it by $E = \{e_1, ..., e_j, ...,\}$ with the distinct elements two by twos, $e_j = (r_j, y_j)$ and for each $e = (t, y) \in E$, denote $S_n(e) = J^n_{t/n}(y) \in H^p$. Obviously, $S_n(e)$ are analytic functions in $\mathbb{D}$, for all $n \in \mathbb{N}$ and all $e \in E$.

According to e.g. [7, p. 35, (3.4)], the point evaluations $\varphi_z(x) = x(z)$, $z \in \mathbb{D}$, are linear and bounded functionals on H^p, $0 < p < 1$ and the following inequality holds

$$|x(re^{i\theta})| \leq 2^{1/p}\|x\|_p(1 - r)^{-1/p}, \text{ for all } x \in H^p \text{ and } z = re^{i\theta}.$$

Together with (5), this implies that for all $z = re^{i\theta}$, $|z| \leq r_0 < 1$, i.e. $0 < r \leq r_0$, $e = (t, y) \in E$, we obtain

$$|S_n(e)(z)| = |\varphi_z[S_n(e)]| \leq 2^{1/p}\|S_n(e)\|_p \frac{1}{(1 - r)^{1/p}} \leq$$

$$2^{1/p}\frac{1}{(1 - r_0)^{1/p}}M(t, p, \omega)\|y\|_p.$$

In other words, for any fixed $e = (t, y) \in E$, the sequence of analytic functions $(S_n(e))_{n\in\mathbb{N}}$, is uniformly bounded on each compact subset of $\mathbb{D}$, which by the classical Montel's theorem implies that it contains a subsequence uniformly convergent on compact subsets of $\mathbb{D}$.

For $e_1 \in E$, there exists a subsequence of $(S_n(e_1))_{n\in\mathbb{N}}$, denoted by $(S_{1,n}(e_1))_{n\in\mathbb{N}}$, which is uniformly convergent on compact subsets of $\mathbb{D}$.

For $e_2 \in E$, reasoning analogously, the sequence $(S_{1,n}(e_2))_{n\in\mathbb{N}}$ contains in turn, a subsequence denoted by $(S_{2,n}(e_2))_{n\in\mathbb{N}}$, which is uniformly convergent on compact subsets of $\mathbb{D}$.

In general, for $e_m \in E$, there exists a subsequence of the previous one, $(S_{m,n}(e_m))_{n\in\mathbb{N}}$, uniformly convergent on compact subsets of $\mathbb{D}$.

Continuing this process gives rise to the infinite array of analytic functions in $\mathbb{D}$,

$$S_{1,1}, \quad S_{1,2}, \quad S_{1,3}, ...,$$

$$S_{2,1}, \quad S_{2,2}, \quad S_{2,3}, ...,$$

$$S_{1,1}, \quad S_{1,2}, \quad S_{3,3}, ...,$$

....

$$S_{1,m}, \quad S_{2,m}, \quad S_{3,m}, \ldots,$$

and so on, such that the first row means that $(S_{1,n}(e_1))_{n\in\mathbb{N}}$ uniformly converges on compact subsets of $\mathbb{D}$, the second row means that $(S_{2,n}(e_j))_{n\in\mathbb{N}}$ is uniformly convergent on compact subsets of $\mathbb{D}$ for $j = 1, 2$, the third row means that $(S_{3,n}(e_j))_{n\in\mathbb{N}}$ is uniformly convergent on compact subsets of $\mathbb{D}$, for $j = 1, 2, 3$, and so on.

As a consequence, we can consider the diagonal sequence $(S_{n,n})_{n\in\mathbb{N}}$, which has the property that $(S_{n,n}(e_j))_{n\in\mathbb{N}}$ is uniformly convergent on compact subsets of $\mathbb{D}$, for all $j \in \mathbb{N}$, that is $(S_{n,n}(e))_{n\in\mathbb{N}}$ is uniformly convergent on compact subsets of $\mathbb{D}$, for all $e \in E$.

Now, let us denote $A = \mathcal{R}_+ \times H^p$. We will show that in fact $(S_{n,n}(e))_{n\in\mathbb{N}}$ is uniformly convergent on compact subsets of $\mathbb{D}$, for all $e \in A$. Indeed, let $e = (r, x) \in A$ and since Y is dense in H^p, let $y_k \in Y, k \in \mathbb{N}$, satisfying $\|x - y_k\|_p \to 0$, when $k \to \infty$. Denoting $a_k = (r, y_k) \in E$, by (4) we have

$$\|S_{n,n}(e) - S_{n,n}(a_k)\|_p \le M(r, p, \omega)\|x - y_k\|_p,$$

for all $k \in \mathbb{N}$. It is enough to show that $(S_{n,n}(e)(z))_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the uniform norm (denoted by $\|\cdot\|$) in each compact disk, $\overline{D_r}$ in $\mathbb{D}$. Indeed, this is immediate by the inequalities

$$\|S_{n,n}(e) - S_{m,m}(e)\| \le \|S_{n,n}(e) - S_{n,n}(a_k)\| + \|S_{n,n}(a_k) - S_{m,m}(a_k)\| + \|S_{m,m}(a_k) - S_{m,m}(e)\| \le$$

$$2M(r, p, \omega)\|x - y_k\|_p + \|S_{n,n}(a_k) - S_{m,m}(a_k)\|$$

and by the above properties.

The theorem is proved.

Remarks. 1) By relation (3.4) in [7, p. 35], it is evident that if $\lim_{n\to\infty} \|x_n - x\|_p = 0$, then $(x_n)_n$ is uniformly convergent on compact subsets of $\mathbb{D}$. In general, the converse is not valid. As a consequence, if we denote by x the uniform limit of $(x_n)_n$ on compact subsets of $\mathbb{D}$, then x is an analytic function in $\mathbb{D}$, but in general it does not belong to H^p, $0 < p < 1$.

2) We can repeat the reasonings in the proof of Theorem 4.1 for the sequence $(J^n_{t/n}(x), n \in \mathbb{N}, n \ne n_k)$, where n_k is the subsequence in Theorem 4.1, so that by mathematically induction we easily obtain that the sequence $(J^n_{t/n}(x))_{n\in\mathbb{N}}$ has at most a countable set of limit points in the locally convex topology of uniform convergence on compact subsets in $\mathbb{D}$. If we denote that set by $T^*(t)(x)$, where $t \in \mathcal{R}_+, x \in H^p$, then for any fixed $t \in \mathcal{R}_+$, an element $a \in T^*(t)$ is in fact a mapping $a : H^p \to Hol(\mathbb{D})$, where $Hol(\mathbb{D})$ denotes the spaces of all holomorphic (analytic) functions in $\mathbb{D}$.

Corollary 4.2. *Let* $A : (H^p, \|\cdot\|_p) \to (H^p, \|\cdot\|_p)$, $0 < p < 1$*, be nonlinear and Lipschitz, such that* $A(0) = 0$*, there exists* $\omega \in \mathbb{R}$ *with* $A - \omega I$ *dissipative and* $A : (H^p, \mathcal{T}) \to (H^p, \mathcal{T})$ *is continuous, where* $\mathcal{T}$ *represents the locally convex topology of uniform convergence on compact subsets in* $\mathbb{D}$*.*

For $x \in H^p$ *and* $t \in \mathcal{R}_+$*, let us consider as in the statement of Theorem 4.1, the sequence in* H^p*,* $u_k(x)(t) = J^{n_k}_{t/n_k}(x), k \in \mathbb{N}$*, uniformly convergent on compacts in* $\mathbb{D}$ *(as* $k \to \infty$*) to* $u(x)(t)$*.*

Let us suppose that for $x \in H^p$ *and all* $z \in \mathbb{D}$*, the complex valued function* $[u(x)(t)](z)$ *is left derivable with respect to the real variable* $t \in \mathcal{R}_+$*, that is there exists (finite)*

$$\frac{\partial[u(x)(t)(z)]_-}{\partial t} = \lim_{h \to 0, h \in \mathcal{R}_+} \frac{[u(x)(t)](z) - [u(x)(t-h)](z)}{h}, t \in \mathcal{R}_+,$$

and also suppose that

$$\lim_{k\to\infty} \frac{[u_k(x)(t)](z) - [u_k(x)(t - t/n_k)](z)}{\frac{t}{n_k}} = \frac{\partial[u(x)(t)(z)]_-}{\partial t},$$

for all $t \in \mathcal{R}_+ \cap [0, \sigma)$*,* $x \in H^p$*,* $z \in \mathbb{D}$*.*

Here, for $s < 0$ *we take by convention* $[u(x)(s)](z) = [u(x)(0)](z)$*,* $[u_k(x)(s)](z) = [u_k(x)(0)](z)$*, for all* $z \in \mathbb{D}$*, which gives sense to* $\frac{\partial[u(x)(0)(z)]_-}{\partial t}$*, for all* $z \in \mathbb{D}$*.*

Then, $v(t) = u(x)(t)$ *is a solution (analytic in* $\mathbb{D}$ *but not necessarily in* H^p*) of the Cauchy problem*

$$\frac{\partial[v(t)(z)]_-}{\partial t} = A[v(t)](z), t \in [0, \sigma) \cap \mathcal{R}_+, z \in \mathbb{D}$$

$$v(0)(z) = x(z), z \in \mathbb{D},$$

where $\frac{\partial[v(t)(z)]_-}{\partial t}$ *is defined as above and v(s)=v(0), for* $s < 0$*.*

Proof. By the considerations from the beginning of the Section 3, it follows that $u_k(x)(t)$ satisfies the difference equation

$$\frac{u_k(x)(t) - u_k(x)(t - t/n_k)}{\frac{t}{n_k}} = A[u_k(x)(t)], t \geq 0.$$

But by Theorem 4.1 and by the continuity assumption on A, we have $\lim_{k\to\infty} A[u_k(x)(t)](z) = A[u(x)(t)](z)$, for all $z \in \mathbb{D}$.

Therefore, passing to limit with $k \to \infty$ in the above difference equation, by the hypothesis we immediately obtain

$$\frac{\partial[u(x)(t)(z)]_-}{\partial t} = A[u(x)(t)](z), t \in [0, \sigma) \cap \mathcal{R}_+, z \in \mathbb{D},$$

$z \in \mathbb{D}$.

Also, obviously $u(x)(0) = x$, which proves the corollary.

5. Nonlinear Semigroups on $L^p[0, 1]$, $0 < p < 1$

In this section we consider the $L^p[0, 1]$ space, $0 < p < 1$, where we denote its p-norm by $\|\cdot\|_p$. The main result is the following.

Theorem 5.1. *Let* $A : (L^p[0,1], \|\cdot\|_p) \to (L^p[0,1], \|\cdot\|_p)$*,* $0 < p < 1$*, be nonlinear and Lipschitz, such that* $A(0) = 0$ *and there exists* $\omega \in \mathbb{R}$ *with* $A - \omega I$ *dissipative. Then, for any fixed* $t \geq 0, x \in L^p[0,1]$*, the sequence in* $L^p[0,1]$ *defined by* $J^n_{t/n}(x) = (I - \frac{t}{n}A)^{-n}(x), n \in \mathbb{N}$*, contains a subsequence* $a_k(t,x) := J^{n_k}_{t/n_k}(x), k \in \mathbb{N}$*, such that*

$$\lim_{k\to\infty} \frac{1}{k^{1/p}} \sum_{i=1}^{k} a_i(t,x)(s) = 0,$$

a.e. $s \in [0,1]$.

Proof. Reasoning exactly as in the proof of Theorem 4.1, relations (4)-(5), we get that

$$\|J^n_{t/n}(x)\|_p \leq M(t,p,\omega)\|x\|_p,$$

where $\|\cdot\|_p$ is the p-norm in $L^p[0,1]$. In other words, for any fixed $t \geq 0$ and $x \in L^p[0,1]$, the sequence $(J^n_{t/n}(x))_n$ is bounded in the p-norm of $L^p[0,1]$, $0 < p < 1$.

According to [2], this implies that for any $t \geq 0$ and $x \in L^p[0,1]$, there exists a subsequence $a_i(t,x) := J^{n_i}_{t/n_i}(x), i \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \frac{1}{n^{1/p}} \sum_{i=1}^{n} a_i(t,x)(s) = 0,$$

a.e. $s \in [0,1]$.

Remark. Unfortunately, a sequence $(a_i(t,x))_{i\in\mathbb{N}}$, satisfying the relation proved by Theorem 5.1, can satisfy (in the sense that does not produce a contradiction) $\lim_{i\to\infty} a_i(t,x,\omega)(s) = \infty$, a.e. $s \in [0,1]$, which is the worst possible divergence result. If to this fact we add that the dual space of $L^p[0,1]$, $0 < p < 1$, is $\{0\}$, then it seems that in this space, in general we cannot derive any result on the convergence of some subsequences of $(J^n_{t/n}(x))_n$.

Acknowledgement. The author thanks the referee for the very useful remarks.

Received: September 2009. Revised: December 2009.

Referencias

[1] A. Bayoumi, *Foundations of Complex Analysis in Non-Locally Convex Spaces*, North-Holland Mathematics Studies, Elsevier, vol. 193, Amsterdam, 2003.

[2] S.D. Chatterji, *A general strong law*, Inventiones Math., 9(1970), 235-245.

[3] M.G. Crandall and T.M. Liggett, *Generation of semigroups of non-linear transformations on general Banach spaces*, Amer. J. Math., 93(1971), 265-298.

[4] N. Dunford and J.T. Schwartz, *Linear Operators, Part I*, Interscience, New York, 1964.

[5] S.G. Gal and J.A. Goldstein, *Semigroups of linear operators on* p*-Fréchet spaces,* $0 < p < 1$, Acta Math. Hungar., 114(1-2)(2007), 13-36.

[6] J.A. Goldstein, S. Oharu and A. Vogt, *Affine semigroups on Banach spaces*, Hiroshima Math. J., 18(1988), No. 2, 433-450.

[7] N.J. Kalton, N.T. Peck and J.W. Roberts, *An* F*-Space Sampler*, London Mathematical Society Lecture Notes Series, vol. 89, Cambridge University Press, 1984.

[8] I. Muntean, *Course and Problems in Functional Analysis, Vol. II* (Romanian), Faculty of Mathematics, "Babes-BolyaiÜniversity Press, Cluj, 1988.

[9] P. Oswald, *On Schauder bases in Hardy spaces*, Proc. Roy. Soc. Edinburg, Sect. A, 93(1982/83), no. 3-4, 259-263.

[10] T. Popoviciu, *Course of Mathematical Analysis, Part 2, Functions. Limits.* (Romanian), "Babes-BolyaiÜniversity, Faculty of Mathematics-Mechanics, Cluj, 1972.

[11] W. Rudin, *Principles of Mathematical Analysis*, Third Edition, McGraw-Hill Inc., 1976.

CUBO A Mathematical Journal
Vol.13, N^o 02, (59–72). June 2011

On λ strong homogeneity existence for cofinality logic

Saharon Shelah [1]
Einstein Institute of Mathematics,
The Hebrew University of Jerusalem,
Jerusalem, 91904, Israel.
email: `shelah@math.huji.ac.il`
url: `http://shelah.logic.at`

ABSTRACT

Let $\mathsf{C} \subset_{\neq}$ Reg be a non-empty class (of regular cardinals). Then the logic $\mathbb{L}(\mathrm{Q}^{\mathrm{cf}}_{\mathsf{C}})$ has additional nice properties: it has the homogeneous model existence property.

RESUMEN

Sea $\mathsf{C} \subset_{\neq}$ Reg una clase no vacía (de cardinales regulares). Entonces la lógica $\mathbb{L}(\mathrm{Q}^{\mathrm{cf}}_{\mathsf{C}})$ tiene propiedades adicionales: Esta tiene la propiedad de modelo existencia homogénea.

Keywords and phrases: Model theory, soft model theory, cofinality quantifier.
Mathematics Subject Classification: 03C95, 03C80.

[1]The author thanks Alice Leonhardt for the beautiful typing. The author would like to thank the Israel Science Foundation for partial support of this research (Grants No. 451/99 and 710/07). Publication 750.

1. Introduction

We deal with logics gotten by strengthening of first order logic by generalized quantifiers, in particular compact ones. We continue [Sh:199] (and [Sh:43])

A natural quantifier is the cofinality quantifier, $Q^{cf}_{\leq\lambda}$ (or Q^{cf}_{C}), introduced in [Sh:43] as the first example of compact logic (stronger than first order logic, of course). Recall that the "uncountably many x's" quantifier $Q^{card}_{\geq\aleph_1}$, is $\aleph_0$-compact but not compact. Also note that $\mathbb{L}(Q^{cf}_{\leq\lambda})$ is a very nice logic, e.g. with a nice axiomatization (in particular finitely many schemes) like the one of $\mathbb{L}(Q^{card}_{\geq\aleph_1})$ of Keisler. By [Sh:199], e.g. for $\lambda = 2^{\aleph_0}$, its Beth closure is compact, giving the first compact logic with the Beth property (i.e. implicit definition implies explicit definition).

Earlier there were indications that having the Beth property is rare for such logic, see e.g. in Makowsky [Mak85]. A weaker version of the Beth property is the weak Beth property dealing with implicit definition which always works; H. Friedman claim that historically this was the question. Mekler-Shelah [MkSh:166] prove that at least consistently, $\mathbb{L}(Q^{card}_{\geq\aleph_1})$ satisfies the weak Beth property. Väänänen in the mid nineties motivated by the result of Mekler-Shelah [MkSh:166] asked whether we can find a parallel proof for $\mathbb{L}(Q^{cf}_{\leq\lambda})$ in ZFC.

A natural property for a logic $\mathcal{L}$ is

Definition 1. A logic $\mathcal{L}$ has the (strong) homogeneous model existence property <u>when</u> every theory $T \subseteq \mathcal{L}(\tau)$, (so has a model) has a strongly $(\mathcal{L}, \aleph_0)$-homogeneous model M, so $\tau_M = \tau$ and M is a model of T and M satisfies: if $\bar{a}, \bar{b} \in {}^{\omega>}M$ realize the same $\mathcal{L}(\tau)$-type in M then there is an automorphism of M mapping $\bar{a}$ to $\bar{b}$.

This property was introduced in [Sh:199] being natural and also as it helps to investigate the weak Beth property.

In §1 we prove that $\mathbb{L}(Q^{cf}_{C})$ has the strongly $\aleph_0$-saturated model existence property. The situation concerning the weak Beth property is not clear.

Question 2. 1) Does the logic $\mathbb{L}(Q^{cf}_{C})$ have the weak Beth property?

2) Does the logic $\mathbb{L}(Q^{cf}_{\leq\lambda_1}, Q^{cf}_{\leq\lambda_2})$ has the homogeneous model existence property?

The first version of this work was done in 1996.

Notation 3. 1) τ denotes a vocabulary, $\mathcal{L}$ a logic, $\mathcal{L}(\tau)$ the language for the logic $\mathcal{L}$ and the vocabulary τ.

2) Let $\mathbb{L}$ be first order logic, $\mathbb{L}(Q_*)$ be first order logic when we add the quantifier Q_*.

3) For a model M and ultrafilter D on a cardinal λ, let M^λ/D be the ultrapower and $\mathbf{j}_{M,D} = \mathbf{j}^\lambda_{M,D}$ be the canonical embedding of M into M^λ/D; of course, we can replace λ by any set.

4) Let LST (theorem/argument) stand for Löwenheim-Skolem-Tarski (on existence of elementary submodels).

Concerning 1, more generally

Definition 4. 1) M is strongly $(\mathcal{L},\theta)$-saturated (in $\mathcal{L}=\mathbb{L}$ we may write just θ) <u>when</u>

(a) it is θ-saturated (i.e. every set of $\mathcal{L}(\tau_M)$-formulas with $<\theta$ parameters from M and $<\theta$ free variables which is finitely satisfiable in M is realized in M

(b) if $\zeta<\theta$ and $\bar{a},\bar{b}\in {}^{\zeta}M$ realize the same $\mathcal{L}(\tau_M)$-type in M, <u>then</u> some automorphism of M maps $\bar{a}$ to $\bar{b}$.

2) M is a strongly sequence $(\mathcal{L},\theta)$-homogeneous <u>when</u> clause (b) above holds.

3) M is sequence (Δ,κ)-homogeneous when: $\Delta\subseteq\mathcal{L}(\tau_M)$ and if $\zeta<\kappa,\bar{a}\in{}^{\zeta}M,\bar{b}\in{}^{\zeta}M$ and $\text{tp}_\Delta(\bar{a},\emptyset,M)=\text{tp}_\Delta(\bar{b},\emptyset,M)$ <u>then</u> for every $c\in M$ for some $d\in M$ we have $\text{tp}_\Delta(\bar{b}\hat{\ }\langle d\rangle,\emptyset,M)=\text{tp}_\Delta(\bar{a}\hat{\ }\langle c\rangle,M)$.

3A) $\Sigma_1(\tau)$ is the set of formulas of the form $\varphi(\bar{x})=(\exists\bar{y})\vartheta(x,\bar{y})$ where $\vartheta(\bar{x},\bar{y})$ is quantifier free first order formula in the vocabulary τ.

4) We may omit "sequence".

Definition 5. 1) The logic $\mathcal{L}$ has "the strong κ-homogeneous existence property"<u>when</u> every theory $T\subseteq\mathcal{L}(\tau_1)$ has a strongly $(\mathcal{L},\kappa)$-homogeneous model.

2) Similarly "the strong κ-saturated existence property", etc.

2. On strongly saturated models

We prove that any theory in $\mathbb{L}(Q^{\text{cf}}_C)$ has strongly $(\mathbb{L}(Q^{\text{cf}}_C),\theta)$-saturated model when $C\backslash\theta\notin\{\emptyset,\text{Reg}\backslash\theta\}$ of course.

Definition 6. Let $\iota\in\{1,2\}$ and C be a class of regular cardinals such that $C\neq\emptyset$, Reg.

1) The quantifier $Q^{\text{cf}(\iota)}_C$ is defined as follows:

<u>syntactically</u>: it bounds two variables, i.e. we can form $(Q^{\text{cf}(\iota)}_C x,y)\varphi$, with its set of free variables being defined as $\text{FVar}(\varphi)\backslash\{x,y\}$.

<u>semantically</u>: $M\models(Q^{\text{cf}(\iota)}_C x,y)\varphi(x,y,\bar{a})$ <u>iff</u> (a) + (b) holds where

(a) relevancy demand:

<u>the case $\iota=1$</u>: the formula $\varphi(-,-;\bar{a})^M$ define in M a linear order with no last element called $\leq^{\varphi}_{M,\bar{a}}$ on the non-empty set $\text{Dom}(\leq^{\varphi}_{M,\bar{a}})=\{b\in M: M\models(\exists y)(\varphi(b,y;\bar{a})\}$

<u>The case $\iota=2$</u>: similarly but $\leq^{\varphi}_{M,\bar{a}}$ is a quasi linear order on its domain

(b) the actual demand: $\leq^{\varphi}_{M,\bar{a}}$ has cofinality $\mathrm{cf}(\leq^{\varphi}_{M,\bar{a}})$, (necessarily an infinite regular cardinal) which belongs to C.

Convention 7. 1) Writing Q^{cf}_C we mean that this holds for $Q^{\mathrm{cf}(\iota)}_C$ for $\iota = 1$ and for $\iota = 2$.

2) Let ι-order mean order when $\iota = 1$ and quasi order when $\iota = 2$; but when we are using $Q^{\mathrm{cf}(\iota)}_C$ then order means ι-order.

Definition 8. 1) As $\{\psi \in \mathbb{L}(Q^{\mathrm{cf}}_C) : \psi$ has a model$\}$ does not depend on C (and is compact, see [Sh:43]) we may use the formal quantifier Q_{cf}, so the syntex is determined but not the semantics, i.e. the satisfaction relation $\models$. We shall write $M \models_C \psi$ or $M \models_C T$ for the interpretation of Q_{cf} as Q^{cf}_C, but also can say "$T \subseteq \mathbb{L}(Q_{\mathrm{cf}})(\tau)$ has model/is consistent".

2) If C is clear from the context, <u>then</u> Q^{cf}_ℓ stands for Q^{cf}_C if $\ell = 1$ and $Q^{\mathrm{cf}}_{\mathrm{Reg}\setminus C}$ if $\ell = 0$.

Convention 9. 1) T^* is a complete (consistent $\equiv$ has models) theory in $\mathbb{L}(Q_{\mathrm{cf}})$ which is closed under definitions i.e. every formula $\varphi = \varphi(\bar{x})$ is equivalent to a predicate $P_\varphi(\bar{x})$ so $P_\varphi \in \tau(T^*)$, i.e. $T^* \vdash (\forall \bar{z})[\varphi(\bar{z}) \equiv P_\varphi(\bar{z})]$.

2) Let $T = T^* \cap$ (first order logic), i.e. $T = T^* \cap \mathbb{L}(\tau_{T^*})$, it is a complete first order theory.

3) $C \subseteq$ Reg, we let $C_1 = C$ and $C_0 =$ Reg$\setminus C$, both non-empty.

Theorem 10. Assume $\chi = \mathrm{cf}(\chi), \mu = \mu^{<\theta} \geq 2^{|T|} + \chi + \kappa, \theta \leq \lambda, \mathrm{cf}(\theta) \leq \min\{\chi, \kappa\}, \chi \neq \kappa = \mathrm{cf}(\kappa)$ and

$$\mu_\ell = \begin{cases} \chi & \ell = 0 \\ \kappa & \ell = 1 \end{cases}$$

<u>Then</u> there is a $\tau(T)$-model M such that

(a) $M \models T, \|M\| = \mu$ and M is θ-saturated

(b) if $\varphi(\bar{z}) = (Q^{\mathrm{cf}}_\ell)\psi(x, y; \bar{z})$ then: $M \models P_{\varphi(\bar{x})}[\bar{a}]$ iff $\varphi(y, z; \bar{a})$ define in M a linear order with no last element and cofinality μ_ℓ

(c) M is strongly[2] θ-saturated model of T^*.

Remark 11. 1) We can now change χ, κ, μ and $\|M\|$ by LST. Almost till the end instead $\mu \geq 2^{|T|} + \chi + \kappa$ just $\mu \geq |T| + \chi + \kappa$ suffice. The proof is broken to a series of definitions and claims. The "$\geq 2^{|T|}$" is necessary for $\aleph_0$-saturativity.

3) We can assume $\mathbf{V}$ satisfies GCH high enough and then use LST. So $\mu^+ = 2^\mu$ below is not a real burden.

[2]as T^* has elimination of quantifiers, doing it for $\mathbb{L}(Q^{\mathrm{cf}}_C)$ or for $\mathbb{L}$ is the same

Definition 12. 0) Mod_T is the class of models of T.

1)

(a) $K = \{(M, N) : M \prec N \text{ are from } \mathrm{Mod}_T\}$

(b) $K_\alpha = \{\bar{M} : \bar{M} = \langle M_i : i < \alpha\rangle$ satisfies $M_i \in \mathrm{Mod}_T$ and $i < j \Rightarrow M_i \prec M_j\}$ (so $K = K_2$)

(c) $K^\alpha_\mu = \{\bar{M} \in K_\alpha : \|M_i\| \leq \mu$ for $i < \alpha\}$, but then we (naturally) assume $\alpha < \mu^+$

(d) let $\tau_\alpha = \tau_T \cup \{P_\beta : \beta < \alpha\} \cup \{R_{\varphi(x,y,\bar{z}),\beta} : \varphi(x, y, \bar{z}) \in \mathbb{L}(\tau_T), \beta < \alpha\}$, each P_β a unary predicate and each $R_{\varphi(x,y,\bar{z}),\beta}$ is an $(\ell g(\bar{z})+1)$-place predicate and no incidental identification (so $P_\alpha \notin \tau$, etc.)

(e) for $\bar{M} \in K_\alpha$ let $\mathbf{m}(\bar{M})$ be the τ_α-model M with

- universe $\cup\{M_\beta : \beta < \alpha\}$
- $M{\restriction}\tau_T = \cup\{M_\beta : \beta < \alpha\}$
- $P^M_\beta = M_\beta$
- $R^M_{\varphi(x,y,\bar{z}),\beta} = \{\langle c\rangle^\frown\bar{a} : \varphi(x, y, \bar{a})$ a linear order, $\bar{a} \in {}^{\ell g(\bar{z})}(P^M_\beta)$ such that $M \models P_{(Q^{cf}_0 x,y)\varphi(x,y,\bar{z})}[\bar{a}]$ and $c \in \mathrm{Dom}(\leq^\varphi_{M,\bar{a}})$ and $[b \in \mathrm{Dom}(\leq^\varphi_{M,\bar{a}}) \cap P^M_\beta \Rightarrow b \leq^\varphi_{M,\bar{a}} c]\}$

(f) let $\mathbf{m}_0(\bar{M})$ be the τ-model $\cup\{M_\beta : \beta < \alpha\}$ so $\mathbf{m}_0(\bar{M}) = \mathbf{m}(\bar{M}){\restriction}\tau$.

2) Assume $(M^\ell, N^\ell) \in K$ for $\ell = 1, 2$ let $(M^1, N^1) \leq (M^2, N^2)$ mean that clauses (a),(b),(c) below hold and let $(M^1, N^1) \leq_K (M^2, N^2)$ mean that in addition clause (d) below holds, where:

(a) $M^1 \prec M^2$

(b) $M^2 \cap N^1 = M^1$

(c) $N^1 \prec N^2$

(d) if $M^1 \models P_{(Q^{cf}_0 x,y)\varphi(x,y,\bar{z})}[\bar{a}], c \in N^1, c \in \mathrm{Dom}(\leq^\varphi_{N^1,\bar{a}})$ and in N^1 the element c is $\leq^\varphi_{N^1,\bar{a}}$-above all $d \in \mathrm{Dom}(\leq^\varphi_{M^1,\bar{a}})$, <u>then</u> in N^2 the element c is $\leq^\varphi_{N^2,\bar{a}}$-above all $d \in \mathrm{Dom}(\leq^\varphi_{M^2,\bar{a}})$.

3) For $\bar{M}^1, \bar{M}^2 \in K_\alpha$ let $\bar{M}^1 \leq \bar{M}^2$ <u>means</u> $\gamma < \beta < \alpha \Rightarrow (M^1_\gamma, M^1_\beta) \leq (M^2_\gamma, M^2_\beta)$; similarly $\bar{M}^1 \leq_{K_\alpha} \bar{M}^2$ means $\bar{M}^1, \bar{M}^2 \in K_\alpha$ and $\gamma < \beta < \alpha \Rightarrow (M^1_\delta, M^1_\beta) \leq_K (M^2_\gamma, M^2_\beta)$.

4) For $\bar{M} \in K_\alpha, D$ an ultrafilter on λ we define $\bar{N} = \bar{M}^\lambda/D, \mathbf{j}_{M,D} = \mathbf{j}^\lambda_{M,D}$ naturally: $N_\beta = M^\lambda_\beta/D$ for $\beta < \alpha$ and $\mathbf{j}_{\bar{M},D} = \cup\{\mathbf{j}_{M_\beta,D} : \beta < \alpha\}$, recalling 3.

Fact 13. 0) For $\bar{M}^1, \bar{M}^2 \in K_\alpha$ we have

(a) $\bar{M}^1 \leq_{K_\alpha} \bar{M}^2$ iff $\mathbf{m}(M^1) \subseteq \mathbf{m}(\bar{M}^2)$

(b) $(\mathbf{m}(\bar{M}^\ell) \restriction P^{M_\beta}) \restriction \tau_T = M^\ell_\beta$

(c) $\bar{M}^1 \leq_{K_\alpha} \bar{M}^1$ implies $\bar{M}^1 \leq \bar{M}^2$.

1) $(K_\alpha, \leq)$ and $(K_\alpha, \leq_{K_\alpha})$ are partial orders.

2a) If $\bar{M}^1 \leq_{K_\alpha} \bar{M}^2$ in K_α and $0 < \gamma < \beta \leq \alpha$ <u>then</u> $(\bigcup_{\varepsilon<\gamma} M^1_\varepsilon, \bigcup_{\varepsilon<\beta} M^1_\varepsilon) \leq (\bigcup_{\varepsilon<\gamma} M^2_\varepsilon, \bigcup_{\varepsilon<\beta} M^2_\varepsilon)$ moreover $\langle \bigcup_{i<1+\varepsilon} M^1_i : 1+\varepsilon \leq \alpha\rangle \leq_{K_\xi} \langle \bigcup_{i<1+\varepsilon} M^2_i : 1+\varepsilon \leq \alpha\rangle$ where ξ is α if $\alpha < \omega$ and is $\alpha+1$ if $\alpha \geq \omega$.

2b) If $\langle \bar{M}^i : i < \delta\rangle$ is a $\leq_{K_\alpha}$-increasing sequence (of members of K_α) and we define $\bar{M}^\delta = \langle M^\delta_\varepsilon : \varepsilon < \alpha\rangle$ by $M^\delta_\varepsilon = \cup\{M^i_\varepsilon : i < \delta\}$ then $i < \delta \Rightarrow \bar{M}^i \leq_{K_\alpha} \bar{M}^\delta$ and the sequence $\langle \bar{M}^i : i \leq \delta\rangle$ is continuous in δ.

3) In part (2b), if in addition $i < \delta \Rightarrow \bar{M}^i \leq_{K_\alpha} \bar{N}$ so $\bar{N} \in K_\alpha$ <u>then</u> $\bar{M}^\delta \leq_{K_\alpha} \bar{N}$.

4) In part (2b), if $\delta < \mu^+$ and $i < \delta \Rightarrow \bar{M}^i \in K^\alpha_\mu$ then $\bar{M}^\delta \in K^\alpha_\mu$.

5) If $\bar{M} \leq_{K_\alpha} \bar{N}$ and $Y_\varepsilon \subseteq N_\varepsilon$ for $\varepsilon < \alpha$ and $\Sigma\{\|M_\varepsilon\| + |Y_\varepsilon| : \varepsilon < \alpha\} + |\tau| + |\alpha| \leq \lambda$ <u>then</u> there is $\bar{N}' \in K^\alpha_\lambda$ such that $\bar{M} \leq_{K_\alpha} \bar{N}' \leq_{K_\alpha} \bar{N}$ and $\varepsilon < \alpha \Rightarrow Y_\varepsilon \subseteq N'_\varepsilon$.

6) Assume $\bar{M}^i \in K^{\alpha(i)}_\mu$ for $i < \delta < \mu^+$, $\langle \alpha(i) : i < \delta\rangle$ is a non-decreasing sequence of ordinals and $i < j < \delta \Rightarrow \bar{M}^i \leq_{K_{\alpha(i)}}, \bar{M}^j \restriction \alpha(i)$ and we define $\alpha(\delta) = \cup\{\alpha(i) : i < \delta\}, \bar{M}^\delta = \langle M^\delta_\beta : \beta < \alpha(\delta)\rangle$ where $M^\delta_\beta = \cup\{M^i_\beta : \beta < \delta$ satisfies $\beta < \alpha(i)\}$ <u>then</u> $\bar{M}^\delta \in K^{\alpha(\delta)}_\mu$ and $i < \delta \Rightarrow \bar{M}^i \leq_{K_{\alpha(i)}} \bar{M}^\delta \restriction \alpha(i)$.

7) If $\bar{M}^\ell \leq_{K_\alpha} \bar{N}$ for $\ell = 1, 2$ and $[a \in \mathbf{m}(\bar{M}^1) \Rightarrow a \in \mathbf{m}(\bar{M}^2)]$ then $\bar{M}^1 \leq_{K_\alpha} \bar{M}^2$.

8) Parts (2)-(7) holds also when we replace $\leq_{K_\alpha}$ by $\leq$.

Demostración. Check. $\square_{13}$

Fact 14. 1) If $(M_0, M_1) \in K^2_\mu$ and $(M_0, M'_1) \in K^2_\mu$ <u>then</u> there are M_2, f such that

(a) $M'_1 \prec M_2 \in K_\mu$

(b) f is an elementary embedding of M_1 into M_2

(c) $f \restriction M_0 = \mathrm{id}_{M_0}$

(d) $(M_0, M'_1) \leq_{K_2} (f(M_1), M_2)$.

2) If $\bar{M} \in K_\alpha, \bar{x} = \langle x_\varepsilon : \varepsilon < \zeta \rangle$ and Γ is a set of first order formulas from $\mathbb{L}(\tau_\alpha^+)$ in the variables $\bar{x}$ with parameters from the model $\mathbf{m}(\bar{M})$ finitely satisfiable in $\mathbf{m}(M)$ such that $\varepsilon < \zeta \Rightarrow \bigvee_{\beta<\alpha} P_\beta(x_\varepsilon) \in \Gamma$, <u>then</u> there is $\bar{N} \in K_\alpha$ such that $\bar{M} \leq_{K_\alpha} \bar{N}$ and Γ is realized in $\mathbf{m}(\bar{N})$.

3) If Γ is a type over $\mathbf{m}_0(\bar{M})$ of cardinality[3] $<$ cf(α) <u>then</u> it is included in some Γ' as in part (2).

4) If $\bar{M} \in K_\mu^\alpha$, D an ultrafilter on θ and $M'_\beta = (M_\beta)^\theta/D$ for $\beta < \alpha$ then

(a) $\bar{M}' = \langle M'_\beta : \beta < \alpha \rangle \in K_\alpha$

(b) $\mathbf{j}^\theta_{\bar{M},D} := \cup\{\mathbf{j}^\theta_{M_\beta,D} : \beta < \alpha\}$ is a $\leq_{K_\alpha}$-embedding of $\bar{M}$ into $\bar{M}'$, i.e.

(b)′ $\langle \mathbf{j}^\theta_{M_\beta,D}(M_\beta) : \beta < \alpha \rangle = \bar{M}' \leq_{K_\alpha} \langle M'_\beta : \beta < \alpha \rangle$, so

(c) for many $Y \in [\cup\{M'_\beta : \beta < \alpha\}]^\mu$ we have $\mathbf{j}^\theta_{\bar{M},D}(\bar{M}) \leq_{K_\alpha} \langle M'_\beta \restriction Y : \beta < \alpha \rangle \in K^\alpha_\mu$; see 13(5), 17(3).

Demostración. 1) See [Sh:199, §4]; just let D be a regular ultrafilter on $\lambda \geq \|M_1\| + |\tau|$, let g an elementary embedding of M_1 into $(M_0)^\lambda/D$ extending $\mathbf{j} = \mathbf{j}^\lambda_{M_0,D}$, necessarily exists.

Lastly, let $M_2 \prec (M'_1)^\lambda/D$ include $\mathbf{j}^\lambda_{M_1,D}(M'_1) \cup g(M_1)$ be of cardinality μ. Identifying M'_1 with $\mathbf{j}^\lambda_{M'_1,D}(M'_1) \prec (M'_1)^\lambda/D$ we are done.

2) Similarly.

3) Trivial.

4) Should be clear. $\square_{14}$

Definition 15. K^{ec}_α is the class of $\bar{M} \in K_\alpha$ such that: if $\bar{M} \leq_{K_\alpha} \bar{N} \in K_\alpha$, <u>then</u> $\mathbf{m}(\bar{M}) \leq_{\Sigma_1} \mathbf{m}(\bar{N})$, i.e. $(*)$ below and $K^{ec,\alpha}_\lambda = K^{ec}_\alpha \cap K_\lambda$ where

$(*)$ if $a_1, \ldots, a_n \in \mathbf{m}(\bar{M}), b_1, \ldots, b_k \in \mathbf{m}(\bar{N}), \varphi \in \mathbb{L}(\tau^+_\alpha)$ is quantifier free and $\mathbf{m}(\bar{N}) \models \varphi[a_1, \ldots, a_n, b_1, \ldots, b_k]$ <u>then</u> for some $b'_1, \ldots, b'_k \in \bigcup_{\beta<\alpha} M_\beta$ we have $\mathbf{m}(\bar{M}) \models \varphi[a_1, \ldots, a_n, b'_1, \ldots, b'_k]$.

Claim 16. 1) $K^{ec,\alpha}_\mu$ is dense in K^α_μ when $\mu \geq |\tau_T| + |\alpha|$ of course.

2) $K^{ec,\alpha}_\mu$ is closed under union of increasing chains of length $< \mu^+$.

3) In Definition 15, if $|\alpha| + |\tau_T| \leq \mu$ and $\bar{M} \in K^\alpha_\mu$ <u>then</u> without loss of generality $\bar{N} \in K^\alpha_\mu$.

Demostración. 1) Given $\bar{M}_0 \in K^\alpha_\mu$ we try to choose $\bar{M}_\varepsilon \in K^\alpha_\mu$ by induction on $\varepsilon < \mu^+$ such that $\langle \bar{M}_\zeta : \zeta \leq \varepsilon \rangle$ is $\leq_{K_\alpha}$-increasing continuous and $\varepsilon = \zeta + 1 \Rightarrow \mathbf{m}(\bar{M}_\zeta) \not\leq_{\Sigma_1} \mathbf{m}(M_\varepsilon)$. For $\varepsilon = 0$ the sequence is given, for ε limit use 13(2), for $\varepsilon = \zeta + 1$ if we cannot choose then by 13(5) we get

[3] also if cf$(\alpha) = 1$, i.e. α is a successor ordinal

$\bar{M}_\zeta \in K^{ec,\alpha}_\mu$ is as required. But if we succeed to choose $\langle \bar{M}_\varepsilon : \varepsilon < \mu^+ \rangle$ we get contradiction by Fodor lemma.

2) Think on the definitions.

3) By LST. $\square_{16}$

Claim 17. 1) If $\bar{M}, \bar{N} \in K^\alpha_\mu$ and $\bar{M} \leq_{\Sigma_1} \bar{N}$ and $\bar{N} \in K^{ec}_\alpha$ <u>then</u> $\bar{M} \in K^{ec}_\alpha$.

2) If $\bar{N} \in K^{ec,\alpha}_\mu, Y \subseteq \mathbf{m}_0(\bar{N})$ and $\lambda = |\tau_T| + |\alpha| + |Y|$ <u>then</u> there is $\bar{M} \in K^{ec,\alpha}_\lambda$ such that $\bar{M} \leq_{K_\alpha} \bar{N}$ and $Y \subseteq \mathbf{m}_0(\bar{M})$.

3) Assume $\bar{M}^\ell \in K^\alpha_\mu$ and $\bar{M}^0 \leq_{K_\alpha} \bar{M}^1$ and $\bar{M}^0 \leq \bar{M}^2$. If $\bar{M}^0 \in K^{ec,\alpha}_\mu, \bar{M}^0 \leq_{K_\alpha} \bar{M}^2$ <u>or</u> $\mathbf{m}(\bar{M}^0) \leq_{\Sigma_1} \mathbf{m}(\bar{M}^2)$, <u>then</u> we can find $(\bar{N}, f_2)$ such that:

$\bar{M}^1 \leq_{K_\alpha} \bar{N} \in K^\alpha_\mu$, moreover $\bar{N} \in K^{ec,\alpha}_\mu$ and f_2 is a $\leq_{K_\alpha}$-embedding of $\bar{M}^2$ into $\bar{N}$ over $\bar{M}^0$.

Demostración. 1) By part (3).

2) By part (1) and the LST argument.

3) By the definition of $\bar{M}^0 \in K^{ec,\alpha}_\mu$ in both cases we can assume $\bar{M}^0 \leq_{\Sigma_1} \bar{M}^2$. Let $\bar{\mathbf{a}} = \langle a_\varepsilon : \varepsilon < \zeta \rangle$ list the elements of $\mathbf{m}(\bar{M}^2)$ and let $\Gamma = \text{tp}_{\text{qf}}(\bar{\mathbf{a}}, \emptyset, \mathbf{m}(\bar{M}^2)) = \{\varphi(x_{\varepsilon_0}, \ldots, x_{\varepsilon_{n-1}}, \bar{b}) : \varphi \in \mathbb{L}(\tau^+_\alpha)$ is quantifier free, $\bar{b} \subseteq \mathbf{m}(\bar{M}^0)$ and $\mathbf{m}(\bar{M}^2) \models \varphi[a_{\varepsilon_0}, \ldots, a_{\varepsilon_{n-1}}, \bar{b}]\}$; note that $P_\beta(x_\varepsilon))^{\mathbf{t}(\varepsilon,\beta)} \in \Gamma$ when $\beta < \alpha, \varepsilon < \zeta$ and $\mathbf{t}(\varepsilon, \beta)$ is the truth value of $a_\varepsilon \in M^2_\beta$.

Now let D be a regular ultrafilter on $\lambda = \|\mathbf{m}(\bar{M}^2)\|$ and use 14(2),(3). This is fine to get $(f_2, \bar{N})$ with $\bar{N} \in K_\alpha$ and by 13(5) without loss of generality $\bar{N} \in K^\alpha_\mu$ and by 16(1) without loss of generality $\bar{N} \in K^{ec,\alpha}_\mu$. $\square_{17}$

Claim 18. 1) $(K^{ec,\alpha}_\mu, \leq_{K^\alpha_\mu})$ has the JEP.

2) Suppose $\bar{M}^1, \bar{M}^2 \in K^\alpha_\mu, \beta \leq \alpha$, f is an elementary embedding of $\bigcup_{\gamma<\beta} M^1_\gamma$ into $\bigcup_{\gamma<\beta} M^2_\gamma$ such that $\langle f(M_\gamma) : \gamma < \beta \rangle \leq_{K_\mu} \langle M^2_\gamma : \gamma < \beta \rangle$, equivalently f is an embedding of $\mathbf{m}(\bar{M}^1 \restriction \beta)$ into $\mathbf{m}(\bar{M}^2 \restriction \beta)$ (so if $\beta = 0$ then $f = \emptyset$ and there is no demand).

<u>Then</u> we can find $\bar{M}^3, f^+$ such that:

(a) $\bar{M}^2 \leq_{K_\mu} \bar{M}^3 \in K^\alpha_\mu$

(b) $f \subseteq f^+$

(c) f^+ is an elementary embedding of $\bigcup_{\gamma<\alpha} M^1_\gamma$ into $\bigcup_{\gamma<\alpha} M^3_\gamma$

(d) $\langle f^+(M^1_\gamma)) : \gamma < \alpha \rangle \leq_{K_\alpha} \langle M^3_\gamma : \gamma < \alpha \rangle$.

Demostración. 1) A special case of part (2) recalling 16(1).

2) By induction on α.

$\underline{\alpha = 0}$: nothing to do.

$\underline{\beta = \alpha}$: nothing to do.

$\underline{\alpha = 1}$: so $\beta = 0$ which is trivial or $\beta = \alpha$, a case done above.

$\underline{\alpha \text{ successor}}$: by the induction hypothesis and transitive nature of conclusion replacing $\bar{M}^2$ without loss of generality $\beta = \alpha - 1$, then use 14(1).

$\underline{\alpha \text{ limit}}$: By $\alpha - \beta$ successive uses of induction hypothesis using 13(2b). $\square_{18}$

Conclusion 19. $(K^{ec}_\alpha, \leq_{K_\alpha})$, or formally $\mathfrak{k} = (K_\mathfrak{k}, \leq_\mathfrak{k})$ defined by $K_\mathfrak{k} := \{\mathbf{m}(\bar{M}) : \bar{M} \in K^{ec}_\alpha\}, \mathbf{m}(\bar{M}^1) \leq_\mathfrak{k} \mathbf{m}(\bar{M}^2) \Leftrightarrow \mathbf{m}(M^1) \subseteq \mathbf{m}(\bar{M}^2)$, is an a.e.c. with amalgamation, the JEP and $\mathrm{LST}(\mathfrak{k}) \leq |\tau_T| + |\alpha| + \aleph_0$.

Demostración. By the above, on a.e.c. see [Sh:h, Ch.I], i.e. [Sh:88r] and history there. $\square_{19}$

Fact 20. Assume $\lambda = \lambda^{<\lambda} > |\tau_T| + \aleph_0 + |\alpha|$. Then there is $\bar{M}$ such that

(a) $\bar{M} \in K^{ec}_\alpha$ is universal for $(K^{ec}_\alpha, \leq_{K_\alpha})$ in cardinality λ

(b) $\mathbf{m}(\bar{M})$ is model homogeneous for $(K^{ec}_\alpha, \leq_{K_\alpha})$ of cardinality λ

(c) $\mathbf{m}(\bar{M})$ is sequence $(\Sigma_1(\tau^+_\alpha), \lambda)$-homogeneous, see 4(3).

Demostración. Clause (a) + (b) are straight by 17 + 18(1), or use 19 and see [Sh:h, Ch.I,§2] = [Sh:88r, §2]. Now clause (c) follows: just think. $\square_{20}$

Fact 21. Assume $\bar{M} \in K^\alpha_\mu, \beta + 1 < \alpha, \ell \in \{0, 1\}$ and $M_\beta \models P_{(Q^{cf}_\ell x,y)\varphi(x,y,\bar{z})}[\bar{a}]$ then there are $\bar{N}, c$ such that $\bar{M} \leq_{K_\alpha} \bar{N} \in K^\alpha_\mu$ and:

$(*)_1$ if $\ell = 1$ then $c \in \mathrm{Dom}(\leq^\varphi_{N_\beta,\bar{a}})$ and c is $\leq^\varphi_{N_\gamma,\bar{a}}$-above $d \in \mathrm{Dom}(\leq^\varphi_{M_\gamma,\bar{a}})$ for any $\gamma \in [\beta, \alpha)$

$(*)_2$ if $\ell = 0$ then $c \in \mathrm{Dom}(\leq^\varphi_{N_{\beta+1},\bar{a}})$ and is $\leq^\varphi_{N_{\beta+1},\bar{a}}$-above any
$d \in \mathrm{Dom}(\leq^\varphi_{N_\beta,\bar{a}})$.

Demostración. First assume $\ell = 1$, without loss of generality $\beta = 0$ as we can let $\bar{N}\restriction\beta = \bar{M}\restriction\beta$.

By 13(2a) wlog $\bar{M}$ is increasing continuous; we prove by induction on α so easily without loss of generality $\alpha = 2$. Now this is obvious by [Sh:43], [Sh:199]; in details by [Sh:43] there is a μ^+-saturated model M_* of T such that $M_1 \prec M_*$ and $M_* \models_{C_*} T^*$ whenever, e.g. $\mu^{++} \in C_* \wedge \mu^+ \notin C_*$. Let $\{\varphi_i(x, y, \bar{a}^*_i) : i < \mu\}$ list $\{\varphi(x, y, \bar{a}') : \varphi \in \mathbb{L}(\tau_T), M_0 \models P_{(Q^{cfx,y}_0)\varphi(x,y;\bar{z})}[\bar{a}']\}$, and for each $i < \mu$ let $\langle c_{i,\varepsilon} : \varepsilon < \mu^+\rangle$ be $\leq^{\varphi_i}_{M_*,\bar{a}^*_i}$-increasing and cofinal. For $\varepsilon < \mu^+$ let f_ε be an elementary embedding of M_1 into M_* over M_0 such that:

(∗) if $c \in \mathrm{Dom}(\leq^{\varphi_i}_{M_*,\bar{a}^*_i})$ is a $\leq^{\varphi_i}_{M_*,\bar{a}^*_i}$–upper bound of $\mathrm{Dom}(\leq^{\varphi_i}_{M_0,\bar{a}^*_i})$, then $c_{i,\varepsilon} \leq^{\varphi_i}_{M_*,\bar{a}^*_i} c$.

Let $c_* \in M_*$ be a $\leq^{\varphi}_{M_*,\bar{a}}$-upper bound of $\mathrm{Dom}(\leq^{\varphi}_{M_0,\bar{a}})$. Choose $N_0 \prec M_*$ of cardinality μ be such that $M_0 \cup \{c_*\} \subseteq N_0$ and choose $\varepsilon < \mu^+$ large enough such that:

(∗) if $i < \mu$ and $d \in N_0$ is a $\leq^{\varphi_i}_{M_*,\bar{a}_i}$-upper bound of $\mathrm{Dom}(\leq^{\varphi_i}_{M_i,\bar{a}_i})$ then $d \leq^{\varphi_i}_{M_*,\bar{a}_i} c_{i,\varepsilon}$.

Let $N_1 \prec M_*$ be of cardinality μ be such that $N_0 \cup f_\varepsilon(M_1) \subseteq N_1$. Renaming, f_ε is the identity and (N_0, N_1) is as required.

Second, assume $\ell = 0$ is even easier (again without loss of generality first, $\alpha = \beta + 2$ and second $\beta = 0, \alpha = 2$ and use $N_0 = M_0, N_1$ satisfies $M_1 \prec N_1$ and $\|N_1\| = \mu$ and N_1 realizes the relevant upper). $\square_{21}$

Conclusion 22. In 20 the model $M^* = \mathbf{m}(\bar{M}^*) = \bigcup\limits_{\beta<\alpha} M^*_\beta$ satisfies

(a) if $M^* \models P_{(Q^{\mathrm{cf}}_1 x,y)\varphi}[\bar{a}]$ then the order $\leq^{\varphi}_{M^*,\bar{a}}$ has cofinality λ

(b) if α is a limit ordinal and $M^* \models P_{(Q^{\mathrm{cf}}_0 x,y)\varphi}[\bar{a}]$ <u>then</u> the linear order $\leq^{\varphi}_{M^*,\bar{a}}$ has cofinality $\mathrm{cf}(\alpha)$

(c) M^* is $\mathrm{cf}(\alpha)$-saturated

(d) if $\lambda \in C$ and $\mathrm{cf}(\alpha) \in \mathrm{Reg}\backslash C$ then M^* is a model of T^*.

Claim 23. Assume $\bar{M} \in K^{\mathrm{ec}}_\alpha$. If $\zeta \leq \mu$ and $\bar{a}, \bar{b} \in {}^\zeta(M^*_0)$ realize the same type (equivalently q.f. type) in M_0 <u>then</u> they realize the same Σ_1-type in $\mathbf{m}(\bar{M})$.

Demostración. We choose $(N_\beta, f_\beta, g_\beta, h_\beta)$ by induction on $\beta < \alpha$ such that:

(a) N_β is a model of T

(b) N_β is $\prec$-increasing continuous with β

(c) f_β, g_β are $\leq_{K_{1+\beta}}$-embedding of $\bar{M}\restriction(1+\beta)$ into $\langle N_\gamma : \gamma < 1+\beta\rangle \in K_{1+\beta}$

(d) $f_0(\bar{a}) = g_0(\bar{b})$

(e) if $\gamma < \beta$ then $f_\gamma \subseteq f_\beta, g_\gamma \subseteq g_\beta$.

For $\beta = 0$ this speaks just on Mod_T.

For β successor use 14.

For β limit as in the successor case, recalling we translated it to the successor case (by 13(2a)).

Having carried the induction $f = \cup\{f_\beta : \beta < \alpha\}$ and $g = \cup\{g_\beta : \beta < \alpha\}$ are $\leq_{K_\alpha}$-embedding of $\bar{M}$ into $\bar{N} = \langle N_\beta : \beta < \alpha\rangle$. By 16(1) there is $\bar{N}' \in K^{\rm ec}_\alpha$ which is $\leq_{K_\alpha}$-above $\bar{N}$. Now as $\bar{M} \in K^{\rm ec}_\alpha$, the Σ_1-type of $\bar{a}$ in $\mathbf{m}(\bar{M})$ is equal to the Σ_1-type of $f(\bar{a})$ in $\mathbf{m}(\bar{N}')$, and the Σ_1-type of $\bar{b}$ in $\mathbf{m}(\bar{M})$ is equal to the Σ_1-type of $f(\bar{a})$ in $\mathbf{m}(\bar{N}')$. But $f(\bar{a}) = f_0(\bar{a}) = g_0(\bar{b}) = g(\bar{b})$, so we have gotten the promised equality of Σ_1-types. $\square_{23}$

Observation 24. 1) If $\bar{M} \in K^{\rm ec}_\alpha$ and $\beta < \alpha$ <u>then</u> $\bar{M}' : \bar{M}{\restriction}[\beta,\alpha) = \langle M_{\beta+\gamma} : \gamma < \alpha - \beta\rangle$ belongs to $K^{\rm ec}_{\alpha-\beta}$.

2) If $\bar{M} \in K_\alpha, \beta < \alpha$ and $\bar{M}{\restriction}[\beta,\alpha) \leq_{K_{\alpha,\beta}} \bar{N}'$ then for some $\bar{N} \in K_\alpha$ we have $\bar{M} \leq_{K_\alpha} \bar{N}$ and $\bar{N}{\restriction}[\beta,\alpha) = \bar{N}'$.

Demostración. 1) If not, then there is $\bar{N}' \in K_{\alpha-\beta}$ such that $\bar{M}' \leq_{K_{\alpha-\beta}} \bar{N}'$ but $\mathbf{m}(\bar{M}') \not\leq_{\Sigma_1} \mathbf{m}(\bar{N}')$. Define $\bar{N} = \langle N_\gamma : \gamma < \alpha\rangle$ by: N_γ is M_γ if $\gamma < \beta$ and is $N'_{\gamma-\beta}$ if $\gamma \in [\beta,\alpha)$. Easily $\bar{M} \leq_{K_\alpha} \bar{N} \in K_\alpha$ but $\mathbf{m}(\bar{M}) \not\leq_{\Sigma_1} \mathbf{m}(\bar{N})$, contradiction to the assumption $\bar{M} \in K^{\rm ec}_\alpha$.

2) The proof is included in the proof of part (1). $\square_{24}$

Claim 25. In 20 for each $\beta < \alpha$ we have

(a) $\langle M^*_{\beta+\gamma} : \gamma < \alpha - \beta\rangle$ is homogeneous universal for $K^{\alpha-\beta}_\mu$

(b) if $\alpha = \alpha - \beta$, i.e. $\beta + \alpha = \alpha$ <u>then</u> there is an isomorphism from $\bar{M}^*$ onto $\langle M^*_{\beta+\gamma} : \gamma < \alpha - \beta\rangle$, in fact, we can determine $f(\bar{a}) = \bar{b}$ if $\bar{a} \in {}^\zeta(M^*_0), \bar{b} \in {}^\zeta(M^*_\beta)$ and $\mathrm{tp}(\bar{a}, \emptyset, M^*_\beta) = \mathrm{tp}(\bar{b}, \emptyset, M^*_\beta)$.

Demostración. Chase arrows as usual recalling 24. $\square_{25}$

Demostración. <u>Proof of Theorem 10</u>:

Without loss of generality there is $\sigma = \sigma^\theta \geq \mu$ such that $2^\sigma = \sigma^+$ (why? let $\sigma = \sigma^\theta > \mu$ be regular, work in $\mathbf{V}^{\mathrm{Levy}(\sigma^+, 2^\sigma)}$ and use absoluteness argument, or choose set A of ordinals such that $\mathcal{P}(\mu) \in \mathbf{L}[A]$ hence $T, T^* \in \mathbf{L}[A]$ and regular θ large enough such that $\mathbf{L}[A] \models$ "$2^\sigma = \sigma^+$", work in $\mathbf{L}[A]$ a little more; and for the desired conclusion (there is a model of cardinality μ such that ...) it makes no difference). Let $\alpha = \kappa$ and let $\bar{M}^* \in K^{\mathrm{ec},\alpha}_\lambda$ be as in 20 for $\lambda := \sigma^+$ and let $M_* = \cup\{M^*_\beta : \beta < \alpha\}$.

Now

$(*)_1$ M_* is a model of T^* by the $\{\mu^+\}$-interpretation.

[Why? By 22.]

$(*)_2$ M_* is θ-saturated.

[Why? Clearly M^*_β is θ-saturated for each $\beta < \theta$. As θ is regular and $\langle M^*_\beta : \beta < \theta\rangle$ is increasing with union M_*, also M_* is θ-saturated.]

$(*)_3$ M_* is strongly $\aleph_0$-saturated and even strongly θ-saturated, see Definition 4(1).

[Why? Let $\zeta < \theta$ and $\bar{a}, \bar{b} \in {}^\zeta(M_*)$ realize the same q.f.-type (equivalently the first order type) in M_*. As $\zeta < \theta$ for some $\beta < \theta$ we have $\bar{a}, \bar{b} \in {}^\zeta(M_\beta)$. Now by 25 we know that $\langle M^*_{\beta+\gamma} : \gamma < \theta\rangle \cong \langle M^*_\gamma : \gamma < \theta\rangle$, and by 23 the sequences $\bar{a}, \bar{b}$ realize the same Σ_1-type in $\mathbf{m}(\langle M^*_{\beta+\gamma} : \gamma < \theta\rangle)$ hence by clause (c) of 20 there is an automorphism π of it mapping $\bar{a}$ to $\bar{b}$. So π is also an automorphism of M_* mapping $\bar{a}$ to $\bar{b}$ as required.]

Lastly, we have to go back to models of cardinality $\mu = \mu^{<\theta} \geq \lambda + \kappa + 2^{|T|}$, this is done by the LST argument recalling 22.

More fully, first let $\langle \bar{M}^\varepsilon : \varepsilon < \lambda\rangle$ be $\leq_{K^\sigma\chi}$-increasing continuous sequence with union $\bar{M}^*$. For $\zeta < \theta$ and $\bar{a}, \bar{b} \in {}^\zeta(M_*)$ let $f_{\bar{a},\bar{b}}$ be an automorphism of M_* mapping $\bar{a}$ to $\bar{b}$. Now the set of $\delta < \lambda$ satisfying $\circledast_\delta$ below is a club of λ hence if $\mathrm{cf}(\delta) = \chi$ then $M = \cup\{M^\varepsilon_\beta : \beta < \lambda\}$ is as required except of being of cardinality μ, where

$\circledast_\delta$ (a) if $\varepsilon < \delta, \zeta < \theta$ and $\bar{a}, \bar{b} \in {}^\zeta(\cup\{M^\varepsilon_\beta : \beta < \alpha\})$ realize the same Σ_1-type in $\bar{M}^\zeta$ then $\cup\{M^\delta_\beta : \beta < \alpha\}$ is closed under $f_{\bar{a},\bar{b}}$ and under $f^{-1}_{\bar{a},\bar{b}}$

(b) the witnesses for the cofinality work, i.e.

- $\bullet_1$ if $\beta < \alpha, \bar{a} \in {}^{\omega>}(M^\delta_\beta), M^\delta_\beta \models P_{(Q^{cf}_0 y,z)\varphi(y,z,\bar{x})}[\bar{a}]$ then for some $\varepsilon < \delta$ we have $\bar{a} \subseteq M^\varepsilon_\beta$ and for every $\gamma \in (\beta, \alpha)$ there is $c = c_{\varphi,\bar{a},\gamma} \in M^\varepsilon_{\gamma+1}$ which is a $\leq^\varphi_{M^\varepsilon_{\gamma+1},\bar{a}}$-upper bound of $\mathrm{Dom}(\leq^\varphi_{M^\varepsilon_\gamma,\bar{a}})$, hence this holds for any $\varepsilon' \in [\varepsilon, \lambda)$
- $\bullet_2$ if $\beta < \alpha, \bar{a} \in {}^{\omega>}(M^\gamma_\beta)$ and $M^\delta_\beta \models P_{(Q^{cf}_1 y,z)\varphi(y,z,\bar{x})}[\bar{a}]$ <u>then</u> for arbitrarily large $\varepsilon < \delta$ we have $\bar{a} \subseteq M^\varepsilon_\beta$ and there is $c = c_{\varphi,\bar{a}} \in M^{\varepsilon+1}_\beta$ which is a $(\leq^\varphi_{M^{\varepsilon+1}_\gamma,\bar{a}})$-upper bound of $\mathrm{Dom}(\leq^\varphi_{M^\varepsilon_\gamma,\bar{a}})$ for every $\gamma \in [\beta, \alpha)$.

By a similar use of the LST argument we get a model of T^* of cardinal μ. $\square_{10}$

Remark 26. If you do not like the use of (set theoretic absoluteness) you may do the following. Use 27 below, which is legitimate as

(a) the class $(K^{ec}_\alpha, \leq_{K_\alpha})$ is an a.e.c. with LST number $\leq |T| + \aleph_0$ and amalgamation, so 27(1) apply

(b) using Σ_1-types, it falls under [Sh:3] more exactly [Sh:54], so 27(3) apply

(c) we can define $K^{ec(\varepsilon)}_\alpha$ by induction on $\varepsilon \leq \omega$

$\underline{\varepsilon = 0}$: K_α

$\underline{\varepsilon = 1}$: K^{ec}_α

$\underline{\varepsilon = n+1}$: $K^{ec(n+1)}_\alpha = \{\bar{M} \in K^{ec(n)}_\alpha$: if $\bar{M} \subseteq N \in K^{ec(n)}_\alpha$ then $\mathbf{m}(M) \leq_{\Sigma_{n+1}} \mathbf{m}(\bar{N})\}$

$\underline{\varepsilon = \omega}$: $K^{ec(\varepsilon)}_\alpha = \cap\{K^{ec(n)}_\alpha : n < \omega\}$.

On $K^{ec(\omega)}_\alpha$ apply 27(2).

Remark 27. 1) Assume $\mathfrak{k} = (K_\mathfrak{k}, \leq_\mathfrak{k})$ is a a.e.c. satisfying amalgamation and the JEP with $\lambda >$ LST$(\mathfrak{k})$ and $\mu = \mu^{<\lambda}$. For any $M \in K_\mu$ there is a strongly model λ-homogeneous $N \in K_\mu$ which $\leq_\mathfrak{k}$-extend M, which means: if $M \in K_\mathfrak{k}$ has cardinality $< \lambda$ and f_1, f_2 are $\leq_\mathfrak{k}$-embedding of M into N then for some automorphism g of N we have $f_2 = g \circ f_1$.

2) Let D be a good finite diagram as in [Sh:3] and let K_D be as below in part (3) for $\Delta = \mathbb{L}(\tau)$. If $\lambda = \lambda^{<\theta} \geq |D|$ and $M \in K_D$ has cardinality λ then there is $N \in K_D$ of cardinality λ which $\prec$-extend M and is strongly (D, θ)-homogenous, i.e.

(a) if $\zeta < \theta, \bar{a}, \bar{b} \in {}^\zeta N$ realizes the same type <u>then</u> some automorphism f of N maps $\bar{a}$ to $\bar{b}$

(b) $D = \{\text{tp}(\bar{a}, \emptyset, N) : \bar{a} \in {}^{\omega>}N\}$.

3) Assume $\Delta \subseteq \mathbb{L}(\tau)$, not necessarily closed under negation, D is a set of Δ-types, K_D is the class of τ-models such that $\bar{a} \in {}^{\omega>}M \Rightarrow \text{tp}_\Delta(\bar{a}, \emptyset, M) \in D$ and $M \leq_D N$ iff $M \subseteq N$ are from K_D and $\bar{a} \in {}^{\omega>}M \Rightarrow \text{tp}_\Delta(\bar{a}, \emptyset, M) = \text{tp}_\Delta(\bar{a}, \emptyset, N)$. Assume further D is good, i.e. for every $M \in K_D$ and λ there is a sequence (D, λ)-homogeneous model $N \in K_D$ which $\leq_D$-extends M. <u>Then</u> for every $\lambda = \lambda^{<\theta} > |T| + \aleph_0$ and $M \in K_D$ of cardinality λ there is a strongly sequence (Δ, λ)-homogeneous.

Conclusion 28. 1) The logic $\mathbb{L}(Q^{cf}_C)$ has the strong $\aleph_0$-saturated model existence property (hence the strong $\aleph_0$-homogeneous model existence property).

2) If $\kappa =$ cf$(\kappa) \leq$ Min(C) and $\kappa \leq$ Min$(\text{Reg} \backslash C)$ <u>then</u> in part (1) we can replace $\aleph_0$ by κ.

Demostración. Choose $\chi \in C, \kappa \in$ Reg$\backslash$C and apply 10. $\square_{28}$

Received: April 2009. Revised: December 2009.

Referencias

[Mak85] Johann A. Makowsky, *Compactnes, embeddings and definability, Model-Theoretic Logics (J. Barwise and S. Feferman, eds.), Springer-Verlag, 1985, pp. 645–716.*

[Sh:h] Saharon Shelah, *Classification Theory for Abstract Elementary Classes, Studies in Logic: Mathematical logic and foundations, vol. 18, College Publications, 2009.*

[Sh:3] *Finite diagrams stable in power, Annals of Mathematical Logic **2** (1970), 69–118.*

[Sh:43] *Generalized quantifiers and compact logic, Transactions of the American Mathematical Society **204** (1975), 342–364.*

[Sh:54] *The lazy model-theoretician's guide to stability, Logique et Analyse **18** (1975), 241–308.*

[Sh:88r] *Abstract elementary classes near* $\aleph_1$, *Chapter I Studies in Logic, College Publ. 18 (2009). 0705.4137. 0705.4137.*

[MkSh:166] Alan H. Mekler and Saharon Shelah, *Stationary logic and its friends. I, Notre Dame Journal of Formal Logic **26** (1985), 129–138, Proceedings of the 1980/1 Jerusalem Model Theory year.*

[Sh:199] Saharon Shelah, *Remarks in abstract model theory, Annals of Pure and Applied Logic **29** (1985), 255–288.*

CUBO A Mathematical Journal
Vol.13, N$^{\underline{o}}$02, (73–84). June 2011

Closure of Pointed Cones and Maximum Principle in Hilbert Spaces

Paolo d'Alessandro
Math. Department,
Third University of Rome,
email: `dalex@mat.uniroma3.it`

ABSTRACT

We prove, in a Hilbert space setting, that all targets of the minimum norm optimal control problems reachable with inputs of minimum norm ρ are support points for the the set reachable by inputs with norm bounded by ρ. This amount to say that the Maximum Principle always holds in Hilbert Spaces.

RESUMEN

En este artículo se demuestra que, para el problema de control óptimo a un nivel mínimo en los espacios de Hilbert, todos los estados alcanzables con un nivel mínimo de entrada de ρ son puntos de apoyo para el conjunto de estados alcanzables por la norma de entrada inferior o igual a ρ. Esto es equivalente a decir que el Principio Máximo siempre es válido en los espacios de Hilbert.

Keywords and phrases: Linear Control Systems in Hilbert Spaces, Norm Optimal Control, Maximum Principle.
Mathematics Subject Classification: 93E20, 93E25.

1. Introduction

Arguing in a Hilbert space setting, suppose that the minimum norm of inputs, which steer the origin to a state ζ in a finite interval of time $[0, \Gamma]$ is ρ. That the Maximum Principle holds, seen through the convex analysis optics, means that the target ζ is a support vector for the set R_ρ of all vectors reachable under an input of norm less than or equal to ρ. In other words the normal cone to R_ρ at ζ is non-trivial. In the literature vectors in the normal cone, which live in the dual space, are also called multipliers, a term that has a more general meaning.

To verify that R_ρ has support at ζ, one might hope to apply to R_ρ and $\{\zeta\}$ the celebrated Separation Theorem for linear topological spaces, stating that, given two non-void convex sets A and B, and assuming that A has interior, they can be can be separated by a continuous linear functional if and only if $B \cap A^i = \phi$.

Unfortunately, this application is not possible in the infinite dimensional case because it is not true in general that R_ρ has interior.

It is very well known, and easy to show, that the set of support points S is contained and is dense in the set $R_\rho^\wedge$ of target points reachable with minimum input norm ρ. This might suggest that some of the above targets are not support point. On the other hand the cited Separation Theorem above is indeed a sufficient condition, in view of the presiding hypothesis (one of the sets has interior), so that it leaves open the problem of determining if separation holds for all targets or if, instead, the dense subset of support points is proper.

Much attention (see [1]) has been devoted to more sophisticated Banach space settings, obtaining a generalization of the Maximum principle, thanks to the definition of a larger linear space of multipliers, which contains the dual space. Similarly, this leaves to determine which multipliers are in the dual space, although sufficient conditions are known.

For details as well as for accurate historical remarks and proper credits, reference can be made to the vast and outstanding work by Fattorini, which covers a variety of Banach space settings. Recent work is, besides [1], [6] and [7].

The purpose here is to answer for Hilbert spaces the question connected to the aforementioned density results: are there vectors of $R_\rho^\wedge$ at which R_ρ has no support?

The question remained open for quite a long time. And the answer is "no". The argument lean on a (very general) result of the theory of cones and on strict convexity of Hilbert space norms. Thus generalization are possible (although beyond the present purposes), but our technique cannot pass the barrier of the requirement of strict convexity of norms.

More specifically, we will show that the tangent cone to R_ρ at any $\zeta \in R_\rho^\wedge$ is pointed. That some vector $\zeta \in R_\rho^\wedge$ is not a support point for R_ρ is equivalent to say that the polar cone to the tangent cone to R_ρ at ζ is trivial. This is in turn equivalent to say that the closure of the tangent cone is the whole space. That the closure of a pointed cone be the whole space is a rather counter-intuitive proposition, and in fact we prove that this is not the case, in any linear topological space.

From this fact it follows, as an immediate consequence, that all $\zeta \in R_\rho^\wedge$ are support points for R_ρ.

No assumptions will be made on either separability or full control. More might be said in the separable case, but this is not dealt with here.

With an apology, our exposition covers some well known basic facts and complements, in order to enhance readability. A more succinct exposition would result in a choppy and difficult to follow narration.

2. Setting

We refer to the Cauchy problem with $u \in L_2([0,\Gamma], H_1)$ as given in [2]. The abstract differential equation has the form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where the spaces H_1, H are real Hilbert spaces, B is an operator (bounded linear transformation) $H_1 \to H$, A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}$ on H and $x(0) = \overline{x}$ is given. The equation is intended in weak sense and its unique solution is expressed by the formula of variation of constants:

$$x(t) = T(t)\overline{x} + \int_{[0,t]} T(t-s)Bu(s)ds$$

which formally recurs also in other settings, like that of differential equations in Banach Spaces. In our specific case the integral is a Pettis integral. The case $H_1 = H$ and $B = I$ is referred to as full control case, but here we do not make any assumption in this respect. For simplicity, dealing with norm optimal control problems, we assume $\overline{x} = 0$. The general case is a variant of this and can be obtained along the lines of [1].

It is assumed that we can reach a certain target vector $\zeta \in H$ at time $\Gamma > 0$ or:

$$\zeta = \int_{[0,\Gamma]} T(\Gamma - s)Bu(s)ds = \mathcal{L}_\Gamma u$$

and we look for the minimum norm input that does the job of reaching ζ. The linear transformation $\mathcal{L}_\Gamma$: $L_2([0,\Gamma], H_1) \to H$ is well known to be continuous. This is all we will need in the sequel as to the properties of this operator.

If the control steering the system from the origin at time 0 to ζ at time Γ is unique, because the operator $\mathcal{L}_\Gamma$ is one to one, then, according to Fattorini, who introduced this concept in sixties, the system is called rigid. Fattorini constructed an example of rigid system showing that this phenomenon can actually occur (see e.g. [1] and references therein). In this case, characterizing the optimum control, becomes obviously pointless. We note that our arguments hold good even when $\mathcal{L}_\Gamma$ is one to one. In this case though, exploiting the fact that ζ is a support point, can only lead to retrieving the unique control that solves the reachability problem.

It is immediate to show that the minimum norm control exists and is unique, since this is a straightforward consequence of the Projection Theorem.

3. Lemmata and Definitions

The environment is, unless otherwise stated, a real Hilbert space H. The symbols $\mathcal{L}(C)$ and $\mathcal{L}^{-}(C)$ denote, respectively, the linear and closed linear hull of the non-void set C, whereas $\mathcal{C}o(C)$ and $\mathcal{C}o^{-}(C)$ denote, respectively, its conical and closed conical hull.

Given a convex cone C in H its polar cone is the (always closed) cone:

$$C^{p} = \{y : (y, x) \leq 0 \colon \forall x \in C\}$$

Definition 1. The *tangent cone to* C *at* $y \in C$ *is the cone* $\mathcal{C}o(-y + C)$, *The normal cone to* C *at* y *is the polar of the tangent cone to* C *at* y.

Consider a convex set C and $\zeta \in C$. We say that ζ is a support point for C if there exists a nonzero vector n (which can be taken with unit norm) such that:

$$(n, \zeta) \geq (n, z), \ \forall z \in C$$

This is equivalent to say that C is contained in the closed half-space defined by the continuous linear functional $(n, .)$:

$$\{y : (n, y) \leq (n, \zeta)\}$$

whose limiting (closed) hyperplane contains $\{\zeta\}$. In still another equivalent terminology, this is equivalent to say that the tangent cone to C at ζ has a non-trivial polar cone (because n is in such cone).

Let n be an unit norm vector. The closed convex set:

$$\{x : a \leq (n, x) \leq b\}$$

(with $b \geq a$) is called a *sandwich. The number* $b - a$ *is the thickness of the sandwich. If the thickness is zero, the sandwich is a closed hyperplane. The sandwich is symmetrical if it has the form*

$$\{x : -a \leq (n, x) \leq a\}$$

for some $a \geq 0$.

The next Lemma regards closed cones in Hilbert spaces

Lemma 2. A closed cone in a Hilbert space H is proper if and only if it is contained in a closed half-space.

Demostración. Let C be a closed proper cone. Then there is a singleton $\{y\}$ disjoint from C. Singletons are convex and compact and therefore the Strong Separation Corollary 14.4 in [4] applies. The rest is immediate. □

We recall briefly some other relevant notions about cones in infinite dimensional Hilbert Spaces. Incidentally, the literature on this topic tend to be either finite dimensional or infinite dimensional, but typically on the footsteps of the seminal work of Choquet, aiming at extending the Krein Milman Theorem in a measure theory setting.

Definition 3. A (convex) cone C is *pointed if* $C \cap -C = \{0\}$. *A cone* C *is blunt if* $\mathcal{L}^{-}(C) = H$

The following Lemma well known.

Lemma 4. If a closed C cone is pointed, its polar cone is blunt, that is, $\mathcal{L}^{-}(C^{p}) = H$.

Demostración. The proof is simple and based on elementary computations that, for brevity, are taken for granted here, but, on the other hand, are rather intuitive, since polarization is the analogous for cones of orthogonal complementation for subspaces. We can write:

$$\{0\} = C \cap -C = [C^{p} + (-C^{p})]^{p} = \mathcal{L}(C^{p})^{p}$$

and, taking polars of the first and last cone the desired conclusion follows. □

If a pointed cone is not closed what can we say of its polar cone? Notice that according to Lemma 2, if the closure of the cone is proper, then the cone has a nontrivial polar cone. For the polar to be trivial instead, again in view of the same Lemma, the closure of the cone must be the whole space, despite the fact that the cone is pointed. But we prove here that this cannot happen even in general, as stated by the following:

Theorem 5. The closure of a pointed cone in a linear topological space is a proper cone.

Demostración. Suppose that it is not true, that is there is a pointed cone C in a linear topological space E, such that $C^{-} = E$. Consider a finite dimensional subspace F, which intersect C in a non trivial, necessarily pointed, cone. Actually we can take instead of F, its subspace $\mathcal{L}(F \cap C)$, without restriction of generality. For simplicity we leave the symbol F unchanged, and equip F with the relative topology. Next notice that, as is well known, because F is the finite dimensional, the pointed convex cone $\Upsilon = F \cap C$ has interior. Thus it can be separated by a continuous linear functional from the origin and therefore it is contained in a closed semi-space. It follows that the closure of Υ in F is contained in a closed half-space and therefore is a proper cone. But by Theorem 1.16 in [5], such closure is $C^{-} \cap F$. By the initial assumption $C^{-} \cap F = E \cap F = F$. This is a contradiction and therefore the proof is finished. □

4. Existence and uniqueness in $L_2(H_1)$

Assume that for some $\zeta \in H$, $\exists u_{\zeta}$ *such that* $x = \zeta = \mathcal{L}_{\Gamma}(u_{\zeta})$. *The set of all* u *satisfying* $\zeta = \mathcal{L}_{\Gamma}(u)$ *is given by* $u_{\zeta} + \mathcal{N}(\mathcal{L}_{\Gamma})$. *This is a closed affine space, because* $\mathcal{L}_{\Gamma}$ *is continuous. If* $\mathcal{N}(\mathcal{L}_{\Gamma})$ *is trivial, the unique* u_{ζ} *solving* $\zeta = \mathcal{L}_{\Gamma}(u_{\zeta})$ *is already optimum. Optimization in this case* *is pointless, but the arguments below hold good anyway. To obtain the minimum norm solution we can apply the projection theorem and project the origin on this closed convex set. Moreover, we know, from the celebrated Projection Theorem for Hilbert spaces, that this projection exists and is unique and hence the minimum norm solution always exists and is unique. We call this unique minimum norm control* u_o.

We can put:

$$\|u_o\|_{L_2(H_1)} = \rho$$

which is the optimum value of the norm. In particular we can say that the optimum control belongs to the closed sphere S_ρ *of radius* ρ*, around the origin in* $L_2([0,\Gamma], H_1)$ *(briefly* $L_2(H_1)$*):*

$$u_o \in S_\rho$$

and so:

$$\zeta \in \mathcal{L}_\Gamma(S_\rho) \subset \mathcal{R}(\mathcal{L}_\Gamma)$$

It is immediate to verify that:

$$\mathcal{L}_\Gamma(S_\rho) = \{z : \text{mín}\{\|u\| : \mathcal{L}_\Gamma(u) = z\} \leq \rho\}\}$$

For notational simplicity we put $\mathcal{L}_\Gamma(S_\rho) = R_\rho$.

5. Reachability

In this section we recast a few well known facts of reachability theory.

We noted that $\zeta \in \mathcal{L}_\Gamma(L_2(H_1)) = \mathcal{R}(\mathcal{L}_\Gamma) = R_\Gamma$. *This is the reachable set at time* Γ *and is a linear subspace of* H*, which is in general not closed. However, we can always argue in the Hilbert space* $\mathcal{L}_\Gamma(L_2(H_1))^-$ *and thus assume, without restriction of generality, that* R_Γ *is a dense linear subspace in* H. *This is equivalent to* $\mathcal{N}(\mathcal{L}_\Gamma^*) = \{0\}$. *To streamline the exposition this assumption will be in force thoroughly.*

Note that $\mathcal{L}_\Gamma^*$ *is the map* $x \to B^*T^*(\Gamma - .)x$. *Thus* $x \in \mathcal{N}(\mathcal{L}_\Gamma^*)$ *if and only if the continuous function* $B^*T^*(\Gamma - .)x$ *is identically zero. Next notice that:*

$$\mathcal{N}(\mathcal{L}_\Gamma^*) = \mathcal{N}(\mathcal{L}_\Gamma\mathcal{L}_\Gamma^*)$$

for $\mathcal{N}(\mathcal{L}_\Gamma^*) \subset \mathcal{N}(\mathcal{L}_\Gamma\mathcal{L}_\Gamma^*)$ *is obvious and, on the other end:*

$$\mathcal{L}_\Gamma\mathcal{L}_\Gamma^*x = 0 \Rightarrow (x, \mathcal{L}_\Gamma\mathcal{L}_\Gamma^*x) = \|\mathcal{L}_\Gamma^*x\|^2 = 0$$

Therefore, under the present hypothesis that $\mathcal{N}(\mathcal{L}_\Gamma^*)^\perp = R_\Gamma^- = H$*, the selfadjoint operator*

$$G_\Gamma = \mathcal{L}_\Gamma\mathcal{L}_\Gamma^*$$

has a trivial kernel, and so it is one to one.

Moreover, obviously $\mathcal{R}(G_\Gamma) \subset \mathcal{R}(\mathcal{L}_\Gamma) = R_\Gamma$.

Indeed, $\mathcal{R}(G_\Gamma)^- = R_\Gamma^-$, *for:*

$$\mathcal{R}(G_\Gamma)^- = \mathcal{N}(G_\Gamma)^\perp = \mathcal{N}(\mathcal{L}_\Gamma^*)^\perp = R_\Gamma^-$$

Thus both $\mathcal{R}(G_\Gamma)$ *and* $\mathcal{R}(\mathcal{L}_\Gamma)$ *are dense, so that for any* $x \in \mathcal{R}(G_\Gamma)$ *and* $\varepsilon > 0$ *there is an* $y \in \mathcal{R}(\mathcal{L}_\Gamma)$ *such that* $\|x - y\| \leq \varepsilon$ *and vice-versa.*

Notice that, by a change of variables, $\forall x \in H$*:*

$$G_\Gamma x = \int_0^\Gamma T(\sigma)BB^*T^*(\sigma)x d\sigma$$

We claim that we can define the integral as a Riemann integral. In fact we can prove that the integrand is a continuous function. To this purpose first note that the function $BB^*T^*(\sigma)x$ *is continuous. Then, to show that the integrand function is continuous, apply the exponential growth property of the semigroup and following well known:*

Theorem 6. Consider a set $\mathcal{T} \subset LC(H,H)$ and suppose that it is bounded in the norm operator topology. Then the evaluation map is jointly continuous in $\mathcal{T} \times H$ where $\mathcal{T}$ is equipped with the (relativized) pointwise topology.

If the state y *is reachable and* $y \in \mathcal{R}(G_\Gamma)$,

$$y = G_\Gamma x$$

has solution in x*, so that*

$$y = \int_0^\Gamma T(\sigma)BB^*T^*(\sigma)d\sigma x = \int_0^\Gamma T(\Gamma - \tau)Bw(\tau)d\tau$$

where $w(\tau) = B^*T^*(\Gamma - \tau)x$. *In this case, there is a smooth control that solves the problem.*

6. The Quasi-Topology of R_ρ and Main Theorem

There are a number of interesting properties of $R_\rho = \mathcal{L}_T(S_\rho)$*, which depend both on the environment (Hilbert space), on continuity of* $\mathcal{L}_T$ *and on the fact that the closed sphere* S_ρ *is convex and weakly compact. In particular recall that* $s - s$ *(strong-strong) continuity is equivalent to* $w - w$ *(weak-weak) continuity and, therefore,* $\mathcal{L}_T$ *is* $w - w$ *continuous.*

The set R_ρ *is obviously:*

- *Convex (as image under an operator of a convex set) with* $0 \in R_\rho$*, and symmetrical.*
- *Weakly compact, as image of a weakly compact set under a* $w-w$ *continuous linear transformation*
- *Weakly closed + convex and hence strongly closed*
- *Bounded (as image under an operator of a bounded set)*
- *Not a convex body in general.*

We now describe what we mean with the quasi-topology of R_ρ*, adding to the above list the fact that* R_ρ *has a "quasi-interior", at whose points it is densely radial and circled, and a quasi-boundary, which is the complement in* R_ρ *of the quasi-interior. These terms are justified because this quasi-topology can be realized as an actual topology, using the topology introduced by Fattorini for* R_ρ*, based on the norm:*

$$p(z) = \inf\{\|u\| : \mathcal{L}_\Gamma u = z\}$$

This concept is of primary importance in more general and complex Banach space settings, but it will not be used here.

Definition 7. The set of all vectors x in R_ρ, such that the minimum norm control to reach x has norm ρ, is called quasi-boundary of R_ρ and denoted by $R_\rho^\wedge$. The set of all vectors x in R_ρ, such that the minimum norm control to reach x has norm $\rho' < \rho$, is called quasi-interior of R_ρ and denoted by $R_\rho^\vee$.

It is well known that no point of the quasi-interior can be a support point for R_ρ*.*

Consider the origin, which belongs to $R_\rho^\vee$*. The set* R_ρ *is densely radial at the origin. In fact consider any point* $z \in R_\Gamma$ *and let* η *be the minimum norm of the unique control* u *that reaches* z*. If* $\eta \le \rho$ *the whole segment* $[0 : z] \subset \mathcal{L}_\Gamma(S_\rho)$*. Otherwise take the point* z' *reached by the control* $\frac{\rho}{\eta}u$*, and* $[0 : z'] \subset \mathcal{L}_\Gamma(S_\rho)$*. Observe that* $\forall z \in R_\Gamma$ *is positively proportional to a vector in* $\mathcal{L}_\Gamma(S_\rho)$*. In other words:*

$$\cup\{\alpha\mathcal{L}_\Gamma(S_\rho) : \alpha > 0\} = R_\Gamma$$

Moreover it is clear that $\alpha\mathcal{L}_\Gamma(S_\rho) \subset \mathcal{L}_\Gamma(S_\rho)$ *for any positive* $\alpha \le 1$*, so that* $\mathcal{L}_\Gamma(S_\rho)$ *is also circled.*

On the other hand, if for $\zeta \in \mathcal{L}_\Gamma(S_\rho)$ *the minimum norm of the corresponding control to reach* ζ *is* ρ*, by a very well known argument, see e.g. [1],* $(1+\varepsilon)\zeta \notin \mathcal{L}_\Gamma(S_\rho)$ *for* $\forall\varepsilon > 0$*.*

All the points of the quasi-interior have the same dense radiality property as the origin. In fact, if $\xi \in R_\rho^\vee$*,* $\xi \ne 0$*, then the minimum norm of the control that steers the origin to the state to* ξ *is some* ρ' *with* $0 < \rho' < \rho$*. Let* u_ξ *be the corresponding minimum norm control. Radiality in the direction of the origin or of* ξ *itself is obvious. Next consider any* $z \in R_\Gamma$ *with* $z \ne 0$ *and* z *not proportional to* ξ*, and let* u_z *be corresponding the minimum norm control. It will be* $\|u_z\| = \gamma > 0$*. Then it is immediate that* $\xi + \alpha z \in R_\rho$ *for any* α*, such that:* $0 < \alpha \le \frac{\rho-\rho'}{\gamma}$*.*

The role of the dense radiality is emphasized by the proof of the anticipated well known result:

Theorem 8. No point of $R_\rho^\vee$ can be a support point of R_ρ.

Demostración. Suppose that a point $y \in R_\rho^\vee$ is a support point. Then the projection of some vector $z \notin R_\rho$ on R_ρ is y. By the Projection Theorem we have:

$$(z-y, x-y) \leq 0,\ \forall x \in R_\rho$$

If equality holds for all $x \in R_\rho$ then R_ρ and hence also R_Γ would be contained in a closed hyperplane, contradicting that R_Γ is dense in the space. So $\exists x \in R_\rho$ such that $(z-y, x-y) < 0$. Because of the radiality property at y we can take in lieu of x, a state $w = y + \alpha(y-x) \in R_\rho$ for some $\alpha > 0$. But then $(z-y, w-y) = \alpha(z-y, y-x) > 0$ contradicting the Projection Theorem. □

At this point we know that any support point is in $R_\rho^\wedge$.

It is well known that support points are dense in the quasi-boundary of R_ρ *(e.g.[2]).*

To show this, one may use the following sequence of support points converging to an arbitrary $\zeta \in R_\rho^\wedge$. *For any positive integer* i, $(1+\frac{1}{i})\zeta \notin R_\rho$. *Because the projection* P_{R_ρ} *on* R_ρ *is continuous, the sequence* $\{P_{R_\rho}((1+\frac{1}{i})\zeta)\}$ *converges strongly to* ζ *and obviously, because the points in the sequence are projections, they are all support points (and hence also lie on the quasi-boundary of* R_ρ*).*

However, more is true for arbitrary vectors of the quasi-boundary of R_ρ *in the present Hilbert space setting, as we show in the next:*

Theorem 9. All vectors in $R_\rho^\wedge$ are extreme. No other point of R_ρ can be extreme

Demostración. Suppose that for the quasi-boundary vector ζ it is true that $\zeta = \frac{\zeta_1+\zeta_2}{2}$ with ζ_1 and ζ_2 in R_ρ, and let $\rho_1 \leq \rho$ be the norm of the minimum norm control u_1 that steers the system to ζ_1 and $\rho_2 \leq \rho$ be the norm of the minimum norm control u_2 that steers the system to ζ_2. The control $\frac{u_1+u_2}{2}$ has norm strictly less than ρ, because the norm of a Hilbert space is strictly convex, and steers the system to ζ. But this contradicts that $\zeta \in R_\rho^\wedge$, and so the proof of the first statement is finished. Points of the quasi-interior cannot be extreme in view of the specific dense radiality property we have illustrated. Therefore we are done. □

Corollary 10. The tangent cone to R_ρ at any $\zeta \in R_\rho^\wedge$ is pointed

Demostración. Obviously if it were not so it would be contradicted that ζ is an extreme point. □

At this point, putting together this Corollary with Lemmata 5 and 2 we have established the following main

Theorem 11. Any point $\zeta \in R_\rho^\wedge$ is a support point of R_ρ. In other words

$$\forall \zeta \in R_\rho^\wedge,\ \exists n \in H^* = H,\ \|n\| = 1$$

such that

$$(n,\zeta) \geq (n,z),\ \forall z \in C$$

Given this result we may expect all the more a large normal fan for R_ρ. Indeed the normal fan is the whole space, but this fact is independent on the main Theorem, as we illustrate in the next section.

7. The Set of Support Points and the Normal Fan

In this Section we collect some, mostly well known, facts about the set of support points S (which we now know to be the same as $R_\rho^\wedge$) and the normal fan. At each support point ζ the tangent cone has a non-trivial polar cone, also called the normal cone at ζ. The union of these cones is the normal fan of R_ρ. Naturally, it is often convenient to normalize vectors in the normal fan.

Suppose ζ is a support point of R_ρ. Using the pairing in H and the CBS inequality, we have a well known expression for ζ. Let n a unit norm vector in the normal cone at ζ. Then it must be for any $y \in R_\rho$:

$$(n, y-\zeta) = (n, \mathcal{L}_\Gamma u_y - \mathcal{L}_\Gamma u_\zeta) \leq 0$$

or

$$(n, \mathcal{L}_\Gamma u_y) \leq (n, \mathcal{L}_\Gamma u_\zeta)$$

or

$$(\mathcal{L}_\Gamma^* n, u_y) \leq (\mathcal{L}_\Gamma^* n, u_\zeta)$$

so that, by the CBS inequality, the optimal control corresponding to ζ has the expression:

$$u_\zeta = \rho \frac{\mathcal{L}_\Gamma^* n}{\|\mathcal{L}_\Gamma^* n\|},\ \text{a.e.}$$

and:

$$\zeta = \mathcal{L}_\Gamma \frac{\rho}{\|\mathcal{L}_\Gamma^* n\|} \mathcal{L}_\Gamma^* n = \frac{\rho}{\|\mathcal{L}_\Gamma^* n\|} G_\Gamma n$$

On the other hand, for an arbitrary unit norm vector n consider the linear program:

$$\text{máx}\{(n,z) : z \in R_\rho\}$$

The existence of solutions of this linear program derive from the fact that R_ρ is convex and weakly compact and the functional is continuous (and hence also weakly continuous). Let ζ be a solution, then for any $y \in R_\rho$:

$$(n, y-\zeta) \leq 0$$

so that ζ is a support point and has the above expression.

Thus

$$S = \{\zeta : \zeta = \frac{\rho}{\|\mathcal{L}_\Gamma^* n\|} G_\Gamma n,\ \|n\| = 1\}$$

Notice that because G_Γ *is one-to-one, to each support* ζ *there corresponds a unique normal* n*. In other words, all normal cones are rays. We collect these facts and more in the following:*

Theorem 12. The normal fan of R_ρ, less the origin, is the whole space $H\backslash\{0\}$. All normal cones at support vectors ζ are rays, under the correspondence between unit normal vectors n and support points ζ:

$$\zeta = \frac{\rho}{\|\mathcal{L}_\Gamma^* n\|} G_\Gamma n$$

The correspondence between unit vectors in the normal fan and the corresponding support points is one-to-one. If ζ is a support point $\{\zeta\}$ is an exposed face of R_ρ. Equivalently , for any pair normal-support point it is true that:

$$(n, y - \zeta) < 0,\ \forall y \in R_\rho$$

Consequently (since $0 \in R_\rho$) the value of the functional $(n, .)$ at ζ is always positive.

Demostración. It remains to be proved the last statement. Suppose a $\zeta \in R_\rho$ is a support point, so that it is necessarily a quasi-boundary and extreme point. Suppose that for some normal n there is z such that:

$$(n, z - \zeta) = 0$$

then z is a support point as well with normal n. In fact, $\forall y \in R_\rho$:

$$(n, y - z) = (n, y - \zeta) + (n, \zeta - z) \leq 0$$

Therefore z is a support point and hence it is an quasi-boundary point, so that the minimum norm control that steers the system to z has norm ρ. Denote by u_z the minimum norm control to reach z and by u_ζ the minimum norm control to reach ζ. By the same argument above all points in $[\zeta : z]$ are support points with normal n. In fact for $0 < \alpha < 1$, $\forall y \in R_\rho$:

$$(n, y - (\zeta + \alpha(z - \zeta))) \leq 0$$

But the point $\frac{\zeta+z}{2}$ can be reached by the control $\frac{u_\zeta+u_z}{2}$ and $\|\frac{u_\zeta+u_z}{2}\| < \rho$. This means that $\frac{\zeta+z}{2}$ is a quasi-interior point, which is a contradiction and the proof is finished. □

Returning to the linear program:

$$\text{máx}\{(n, z) : z \in R_\rho\}$$

Call the maximum m*, then*

$$m = \frac{\rho}{\|\mathcal{L}_\Gamma^* n\|}(n, G_\Gamma n) = \rho(n, G_\Gamma n)^{1/2} = \rho\|\mathcal{L}_\Gamma^* n\|^{1/2}$$

Recall that $m > 0$. *Because* R_ρ *is symmetric, it is contained in the symmetrical sandwich:*

$$R_\rho \subset \{y : -m \leq (n, y) \leq m\}$$

and the limiting hyperplanes of the sandwich meet R_ρ *only in the two points* ζ *and* $-\zeta$. *Also the sandwich cannot degenerate to a hyperplane, because its thickness is* $2m > 0$.

Received: October 2009. Revised: January 2010.

Referencias

[1] H.O. Fattorini, *Infinite Dimensional Linear Control Systems, Elsevier, Amsterdam 2005.*

[2] A.V. Balakrishnan, *Applied Functional Analysis,Springer-Verlag Berlin-Heidelberg-New York 1976.*

[3] P.R. Halmos, *A Hilbert Space Problem Book Van Nostrand, New York, 1967.*

[4] J.L. Kelley and I. Namioka, *Linear Topological Spaces, Springer, New York, 1963*

[5] J.L. Kelley, *General Topology, Springer, New York, 1955.*

[6] H.O. Fattorini *Smoothness of the Costate and the Target in the Time and Norm Optimal Problems Optimization, Vol.55, No.2, 2006, 19-36*

[7] H.O. Fattorini *Regular and Strongly Regular Time and Norm Optimal Controls to appear*

CUBO A Mathematical Journal
Vol.13, N^{o}02, (85–117). June 2011

Differential forms versus multi-vector functions in Hermitean Clifford analysis

F. Brackx, H. De Schepper and V. Souček
Ghent University
Faculty of Engineering
Department of Mathematical Analysis
Gent, Belgium
email: fb@cage.ugent.be
and
F. Brackx, H. De Schepper and V. Souček
Charles University
Faculty of Mathematics and Physics
Praha, Czech Republic

ABSTRACT

Similarities are shown between the algebras of complex differential forms and of complex Clifford algebra-valued multi-vector functions in an open region of Euclidean space of even dimension.

RESUMEN

Se presentan las similitudes entre las álgebras de formas diferenciales complejas y de las funciones de álgebras de Clifford complejas con valores de múltiples vectores aplicados en una región abierta del espacio euclidiano de dimensión par.

Keywords and phrases: complex differential forms, multi-vector functions, Hermitean Clifford analysis.

Mathematics Subject Classification: 30G35

1 Introduction

Usually Clifford analysis is understood to be the study of the solutions of the Dirac equation for functions defined on the (anti-)Euclidean vector space $\mathbb{R}^{0,m}$ and taking values in the corresponding Clifford algebra $\mathbb{R}_{0,m}$. It thus offers a proper analogue to the Cauchy-Riemann equations for holomorphic functions in the complex plane. For a thorough study of the so-called monogenic functions of Clifford analysis we refer to the standard textbooks [5, 15, 17, 18].

The symmetry group of the Dirac equation is either SO(m) or Spin(m), according to the definition of the group action on the values taken by the functions under consideration. If these values are in the Clifford algebra with left multiplication, the symmetry group is Spin(m), which then usually is realized inside the Clifford algebra. In the case of functions with values in the Clifford algebra with both side action, it is more natural to identify the Dirac equation with the equation $(d + d^*)f = 0$, and to identify the space of values, in casu the Clifford algebra, as a vector space, with the Grassmann algebra of $\mathbb{R}^m$. This Grassmann algebra may then be decomposed into the direct sum of its homogeneous parts, which is a decomposition into irreducible parts under the action of SO(m). In this framework it was shown (see [13]) that, on the polynomial level, the space of monogenic functions can be split into a direct sum of solutions of the Hodge-de Rham equations for homogeneous differential forms. This entails a finer structure of the space of monogenic functions, which manifests itself explicitly in a finer form of the corresponding Fischer decomposition (see [14]).

An important ingredient in the latter approach is the translation of spaces and operators from the language of multivector functions with values in a Clifford algebra to the language of real differential forms, as was described in detail in [6]. Let us give a very brief overview. On the one hand we have the Cartan algebra $\bigwedge(G)$ of smooth real differential forms in an open subset G of Euclidean space $\mathbb{R}^m$, endowed with exterior multiplication. A fundamental operator on $\bigwedge(G)$ is the exterior derivative d with its important property that for any differential form ω, $d^2\omega = d(d\omega) = 0$. Introducing the Hodge co-derivative d^* leads to the differential operator $D = d + d^*$, by means of which the so-called "harmonic" r-forms $(0 < r < m)$ are characterized as smooth differential r-forms ω^r satisfying $D\omega^r = 0$. On the other hand we have the algebra $\mathcal{E}(G)$ of smooth multi-vector functions in G. Multi-vector functions arise in a natural way when considering functions defined in G and taking values in the universal real Clifford algebra $\mathbb{R}_{0,m}$ constructed over $\mathbb{R}^{0,m}$, i.e. $\mathbb{R}^m$ equipped with an anti-Euclidean metric. If $\mathbb{R}^r_{0,m}$ $(0 \leq r \leq m)$ denotes the space of r-vectors, then the Clifford algebra $\mathbb{R}_{0,m}$ is precisely the associative algebra $\mathbb{R}_{0,m} = \bigoplus_{r=0}^m \mathbb{R}^r_{0,m}$, and an r-vector function F_r is a map $F_r : G \rightarrow \mathbb{R}^r_{0,m}$. A fundamental operator on the space of smooth multi-vector functions, is the rotation-invariant Dirac operator $\partial_{\underline{X}}$, by means of which the so-called monogenic functions are characterized as the smooth functions f satisfying $\partial_{\underline{X}} f = 0$, as already mentioned above. The spaces of smooth differential forms and of smooth multi-vector functions were shown to be isomorphic in a natural way: a smooth r-form is identified with a smooth r-vector function, and the action of the differential operator $d + d^*$ on the space $\bigwedge^r(G)$ of smooth r-forms, is identified with the action of the Dirac operator $\partial_{\underline{X}}$ on the space $\mathcal{E}_r(G)$ of smooth r-vector functions. Also other correspondences were studied in detail in [6].

When the dimension is taken to be even ($m = 2n$), one can make the framework of Clifford analysis closer to complex analysis by introducing on $\mathbb{R}^{2n}$ a complex structure J. The symmetry group then reduces to the subgroup $U(n) \subset SO(2n)$ preserving the chosen complex structure J. This is the basic setting for so-called Hermitean Clifford analysis, which recently has emerged as a new and successful branch of Clifford analysis, offering yet a refinement of the Euclidean case. The functions studied are defined in open regions of $\mathbb{C}^n$ and take their values in the complex Clifford algebra $\mathbb{C}_{2n}$. More particularly Hermitean Clifford analysis focusses on the simultaneous null solutions, called Hermitean (or h–) monogenic functions, of two Hermitean Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}^\dagger}$. A systematic development of this function theory, including the invariance properties with respect to the underlying Lie groups and Lie algebras, is still in full progress, see e.g. [9, 1, 2, 7, 8, 3, 4, 23, 12]. Part of this program also concerns the study of the finer structure induced on the space of monogenic functions by the choice of the complex structure J.

When studying the Dirac equation for functions with values in a Clifford algebra, it is well known that the Clifford algebra can be split into the direct sum of a number of isomorphic copies of the basic spinor representation. Accordingly, the set of equations will split into a number of independent subsets of equations for functions with values in the various copies of spinor space. It is a trivial observation that all these subsystems are equivalent to each other and their solutions will have the same properties, whence, without any loss of generality, we can restrict the study to functions with values in the space of spinors (or half-spinors in even dimension). In the standard situation, this space of values cannot be split further since they are already irreducible under the (left) action of the $\mathrm{Spin}(m)$ group. However, after having fixed the complex structure J, the symmetry group is reduced, as explained above, and the spinor space decomposes further into smaller pieces. If it is realized in a standard way as the Grassmann algebra over the maximal isotropic subspace in $\mathbb{C}^{2n}$, then this splitting is just the splitting into homogeneous components; for details see e.g. [2].

Our final aim is to understand the finer structure of the space of monogenic functions induced by this splitting. A first step towards that goal is to establish a scheme for the translation of spaces and operators between the language of complex Clifford algebra and the language of complex differential forms. In fact this is the complex analogue of the translation in the Euclidean situation mentioned above, see [6]. The purpose of the underlying paper is precisely to describe in a rather formal, yet detailed, way the similarities between complex differential forms in open regions of $\mathbb{C}^n$ on the one hand and multivector functions in the Hermitean Clifford analysis setting on the other. Crucial to this description is the detailed analysis of the structure of complex Clifford algebra as carried out in [10]. The Hermitean Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}^\dagger}$ and the associated operators $\partial_{\underline{z}}\bullet$, $\partial_{\underline{z}}\wedge$, $\partial_{\underline{z}^\dagger}\bullet$ and $\partial_{\underline{z}^\dagger}\wedge$, originating by splitting the Clifford or geometric product into its "inner" or "dot" and "outer" or "wedge" parts, are identified with well-known differential operators for complex differential forms on Kählerian manifolds in $\mathbb{C}^n$. However it should be emphasized that, in this paper, we restrict ourselves to the flat Kählerian metric on $\mathbb{C}^n$ with fundamental form $\Omega = \frac{i}{2}\,\partial\overline{\partial}|\underline{z}|^2$. The more general approach of Hermitean Clifford analysis on complex Hermitean manifolds and its comparison with complex analysis on Kählerian manifolds is the subject of the forthcoming paper [11].

The paper is structured as follows. Sections 2 and 3 are introductory, fixing our definitions and notations. An identification of all differential operators and forms under consideration in both pictures is described in Section 4. The relation to the operators which are standard in Kählerian geometry is clarified in Section 5. The last section adds some remarks on the Hodge operator.

2 Multi-vector functions: preliminaries

In this section we recall some basic notions and results from Clifford algebra and Clifford analysis.

The construction of the universal real Clifford algebra is well-known; for an in-depth study we refer the reader to e.g. [22]. Here we restrict ourselves to a schematic approach. Let $\mathbb{R}^{0,m}$ be the real vector space $\mathbb{R}^m$ $(m \geq 1)$ endowed with a non-degenerate symmetric bilinear form $\mathcal{B}$ of signature $(0, m)$, and let $(e_1, \ldots, e_m)$ be an associated orthonormal basis, i.e.

$$\mathcal{B}(e_i, e_j) = \begin{cases} -1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \qquad (1 \leq i, j \leq m)$$

then the anti-Euclidean metric on $\mathbb{R}^{0,m}$ is induced by the scalar product

$$\langle e_i, e_j \rangle = -\mathcal{B}(e_i, e_j) = \delta_{ij}, \quad 1 \leq i, j \leq m$$

We first introduce the anti-symmetric outer product by the rules

$$\begin{aligned} e_i \wedge e_i &= 0, \quad 1 \leq i \leq m \\ e_i \wedge e_j + e_j \wedge e_i &= 0, \quad 1 \leq i \neq j \leq m \end{aligned}$$

and for each $A = \{i_1, i_2, \ldots, i_r\} \subset M = \{1, \ldots, m\}$, with $1 \leq i_1 < i_2 < \ldots < i_r \leq m$, i.e. ordered in the natural way, we put

$$e_A = e_{i_1} \wedge e_{i_2} \wedge \; \ldots \; \wedge e_{i_r}$$

while $e_\emptyset = 1$. Then for each $r = 0, 1, ..., m$, the set $\{e_A : A \subset M \text{ and } |A| = r\}$ is a basis for the space $\mathbb{R}^r_{0,m}$ of so-called r-*vectors*. Next, we introduce the inner product

$$e_i \bullet e_j = -\langle e_i, e_j \rangle = \mathcal{B}(e_i, e_j) = -\delta_{ij}, \quad 1 \leq i, j \leq m$$

leading to the so-called *geometric product* of vectors in the Clifford algebra:

$$e_i e_j = e_i \bullet e_j + e_i \wedge e_j, \quad 1 \leq i, j \leq m$$

The respective definitions of the inner, the outer and the geometric product are then extended to r-vectors as follows: for the inner product, we have

$$e_j \bullet e_A = e_j \bullet (e_{i_1} \wedge \ ... \ \wedge e_{i_r}) = \sum_{k=1}^{r} (-1)^k \delta_{j i_k} e_{A \setminus \{i_k\}}$$

with

$$e_{A \setminus \{i_k\}} = e_{i_1} \wedge \ ... \ \wedge e_{i_{k-1}} \wedge \ [e_{i_k} \wedge] \ e_{i_{k+1}} \wedge \ ... \ \wedge e_{i_r}$$

while for the outer product

$$\begin{cases} e_j \wedge e_A & = \quad e_j \wedge (e_{i_1} \wedge \ ... \ \wedge e_{i_r}) = e_j \wedge e_{i_1} \wedge \ ... \ \wedge e_{i_r}, \qquad \text{if } j \notin A \\ e_j \wedge e_A & = \quad 0, \qquad \text{if } j \in A \end{cases}$$

and finally, for the geometric product (or product for short)

$$e_j e_A = e_j \bullet e_A + e_j \wedge e_A$$

Finally, these definitions are linearly extended to the whole of the Clifford algebra $\mathbb{R}_{0,m}$, which is the associative algebra

$$\mathbb{R}_{0,m} = \bigoplus_{r=0}^{m} \mathbb{R}^r_{0,m}$$

If $[\cdot]_r : \mathbb{R}_{0,m} \to \mathbb{R}^r_{0,m}$ denotes the projection operator from $\mathbb{R}_{0,m}$ onto $\mathbb{R}^r_{0,m}$, then each Clifford number $a \in \mathbb{R}_{0,m}$ may be written as

$$a = \sum_{r=0}^{m} [a]_r$$

Note that in particular for a 1-vector u and an r-vector v_r, one has

$$u v_r = u \bullet v_r + u \wedge v_r$$

with

$$\begin{aligned} u \bullet v_r &= [u v_r]_{r-1} = \frac{1}{2}\Big(u v_r - (-1)^r v_r u\Big) \\ u \wedge v_r &= [u v_r]_{r+1} = \frac{1}{2}\Big(u v_r + (-1)^r v_r u\Big) \end{aligned}$$

Usually $\mathbb{R}$ and $\mathbb{R}^m$ are identified with $\mathbb{R}^0_{0,m}$ and $\mathbb{R}^1_{0,m}$ respectively. An element $X = (X_1, \ldots, X_m) \in \mathbb{R}^m$ is thus identified with the 1-vector $\underline{X} = \sum_{j=1}^m X_j e_j$.

Now let G be an open region in $\mathbb{R}^m$. A smooth r-*vector function* F_r is a map

$$F_r : G \longrightarrow \mathbb{R}^r_{0,m}, \ \underline{X} \mapsto \sum_{|A|=r} F_{r,A}(\underline{X})\, e_A$$

where for each A, $F_{r,A}$ is a smooth real valued function in G. We denote by $\mathcal{E}_r(G)$ the space of smooth r-vector functions in G, and we put

$$\mathcal{E}(G) = \bigoplus_{r=0}^{m} \mathcal{E}_r(G)$$

The projection operator from $\mathcal{E}(G)$ onto $\mathcal{E}_r(G)$ is denoted by $[\ .\ ]_r$.

A fundamental operator in Clifford analysis is the so-called *Dirac operator*, a first order vector valued differential operator given by

$$\partial_{\underline{X}} = \sum_{j=1}^{m} e_j\, \partial_{X_j}$$

Since the multiplication in the Clifford algebra is non-commutative, operators can act from the left or from the right on a function. For the Dirac operator and a function $F = \sum_A e_A F_A \in \mathcal{E}(G)$, these actions are given by

$$\partial_{\underline{X}} F = \sum_{j=1}^{m} \sum_{A} e_j e_A\, \partial_{X_j} F_A \quad \text{and} \quad F \partial_{\underline{X}} = \sum_{j=1}^{m} \sum_{A} e_A e_j\, \partial_{X_j} F_A$$

A function $F \in \mathcal{E}(G)$ is called *left* (resp. *right*) *monogenic* in G if and only if it satisfies in G the equation $\partial_{\underline{X}} F = 0$ (resp. $F\partial_{\underline{X}} = 0$).

Restricting the Dirac operator $\partial_{\underline{X}}$ to the space $\mathcal{E}_r(G)$, we find for an r-vector function F_r that $\partial_{\underline{X}} F_r$ and $F_r \partial_{\underline{X}}$ split into an $(r-1)$-vector part and an $(r+1)$-vector part:

$$\begin{aligned} \partial_{\underline{X}} F_r &= \sum_{j=1}^{m} e_j\, \partial_{X_j} F_r = \sum_{j=1}^{m} e_j \bullet \partial_{X_j} F_r + \sum_{j=1}^{m} e_j \wedge \partial_{X_j} F_r \\ F_r \partial_{\underline{X}} &= \sum_{j=1}^{m} \partial_{X_j} F_r\, e_j = \sum_{j=1}^{m} \partial_{X_j} F_r \bullet e_j + \sum_{j=1}^{m} \partial_{X_j} F_r \wedge e_j \end{aligned}$$

It readily follows that

$$\begin{aligned}\left[\partial_{\underline{X}} F_r\right]_{r-1} &= \sum_{j=1}^{m} e_j \bullet \partial_{X_j} F_r = (-1)^{r+1} \sum_{j=1}^{m} \partial_{X_j} F_r \bullet e_j = (-1)^{r+1} \left[F_r \partial_{\underline{X}}\right]_{r-1} \\ \left[\partial_{\underline{X}} F_r\right]_{r+1} &= \sum_{j=1}^{m} e_j \wedge \partial_{X_j} F_r = (-1)^{r} \sum_{j=1}^{m} \partial_{X_j} F_r \wedge e_j = (-1)^{r} \left[F_r \partial_{\underline{X}}\right]_{r+1}\end{aligned}$$

Usually one introduces the notations

$$\begin{aligned}\partial_{\underline{X}} \bullet F_r &= [\partial_{\underline{X}} F_r]_{r-1}, \quad \partial_{\underline{X}} \wedge F_r = [\partial_{\underline{X}} F_r]_{r+1} \\ F_r \bullet \partial_{\underline{X}} &= [F_r \partial_{\underline{X}}]_{r-1}, \quad F_r \wedge \partial_{\underline{X}} = [F_r \partial_{\underline{X}}]_{r+1}\end{aligned}$$

The action of the Dirac operator $\partial_{\underline{X}}$ on $\mathcal{E}_r(G)$ thus gives rise to two auxiliary differential operators

$$\begin{aligned}\partial_{\underline{X}} \bullet : \mathcal{E}_r(G) \to \mathcal{E}_{r-1}(G);\ F_r &\mapsto (\partial_{\underline{X}} \bullet) F_r = \partial_{\underline{X}} \bullet F_r = [\partial_{\underline{X}} F_r]_{r-1} \\ \partial_{\underline{X}} \wedge : \mathcal{E}_r(G) \to \mathcal{E}_{r+1}(G);\ F_r &\mapsto (\partial_{\underline{X}} \wedge) F_r = \partial_{\underline{X}} \wedge F_r = [\partial_{\underline{X}} F_r]_{r+1}\end{aligned}$$

for which it holds that

$$\partial_{\underline{X}} = \partial_{\underline{X}} \bullet + \partial_{\underline{X}} \wedge$$

Symbolically these operators may be written as

$$\begin{aligned}(\partial_{\underline{X}} \bullet) &= \sum_{j=1}^{m} (e_j \bullet) \partial_{x_j} \\ (\partial_{\underline{X}} \wedge) &= \sum_{j=1}^{m} (e_j \wedge) \partial_{x_j}\end{aligned}$$

Their action on $\mathcal{E}_r(G)$ is two-fold in the sense that they act on the multi-vector by means of the inner and outer product with basis vectors, and at the same time on the function coefficients by partial differentiation. We thus have that, for a smooth r-vector function F_r, the notions of left monogenicity and right monogenicity coincide, and moreover that F_r is left as well as right monogenic in G if and only if in G

$$\partial_{\underline{X}} F_r = 0 \iff F_r \partial_{\underline{X}} = 0 \iff \begin{cases} \partial_{\underline{X}} \bullet F_r = 0 \\ \partial_{\underline{X}} \wedge F_r = 0 \end{cases}$$

As the Dirac operator $\partial_{\underline{X}}$ factorizes the Laplace operator, viz

$$\partial_{\underline{X}}^2 = \partial_{\underline{X}} \bullet \partial_{\underline{X}} + \partial_{\underline{X}} \wedge \partial_{\underline{X}} = \partial_{\underline{X}} \bullet \partial_{\underline{X}} = -\langle \partial_{\underline{X}}, \partial_{\underline{X}} \rangle = -\Delta_m$$

a monogenic function in G is also harmonic in G, but the converse clearly is not true. As moreover

$$(\partial_{\underline{X}} \bullet)^2 = (\partial_{\underline{X}} \wedge)^2 = 0$$

we have that

$$-\Delta_m = (\partial_{\underline{X}} \bullet + \partial_{\underline{X}} \wedge)^2 = \partial_{\underline{X}} \bullet \partial_{\underline{X}} \wedge + \partial_{\underline{X}} \wedge \partial_{\underline{X}} \bullet$$

the two second order differential operators $\left(\partial_{\underline{X}} \bullet \partial_{\underline{X}} \wedge\right)$ and $\left(\partial_{\underline{X}} \wedge \partial_{\underline{X}} \bullet\right)$ arising above being scalar operators in the sense that they keep the order of the multi-vector function invariant. However the function coefficients, while being differentiated, are interchanged w.r.t. the basis multi-vectors.

When allowing for complex constants and moreover taking the dimension to be even: $m = 2n$, the same generators $(e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n})$ produce the complex Clifford algebra $\mathbb{C}_{2n}$, which is the complexification of the real Clifford algebra $\mathbb{R}_{0,2n}$, i.e. $\mathbb{C}_{2n} = \mathbb{R}_{0,2n} \oplus i\,\mathbb{R}_{0,2n}$. Any complex Clifford number $\lambda \in \mathbb{C}_{2n}$ may thus be written as $\lambda = a + ib$, $a, b \in \mathbb{R}_{0,2n}$, an observation leading to the definition of the Hermitean conjugation $\lambda^\dagger = (a + ib)^\dagger = \overline{a} - i\overline{b}$, where the bar notation stands for the usual Clifford conjugation in $\mathbb{R}_{0,2n}$, i.e. the main anti–involution for which $\overline{e}_j = -e_j$, $j = 1, \ldots, 2n$. This Hermitean conjugation also leads to a Hermitean inner product and its associated norm on $\mathbb{C}_{2n}$ given by $(\lambda, \mu) = [\lambda^\dagger \mu]_0$ and $|\lambda| = \sqrt{[\lambda^\dagger \lambda]_0} = (\sum_A |\lambda_A|^2)^{1/2}$.

This is the framework for so–called Hermitean Clifford analysis, a refinement of Euclidean Clifford analysis. An elegant way of introducing this setting consists in considering a so–called complex structure, i.e. a specific $\mathrm{SO}(2n;\mathbb{R})$–element J for which it holds that $J^2 = -\mathbf{1}$ (see [1, 2]). Here, J is chosen to act upon the generators $e_1, \ldots, e_{2n}$ of the Clifford algebra as

$$J[e_j] = -e_{n+j} \qquad \text{and} \qquad J[e_{n+j}] = e_j, \qquad j = 1, \ldots, n$$

With J one may associate two projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ which produce the main objects of the Hermitean setting by acting upon the corresponding objects in the Euclidean framework. First of all, the so–called Witt basis elements $(\mathfrak{f}_j, \mathfrak{f}_j^\dagger)_{j=1}^n$ for $\mathbb{C}_{2n}$ are obtained through the action of $\pm\frac{1}{2}(\mathbf{1} \pm iJ)$ on the original orthogonal basis:

$$\begin{aligned} \mathfrak{f}_j &= \frac{1}{2}(\mathbf{1} + iJ)[e_j] = \frac{1}{2}(e_j - i\,e_{n+j}), \quad j = 1, \ldots, n \\ \mathfrak{f}_j^\dagger &= -\frac{1}{2}(\mathbf{1} - iJ)[e_j] = -\frac{1}{2}(e_j + i\,e_{n+j}), \quad j = 1, \ldots, n \end{aligned}$$

The Witt basis elements satisfy the Grassmann identities

$$\mathfrak{f}_j \mathfrak{f}_k + \mathfrak{f}_k \mathfrak{f}_j = \mathfrak{f}_j^\dagger \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j^\dagger = 0 \quad , \quad j, k = 1, \ldots, n$$

including their isotropy: $\mathfrak{f}_j^2 = {\mathfrak{f}_j^\dagger}^2 = 0$, $j = 1, \ldots, n$, as well as the duality identities

$$\mathfrak{f}_j \mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger \mathfrak{f}_j = \delta_{jk} \quad , \quad j, k = 1, \ldots, n$$

The Witt basis of the complex Clifford algebra $\mathbb{C}_{2n}$ is then obtained, in much the same way as is done for the basis of the real Clifford algebra, by taking all possible products of Witt basis vectors.

Introducing the inner and outer products for the Witt basis vectors we have, see also [9],

$$\begin{aligned} \mathfrak{f}_j \bullet \mathfrak{f}_k &= \mathfrak{f}_j^\dagger \bullet \mathfrak{f}_k^\dagger = 0, \qquad j,k=1,\ldots,n \\ \mathfrak{f}_j \bullet \mathfrak{f}_k^\dagger &= \mathfrak{f}_j^\dagger \bullet \mathfrak{f}_k = \frac{1}{2}\delta_{jk}, \qquad j,k=1,\ldots,n \end{aligned}$$

and

$$\begin{aligned} \mathfrak{f}_j \wedge \mathfrak{f}_k &= -\mathfrak{f}_k \wedge \mathfrak{f}_j, \qquad j,k=1,\ldots,n \\ \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_k^\dagger &= -\mathfrak{f}_k^\dagger \wedge \mathfrak{f}_j^\dagger, \qquad j,k=1,\ldots,n \end{aligned}$$

eventually yielding

$$\begin{aligned} \mathfrak{f}_j\mathfrak{f}_k &= \mathfrak{f}_j \bullet \mathfrak{f}_k + \mathfrak{f}_j \wedge \mathfrak{f}_k = \mathfrak{f}_j \wedge \mathfrak{f}_k, \qquad j,k=1,\ldots,n \\ \mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger &= \mathfrak{f}_j^\dagger \bullet \mathfrak{f}_k^\dagger + \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_k^\dagger = \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_k^\dagger, \qquad j,k=1,\ldots,n \\ \mathfrak{f}_j\mathfrak{f}_k^\dagger &= \mathfrak{f}_j \bullet \mathfrak{f}_k^\dagger + \mathfrak{f}_j \wedge \mathfrak{f}_k^\dagger = \frac{1}{2}\delta_{jk} + \mathfrak{f}_j \wedge \mathfrak{f}_k^\dagger, \qquad j,k=1,\ldots,n \end{aligned}$$

This leads to the Grassmann structure of the complex Clifford algebra

$$\mathbb{C}_{2n} \cong \bigoplus_{p=0}^{n} \bigoplus_{q=0}^{n} \bigwedge\nolimits_{2n}^{p,q}$$

where

$$\bigwedge\nolimits_{2n}^{p,q} = \operatorname{span}_{\mathbb{C}} \left\{ \mathfrak{f}_{j_1}^\dagger \wedge \ldots \wedge \mathfrak{f}_{j_p}^\dagger \wedge \mathfrak{f}_{k_1} \wedge \ldots \wedge \mathfrak{f}_{k_q} | j_1 < j_2 < \ldots < j_p,\ k_1 < k_2 < \ldots < k_q \right\}$$

A vector $(X_1, \ldots, X_{2n})$ in $\mathbb{R}^{0,2n}$ is now denoted by $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ and is identified with the Clifford vector $\underline{X} = \sum_{j=1}^{n}(e_j\, x_j + e_{n+j}\, y_j)$; the action of the complex structure J on $\underline{X}$ yields the twisted vector

$$\underline{X}| = J[\underline{X}] = \sum_{j=1}^{n} (e_j\, y_j - e_{n+j}\, x_j)$$

Note that $\underline{X}$ and $\underline{X}|$ anti-commute, since they are orthogonal w.r.t. the standard Euclidean scalar product; more precisely they satisfy the following properties.

Lemma 2.1. One has

(i) $\underline{X} \bullet \underline{X}| = 0$

(ii) $\underline{X} \wedge \underline{X}| = \sum_{j\neq k} x_j y_k (e_j e_k - e_{n+k} e_{n+j}) - \sum_{j,k} e_j e_{n+k}(x_j x_k + y_j y_k)$

(iii) $\underline{X}| \wedge \underline{X} = \sum_{j\neq k} x_j y_k (e_k e_j - e_{n+j} e_{n+k}) - \sum_{j,k} e_{n+k} e_j (x_j x_k + y_j y_k)$

(iv) $\underline{X}\,\underline{X}| + \underline{X}|\,\underline{X} = \underline{X} \wedge \underline{X}| + \underline{X}| \wedge \underline{X} = 0$

The actions of the projection operators on the Clifford vector $\underline{X}$ then produce the mutually Hermitean conjugate Hermitean Clifford variables $\underline{z}$ and $\underline{z}^\dagger$, i.e.

$$\begin{array}{rcccl} \underline{z} & = & \frac{1}{2}(\mathbf{1}+iJ)[\underline{X}] & = & \frac{1}{2}(\underline{X}+i\,\underline{X}|) \\ \underline{z}^\dagger & = & -\frac{1}{2}(\mathbf{1}-iJ)[\underline{X}] & = & -\frac{1}{2}(\underline{X}-i\,\underline{X}|) \end{array}$$

which may also be rewritten in terms of the Witt basis elements as

$$\underline{z} = \sum_{j=1}^{n} \mathfrak{f}_j\, z_j \quad \text{and} \quad \underline{z}^\dagger = (\underline{z})^\dagger = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger\, z_j^c$$

where n complex variables $z_j = x_j + iy_j$ have been introduced, with complex conjugates $z_j^c = x_j - iy_j$, $j = 1, \ldots, n$. Finally, the Hermitean Dirac operators $\partial_{\underline{z}}$ and $\partial_{\underline{z}^\dagger}$ are obtained from the Euclidean Dirac operator $\partial_{\underline{X}}$:

$$\begin{array}{rcccl} \partial_{\underline{z}^\dagger} & = & \frac{1}{4}(\mathbf{1}+iJ)[\partial_{\underline{X}}] & = & \frac{1}{4}(\partial_{\underline{X}}+i\,\partial_{\underline{X}|}) \\ \partial_{\underline{z}} & = & -\frac{1}{4}(\mathbf{1}-iJ)[\partial_{\underline{X}}] & = & -\frac{1}{4}(\partial_{\underline{X}}-i\,\partial_{\underline{X}|}) \end{array}$$

where also the so–called twisted Dirac operator arises:

$$\partial_{\underline{X}|} = J[\partial_{\underline{X}}] = \sum_{j=1}^{n} \left(e_j\, \partial_{y_j} - e_{n+j}\, \partial_{x_j}\right)$$

As for $\partial_{\underline{X}}$, a notion of (twisted) monogenicity may be associated in a natural way to $\partial_{\underline{X}|}$ as well. Passing to the Witt basis, the Hermitean Dirac operators are expressed as

$$\partial_{\underline{z}} = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger\, \partial_{z_j} \quad \text{and} \quad \partial_{\underline{z}^\dagger} = (\partial_{\underline{z}})^\dagger = \sum_{j=1}^{n} \mathfrak{f}_j\, \partial_{z_j^c}$$

involving the classical Cauchy–Riemann operators $\partial_{z_j^c} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ and their complex conjugates $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ in the complex z_j-planes, $j = 1, \ldots, n$. As a consequence of the isotropy of the Witt basis vectors, the Hermitean vector variables and Dirac operators are isotropic, i.e.

$$(\underline{z})^2 = (\underline{z}^\dagger)^2 = 0 \quad \text{and} \quad (\partial_{\underline{z}})^2 = (\partial_{\underline{z}^\dagger})^2 = 0$$

whence the Laplacian $\Delta_{2n} = -\partial_{\underline{X}}^2 = -\partial_{\underline{X}|}^2$ allows for the decomposition

$$\Delta_{2n} = 4(\partial_{\underline{z}}\partial_{\underline{z}^\dagger} + \partial_{\underline{z}^\dagger}\partial_{\underline{z}}) = 4(\partial_{\underline{z}} + \partial_{\underline{z}^\dagger})^2$$

while also

$$(\underline{z}+\underline{z}^\dagger)^2 = \underline{z}\,\underline{z}^\dagger + \underline{z}^\dagger\underline{z} = |\underline{z}|^2 = |\underline{z}^\dagger|^2 = |\underline{X}|^2 = |\underline{X}|\,|^2$$

The central notion in Hermitean Clifford analysis is that of Hermitean monogenicity. A continuously differentiable function g on an open region G of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with values in the complex

Clifford algebra $\mathbb{C}_{2n}$ is called (left) Hermitean monogenic (or h–monogenic for short) in G if and only if it simultaneously is $\partial_{\underline{X}}$– and $\partial_{\underline{X}|}$–monogenic in G, i.e. it satisfies in G the system

$$\partial_{\underline{X}}\, g = 0 = \partial_{\underline{X}|}\, g$$

which is equivalent with the system

$$\partial_{\underline{z}}\, g = 0 = \partial_{\underline{z}^\dagger}\, g$$

Now the multivector functions in the Hermitean Clifford analysis setting are smooth functions defined in an open region G of $\mathbb{R}^{2n} \cong \mathbb{C}^n$ and taking their values in the Grassmann subspaces $\bigwedge_{2n}^{p,q}$. They thus take the form

$$F^{p,q}(x_1,\ldots,x_n,y_1,\ldots,y_n) = \sum \phi_{j_1\ldots j_p k_1\ldots k_q}\, \mathfrak{f}_{j_1}^\dagger \wedge \ldots \wedge \mathfrak{f}_{j_p}^\dagger \wedge \mathfrak{f}_{k_1} \wedge \ldots \wedge \mathfrak{f}_{k_q}$$

where the scalar functions $\phi_{j_1\ldots j_p k_1\ldots k_q}$ are assumed to be smooth functions in G. The space of these multivector functions is denoted by $\mathcal{E}^{p,q}(G)$, and we have

$$\mathcal{E}^r(G) = \bigoplus_{p+q=r} \mathcal{E}^{p,q}(G)$$

Similarly to what was done for the Euclidean Dirac operator $\partial_{\underline{X}}$ (and holds for $\partial_{\underline{X}|}$ as well), also the Hermitean Dirac operators may be split into their scalar or "dot" part and their bivector or "wedge" part, leading to

$$\begin{aligned}
\partial_{\underline{z}}\wedge &= \sum_{i=1}^{n} \partial_{z_i}\, \mathfrak{f}_i^\dagger \wedge \\
\partial_{\underline{z}}\bullet &= \sum_{i=1}^{n} \partial_{z_i}\, \mathfrak{f}_i^\dagger \bullet \\
\partial_{\underline{z}^\dagger}\wedge &= \sum_{i=1}^{n} \partial_{z_i^c}\, \mathfrak{f}_i \wedge \\
\partial_{\underline{z}^\dagger}\bullet &= \sum_{i=1}^{n} \partial_{z_i^c}\, \mathfrak{f}_i \bullet
\end{aligned}$$

for which it thus holds that

$$\partial_{\underline{z}} \wedge + \partial_{\underline{z}} \bullet = \partial_{\underline{z}}, \quad \partial_{\underline{z}^\dagger} \wedge + \partial_{\underline{z}^\dagger} \bullet = \partial_{\underline{z}^\dagger}$$

These operators have a two-fold action on $\mathcal{E}^{p,q}(G)$ in the sense that they act on the multi-vector by means of the inner and outer product with Witt basis vectors, and at the same time on the function coefficients by partial differentiation. They enjoy the following properties, which can be obtained through direct calculation.

Property 2.2. The Hermitean Dirac dot and wedge operators are interrelated by complex conjugation as follows:

(i) $(\partial_{\underline{z}}\wedge)^c = -\partial_{\underline{z}^\dagger}\wedge$

(ii) $(\partial_{\underline{z}}\bullet)^c = -\partial_{\underline{z}^\dagger}\bullet$

Property 2.3. The Hermitean Dirac dot and wedge operators act as follows on the spaces $\mathcal{E}^{p,q}$:

(i) $\partial_{\underline{z}}\wedge : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p+1,q}$

(ii) $\partial_{\underline{z}}\bullet : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p,q-1}$

(iii) $\partial_{\underline{z}^\dagger}\wedge : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p,q+1}$

(iv) $\partial_{\underline{z}^\dagger}\bullet : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p-1,q}$

Property 2.4. The Hermitean Dirac dot and wedge operators are isotropic:

(i) $(\partial_{\underline{z}}\wedge)^2 = (\partial_{\underline{z}}\bullet)^2 = (\partial_{\underline{z}^\dagger}\wedge)^2 = (\partial_{\underline{z}^\dagger}\bullet)^2 = 0$

and they show the following anticommutation relations:

(ii) $(\partial_{\underline{z}}\wedge)(\partial_{\underline{z}}\bullet) + (\partial_{\underline{z}}\bullet)(\partial_{\underline{z}}\wedge) = 0$

(iii) $(\partial_{\underline{z}^\dagger}\wedge)(\partial_{\underline{z}^\dagger}\bullet) + (\partial_{\underline{z}^\dagger}\bullet)(\partial_{\underline{z}^\dagger}\wedge) = 0$

(iv) $(\partial_{\underline{z}}\wedge)(\partial_{\underline{z}^\dagger}\wedge) + (\partial_{\underline{z}^\dagger}\wedge)(\partial_{\underline{z}}\wedge) = 0$

(v) $(\partial_{\underline{z}}\bullet)(\partial_{\underline{z}^\dagger}\bullet) + (\partial_{\underline{z}^\dagger}\bullet)(\partial_{\underline{z}}\bullet) = 0$

Property 2.5. Composition of the Hermitean Dirac dot and wedge operators yields the following actions on the spaces $\mathcal{E}^{p,q}$:

(i) $\overline{(\partial_{\underline{z}}\wedge)(\partial_{\underline{z}}\bullet)} = -(\partial_{\underline{z}}\wedge)(\partial_{\underline{z}}\bullet) = (\partial_{\underline{z}}\bullet)(\partial_{\underline{z}}\wedge) : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p+1,q-1}$

(ii) $\overline{(\partial_{\underline{z}^\dagger}\wedge)(\partial_{\underline{z}^\dagger}\bullet)} = -(\partial_{\underline{z}^\dagger}\wedge)(\partial_{\underline{z}^\dagger}\bullet) = (\partial_{\underline{z}^\dagger}\bullet)(\partial_{\underline{z}^\dagger}\wedge) : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p-1,q+1}$

(iii) $(\partial_{\underline{z}}\wedge)(\partial_{\underline{z}^\dagger}\wedge) = -(\partial_{\underline{z}^\dagger}\wedge)(\partial_{\underline{z}}\wedge) : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p+1,q+1}$

(iv) $(\partial_{\underline{z}}\bullet)(\partial_{\underline{z}^\dagger}\bullet) = -(\partial_{\underline{z}^\dagger}\bullet)(\partial_{\underline{z}}\bullet) : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p-1,q-1}$

Property 2.6. The Hermitean Dirac dot and wedge operators establish a decomposition of the Laplacian in the following ways:

(i) $(\partial_{\underline{z}}\wedge)(\partial_{\underline{z}^\dagger}\bullet) + (\partial_{\underline{z}^\dagger}\bullet)(\partial_{\underline{z}}\wedge) = \frac{1}{8}\,\Delta_{2n}$

(ii) $(\partial_{\underline{z}^\dagger}\wedge)(\partial_{\underline{z}}\bullet) + (\partial_{\underline{z}}\bullet)(\partial_{\underline{z}^\dagger}\wedge) = \frac{1}{8}\,\Delta_{2n}$

Property 2.7. The Hermitean Dirac dot and wedge operators establish decompositions of the corresponding Euclidean ones as follows:

(i) $(\partial_{\underline{z}^\dagger}\wedge) - (\partial_{\underline{z}}\wedge) = \frac{1}{2}\partial_{\underline{X}}\wedge$, $(\partial_{\underline{z}^\dagger}\bullet) - (\partial_{\underline{z}}\bullet) = \frac{1}{2}\partial_{\underline{X}}\bullet$

(ii) $(\partial_{\underline{z}^\dagger}\wedge) + (\partial_{\underline{z}}\wedge) = \frac{i}{2}\partial_{\underline{X}|}\wedge$, $(\partial_{\underline{z}^\dagger}\bullet) + (\partial_{\underline{z}}\bullet) = \frac{i}{2}\partial_{\underline{X}|}\bullet$

whence they also decompose the actual Euclidean Dirac operators as follows:

(iii) $(\partial_{\underline{z}^\dagger}\wedge) - (\partial_{\underline{z}}\wedge) + (\partial_{\underline{z}^\dagger}\bullet) - (\partial_{\underline{z}}\bullet) = \frac{1}{2}\partial_{\underline{X}}$

(iv) $(\partial_{\underline{z}^\dagger}\wedge) + (\partial_{\underline{z}}\wedge) + (\partial_{\underline{z}^\dagger}\bullet) + (\partial_{\underline{z}}\bullet) = \frac{i}{2}\partial_{\underline{X}|}$

Now, let us come back for a moment to the notion of Hermitean monogenicity for multivector functions. A multivector function $F^{p,q}$ is h-monogenic if and only if simultaneously $\partial_{\underline{z}}F^{p,q} = (\partial_{\underline{z}}\bullet + \partial_{\underline{z}}\wedge)F^{p,q} = 0$ and $\partial_{\underline{z}^\dagger}F^{p,q} = (\partial_{\underline{z}^\dagger}\bullet + \partial_{\underline{z}^\dagger}\wedge)F^{p,q} = 0$, which, due to Property 2.3, is equivalent with the system

$$\left\{\partial_{\underline{z}} \bullet F^{p,q} = 0,\ \partial_{\underline{z}} \wedge F^{p,q} = 0,\ \partial_{\underline{z}^\dagger} \bullet F^{p,q} = 0,\ \partial_{\underline{z}^\dagger} \wedge F^{p,q} = 0\right\}$$

In view of Property 2.7 we then obtain the following remarkable result.

Proposition 2.8. For a multivector function $F^{p,q}$ the notions of $\partial_{\underline{X}}$-monogenicity, $\partial_{\underline{X}|}$-monogenicity and Hermitean monogenicity coincide.

Remark 2.9. Obviously the system of equations describing Hermitean monogenicity will take particular forms according to the values of the functions considered. In [2] we have shown e.g. that, if the function takes its values in the subspace of spinor space corresponding to minimal or maximal degree of homogeneity, then Hermitean monogenicity reduces to (anti-)holomorphy for a function of several complex variables. In that sense Proposition 2.8 now reveals that one particular Grassmann cell $\bigwedge_{2n}^{p,q}$ can not be considered as an appropriate value space to study Hermitean monogenicity, since in that case it coincides with Euclidean monogenicity. It remains an interesting problem to discover appropriate value spaces in order to see the Hermitean monogenicity system reduce to a significant system of differential equations. To that end we have investigated in [10] how the complex Clifford algebra $\mathbb{C}_{2n}$ decomposes into subspaces leading to exact sequences for the multiplicative action of the Witt basis vectors.

3 Differential forms: preliminaries

There exists a vast literature on differential forms; in particular we refer to e.g. [19, 24] for real differential forms and to [20, 21] for complex differential forms. Here we will only recall the basic concepts needed.

Let $\mathbb{R}^m$ be endowed with the standard Euclidean metric. Denoting by $\bigwedge^r \mathbb{R}^m$ the space of alternating (or skew multilinear) real valued r-forms ($0 \leq r \leq m$), the Grassmann algebra or exterior algebra over $\mathbb{R}^m$ is the graded associative algebra

$$\bigwedge \mathbb{R}^m = \bigoplus_{r=0}^{m} \bigwedge^r \mathbb{R}^m$$

endowed with the exterior multiplication. A basis for $\bigwedge^r \mathbb{R}^m$ is obtained as follows. Let $\{dX^1, ..., dX^m\}$ be a basis for the dual space $(\mathbb{R}^m)^*$ of $\mathbb{R}^m$. If, as before, the set $A = \{i_1, \ldots, i_r\} \subset M = \{0, 1, ..., m\}$ is ordered in the natural way, put

$$dX^A = dX^{i_1} \wedge dX^{i_2} \wedge \ ... \ \wedge dX^{i_r}$$

and $dX^{\emptyset} = 1$. Then for each $r = 0, 1, ..., m$, the set $\{dX^A : \ A \subset M \text{ and } |A| = r\}$ is a basis for $\bigwedge^r \mathbb{R}^m$. Note that in particular

$$dX^i \wedge dX^i = 0, \qquad i = 0, 1, \ldots, m$$

and

$$dX^i \wedge dX^j + dX^j \wedge dX^i = 0, \qquad 0 \leq i \neq j \leq m$$

A smooth r-form in an open region G of $\mathbb{R}^m$ is a map

$$\omega^r : G \to \bigwedge^r \mathbb{R}^m, \quad X \mapsto \sum_{|A|=r} \omega^r_A(X_1, \ldots, X_m)\, dX^A$$

where, for each A, ω^r_A is a smooth real valued function in G. We denote by $\bigwedge^r(G)$ the space of smooth r-forms in G and we put

$$\bigwedge(G) = \bigoplus_{r=0}^{m} \bigwedge^r(G)$$

The projection operator from $\bigwedge(G)$ onto $\bigwedge^r(G)$ is denoted by $[\,\cdot\,]^r$. A fundamental linear operator on the space of smooth forms is the *exterior derivative* d. It is first defined as $d : \bigwedge^r(G) \to \bigwedge^{r+1}(G)$ ($r < m$), by

$$\omega^r = \sum_{|A|=r} \omega^r_A\, dX^A \quad \longmapsto \quad d\omega^r = \sum_A \sum_j \partial_{X_j} \omega^r_A\, dX^j \wedge dX^A$$

a definition which is then extended to $\bigwedge(G)$ by linearity. A second fundamental linear operator on the space of smooth forms is the *Hodge co-derivative* d^*. For $A = \{i_i, ..., i_r\} \subset M$ we denote

$$dX^{A\setminus\{i_j\}} = dX^{i_1} \wedge \ ... \ \wedge dX^{i_{j-1}} \wedge [dX^{i_j} \wedge]\ dX^{i_{j+1}} \wedge \ ... \ \wedge dX^{i_r}$$

and in a first step we put

$$d^*(\omega^r_A\, dX^A) = \sum_{j=1}^{r} (-1)^j\, \partial_{X_{i_j}} \omega^r_A\, dX^{A\setminus\{i_j\}}$$

Then d^* is defined as $d^* : \bigwedge^r(G) \to \bigwedge^{r-1}(G)$ $(r > 0)$, by

$$\omega^r = \sum_{|A|=r} \omega^r_A \, dX^A \quad \longmapsto \quad d^*(\omega^r) = \sum_{|A|=r} d^*(\omega^r_A \, dX^A)$$

and this definition again is extended to the whole of $\bigwedge(G)$ by linearity. A smooth r-form ω^r in G is called *closed* if and only if $d\omega^r = 0$; it is called *co-closed* if and only if $d^*\omega^r = 0$; and it is called *harmonic* (in the sense of Hodge) when it is at the same time closed and co-closed. Introducing the operator $D = d + d^*$, a necessary and sufficient condition for a smooth r-form ω^r in G to be harmonic thus reads

$$D\omega^r = (d + d^*)\omega^r = 0 \iff \begin{cases} d\omega^r &= 0 \\ d^*\omega^r &= 0 \end{cases} \tag{$*$}$$

The system $(*)$ is called the *Hodge-de Rham system*. Note that if ω^r is harmonic in an open region G of $\mathbb{R}^m$ then automatically ω^r satisfies $\Delta_m \omega^r = 0$ in G, since

$$D^2 = (d + d^*)^2 = d \; d^* + d^* \; d = -\Delta_m$$

The converse, however, is not true.

The action of the operators d and d^* on differential forms is two-fold in the sense that they act on the form itself as well as on the function coefficients by partial differentiation. In order to make this double action explicit we introduce the following symbolic notations for the operators d and d^*:

$$\begin{aligned} d &= \sum_{j=1}^{m} (dX^j \wedge) \, \partial_{X_j} \\ d^* &= \sum_{j=1}^{m} (dX^j \bullet) \, \partial_{X_j} \end{aligned}$$

with

$$dX^j \bullet dX^A = dX^j \bullet (dX^{i_1} \wedge \ \dots \ \wedge \ dX^{i_r}) = \sum_{k=1}^{r} (-1)^k \delta_{j i_k} dX^{A \setminus \{i_k\}}$$

In this last action we recognize the contraction operators $\partial_{X_j} \rfloor$, $j = 1, \dots m$, given by

$$\partial_{X_j} \rfloor dX^A = \partial_{X_j} \rfloor \left(dX^{i_1} \wedge \ \dots \ \wedge dX^{i_r} \right) = \sum_{k=1}^{r} (-1)^{k-1} \delta_{j i_k} \, dX^{A \setminus \{i_k\}}$$

acting only on the basis elements of the differential form, and not on the function coefficients. Apparently the contraction operator $\partial_{X_j} \rfloor$ coincides with the "inner product"-operator $dX^j \bullet$ up to a minus sign:

$$\partial_{X_j} \rfloor = (-dX^j \bullet), \quad j = 0, 1, \dots, m$$

However bear in mind that contractions are more fundamental than dot products. Indeed, they can be introduced independently of a scalar product, and their behaviour is invariant under diffeomorphisms, which is not the case for the dot product. We then indeed have for the operators d and d^*

$$\begin{aligned}\left(\sum_{j=1}^{m}(dX^j\wedge)\partial_{X_j}\right)\left(\sum_{|A|=r}\omega_A^r\, dX^A\right) &= \sum_{|A|=r}\sum_{j=1}^{m}(\partial_{X_j}\omega_A^r)\, dX^j\wedge dX^A \;=\; d\omega^r \\ \left(\sum_{j=1}^{m}(dX^j\,\bullet)\partial_{X_j}\right)\left(\sum_{|A|=r}\omega_A^r\, dX^A\right) &= \sum_{|A|=r}\sum_{k=1}^{r}(-1)^k(\partial_{X_{i_k}}\omega_A^r)\, dX^{A\setminus\{i_k\}} = d^*\omega^r\end{aligned}$$

At this moment we make the transition from the Euclidean to the Hermitean Clifford setting, which, as above, is established by the introduction of the complex structure J, forcing the dimension to be even: $m = 2n$. We may now also consider a twisted exterior derivative $d|$ and a twisted co-derivative $d^*|$, satisfying the following identities.

Property 3.1. It holds that

(i) $dd| + d|d = 0 = d^*d|^* + d|^*d^* = 0$

(ii) $dd|^* + d|^*d = 0 = d^*d| + d|d^* = 0$

Appropriate complex linear combinations of these real operators will give rise to complex exterior derivatives and co-derivatives, but we will first consider the traditional complex differential forms in $\mathbb{C}^n$ or in an open region G of $\mathbb{C}^n$. We call $\bigwedge^{p,q}(G)$ the space of complex differential forms of bidegree (p,q) in G; it contains elements $\omega^{p,q}$ of the form

$$\omega^{p,q} = \sum_{|J|=p}\sum_{|K|=q}\omega_{J,K}(\underline{z},\underline{z}^\dagger)\, dz_J\wedge dz_K^c$$

where $\omega_{K,L}(z_1,\ldots,z_n,z_1^c,\ldots,z_n^c)$ are smooth functions in G and

$$\begin{aligned} dz_J &= dz_{j_1}\wedge\ldots\wedge dz_{j_p}, \quad j_1<j_2<\ldots<j_p \\ dz_K^c &= dz_{k_1}^c\wedge\ldots\wedge dz_{k_q}^c, \quad k_1<k_2<\ldots<k_p\end{aligned}$$

The traditional derivatives in this setting are ∂, ∂^c, ∂^* and ∂^{*c}. They are defined as follows on a complex differential form of bidegree (p,q), definition which is then extended by linearity to an arbitrary complex differential form:

$$\begin{aligned}
\partial \omega^{p,q} &= \sum_{|J|=p} \sum_{|K|=q} \partial \omega_{J,K} \wedge dz_J \wedge dz_K^c \\
\partial^c \omega^{p,q} &= \sum_{|J|=p} \sum_{|K|=q} \partial^c \omega_{J,K} \wedge dz_J \wedge dz_K^c \\
\partial^* \omega^{p,q} &= \sum_{|J|=p} \sum_{|K|=q} \partial^* \left(\omega_{J,K} dz_J \wedge dz_K^c\right) \\
\partial^{*c} \omega^{p,q} &= \sum_{|J|=p} \sum_{|K|=q} \partial^{*c} \left(\omega_{J,K} dz_J \wedge dz_K^c\right)
\end{aligned}$$

with

$$\begin{aligned}
\partial \omega_{J,K} &= \sum_{i=1}^{n} (\partial_{z_i} \omega_{J,K})\, dz_i \\
\partial^c \omega_{J,K} &= \sum_{i=1}^{n} (\partial_{z_i^c} \omega_{J,K})\, dz_i^c \\
\partial^* \left(\omega_{J,K} dz_J \wedge dz_K^c\right) &= \sum_{i=1}^{n} (\partial_{z_i^c} \omega_{J,K})\, dz_i^c \bullet (dz_J \wedge dz_K^c) \\
\partial^{*c} \left(\omega_{J,K} dz_J \wedge dz_K^c\right) &= \sum_{i=1}^{n} (\partial_{z_i} \omega_{J,K})\, dz_i \bullet (dz_J \wedge dz_K^c)
\end{aligned}$$

Here we have introduced, for $j = 1, \ldots, n$, the not commonly used operators $dz_j \bullet$ and $dz_j^c \bullet$, which, via their Euclidean counterparts, are in fact complex contraction operators. We have indeed, for all $j = 1, \ldots, n$, that

$$\begin{aligned}
dz_j \bullet &= (dx_j + i dy_j) \bullet = dx_j \bullet + i dy_j \bullet = -\left(\partial_{x_j} \rfloor + i \partial_{y_j} \rfloor\right) = -2 \partial_{z_j^c} \rfloor \\
dz_j^c \bullet &= (dx_j - i dy_j) \bullet = dx_j \bullet - i dy_j \bullet = -\left(\partial_{x_j} \rfloor - i \partial_{y_j} \rfloor\right) = -2 \partial_{z_j} \rfloor
\end{aligned}$$

The four complex derivatives may thus be written symbolically as

$$\begin{aligned}
\partial &= \sum_{i=1}^{n} \partial_{z_i}\, dz_i \wedge \\
\partial^c &= \sum_{i=1}^{n} \partial_{z_i^c}\, dz_i^c \wedge \\
\partial^* &= \sum_{i=1}^{n} \partial_{z_i^c}\, dz_i^c \bullet \\
\partial^{*c} &= \sum_{i=1}^{n} \partial_{z_i}\, dz_i \bullet
\end{aligned}$$

where it is explicitly shown that ∂ and ∂^c act with a wedge product and ∂^* and ∂^{*c} with a dot product or contraction. They enjoy the following properties.

Property 3.2. The complex derivatives ∂, ∂^c, ∂^* and ∂^{*c} act as follows on the spaces $\bigwedge^{p,q}(G)$ of complex differential forms of bidegree (p, q) in G:

(i) $\partial : \bigwedge^{p,q}(G) \longrightarrow \bigwedge^{p+1,q}(G)$

(ii) $\partial^c : \bigwedge^{p,q}(G) \longrightarrow \bigwedge^{p,q+1}(G)$

(iii) $\partial^* : \bigwedge^{p,q}(G) \longrightarrow \bigwedge^{p-1,q}(G)$

(iv) $\partial^{*c} : \bigwedge^{p,q}(G) \longrightarrow \bigwedge^{p,q-1}(G)$

Property 3.3. The complex derivatives ∂, ∂^c, ∂^* and ∂^{*c} satisfy the Kähler identities

(i) $\partial\partial^{*c} + \partial^{*c}\partial = 0 = \partial^*\partial^c + \partial^c\partial^*$

(ii) $\partial\partial^c + \partial^c\partial = 0 = \partial^*\partial^{*c} + \partial^{*c}\partial^*$

(iii) $\partial\partial^* + \partial^*\partial = -\frac{1}{2}\Delta_{2n} = \partial^c\partial^{*c} + \partial^{*c}\partial^c$

In a very similar way as the Hermitean variables and Dirac operators are linked to their Euclidean counterparts, the Kählerian derivatives $\partial, \partial^c, \partial^*$ and ∂^{*c} are linked to the exterior derivative and co-derivative and their twisted analogues.

Property 3.4. It holds that

(i) $\partial^c + \partial = d$, $\partial^c - \partial = id|$

(ii) $\partial^* + \partial^{*c} = d^*$, $\partial^* - \partial^{*c} = id|^*$

whence we may also write

(iii) $\partial^c = \frac{1}{2}(d + id|)$, $\partial = \frac{1}{2}(d - id|)$

(iv) $\partial^* = \frac{1}{2}(d^* + id|^*)$, $\partial^{*c} = \frac{1}{2}(d^* - id|^*)$

4 Differential forms and multi-vector functions: an identification

In [6] it is shown how the world of real differential forms in an open region G of $\mathbb{R}^m$ and the world of Clifford algebra valued multi-vector functions in G may be naturally identified. The fundamental identification, adapted to the Hermitean setting, reads

$$e_i \longleftrightarrow dx^i, \quad e_{n+i} \longleftrightarrow dy^i, \qquad i = 1, \ldots, n$$

resulting in the identifications listed in Table 1. Note that we have listed here only a few of these identifications; for more details we refer the reader to [6].

$d = \sum_{i=1}^{n} (dx^i \wedge)\partial_{x_i} + (dy^i \wedge)\partial_{y_i}$	$\partial_{\underline{X}} \wedge = \sum_{i=1}^{n} (e_i \wedge)\partial_{x_i} + (e_{n+i} \wedge)\partial_{y_i}$
$d^* = \sum_{i=0}^{n} (dx^i \bullet)\partial_{x_i} + (dy^i \bullet)\partial_{y_i}$	$\partial_{\underline{X}} \bullet = \sum_{i=1}^{n} (e_i \bullet)\partial_{x_i} + (e_{n+i} \bullet)\partial_{y_i}$
$d\vert = \sum_{i=1}^{n} (dx^i \wedge)\partial_{y_i} - (dy^i \wedge)\partial_{x_i}$	$\partial_{\underline{X}\vert} \wedge = \sum_{i=1}^{n} (e_i \wedge)\partial_{y_i} - (e_{n+i} \wedge)\partial_{x_i}$
$d\vert^* = \sum_{i=0}^{n} (dx^i \bullet)\partial_{y_i} - (dy^i \bullet)\partial_{x_i}$	$\partial_{\underline{X}\vert} \bullet = \sum_{i=1}^{n} (e_i \bullet)\partial_{y_i} - (e_{n+i} \bullet)\partial_{x_i}$

Table 1: Identification of the Euclidean Dirac operators

This identification is now further developed in the Hermitean setting. For the Witt basis vectors one explicitly obtains the identifications

$$\begin{aligned}
f_j^\dagger \wedge = -\frac{1}{2}(e_j + i e_{n+j})\wedge = -\frac{1}{2}(e_j \wedge + i e_{n+j} \wedge) &\longleftrightarrow -\frac{1}{2}(dx^j \wedge + i dy^j \wedge) = -\frac{1}{2}(dz_j \wedge) \\
f_j^\dagger \bullet = -\frac{1}{2}(e_j + i e_{n+j})\bullet = -\frac{1}{2}(e_j \bullet + i e_{n+j} \bullet) &\longleftrightarrow -\frac{1}{2}(dx^j \bullet + i dy^j \bullet) = -\frac{1}{2}(dz_j \bullet) \\
f_j \wedge = \frac{1}{2}(e_j - i e_{n+j})\wedge = \frac{1}{2}(e_j \wedge - i e_{n+j} \wedge) &\longleftrightarrow \frac{1}{2}(dx^j \wedge - i dy^j \wedge) = \frac{1}{2}(dz_j^c \wedge)
\end{aligned}$$

and

$$f_j \bullet = \frac{1}{2}(e_j - i e_{n+j})\bullet = \frac{1}{2}(e_j \bullet - i e_{n+j} \bullet) \longleftrightarrow \frac{1}{2}(dx^j \bullet - i dy^j \bullet) = \frac{1}{2}(dz_j^c \bullet)$$

listed in Table 2. The so-called inflation operator, denoted $\cdot\rceil$, is introduced below.

$f_j^\dagger \wedge$	$-\frac{1}{2}(dz_j \wedge) = \partial_{z_j^c} \rceil$
$f_j^\dagger \bullet$	$-\frac{1}{2}(dz_j \bullet) = \partial_{z_j^c} \rfloor$
$f_j \wedge$	$\frac{1}{2}(dz_j^c \wedge) = -\partial_{z_j} \rceil$
$f_j \bullet$	$\frac{1}{2}(dz_j^c \bullet) = -\partial_{z_j} \rfloor$

Table 2: Identification of the Witt basis vectors

In the same order of ideas one explicitly obtains for the Hermitean Dirac operators

$$\partial_{\underline{z}^\dagger} \wedge = \sum_{j=1}^{n} \partial_{z_j^c} \, \mathfrak{f}_j \wedge \quad \longleftrightarrow \quad \sum_{j=1}^{n} \partial_{z_j^c} \, \frac{1}{2} (dz_j^c \wedge) = \frac{1}{2} (\partial^c \wedge)$$

$$\partial_{\underline{z}^\dagger} \bullet = \sum_{j=1}^{n} \partial_{z_j^c} \, \mathfrak{f}_j \bullet \quad \longleftrightarrow \quad \sum_{j=1}^{n} \partial_{z_j^c} \, \frac{1}{2} (dz_j^c \bullet) = \frac{1}{2} (\partial^* \bullet)$$

$$\partial_{\underline{z}} \wedge = \sum_{j=1}^{n} \partial_{z_j} \, \mathfrak{f}_j^\dagger \wedge \quad \longleftrightarrow \quad \sum_{j=1}^{n} \partial_{z_j} \, (-\frac{1}{2}) (dz_j \wedge) = (-\frac{1}{2}) (\partial \wedge)$$

$$\partial_{\underline{z}} \bullet = \sum_{j=1}^{n} \partial_{z_j} \, \mathfrak{f}_j^\dagger \bullet \quad \longleftrightarrow \quad \sum_{j=1}^{n} \partial_{z_j} \, (-\frac{1}{2}) (dz_j \bullet) = (-\frac{1}{2}) (\partial^{*c} \bullet)$$

as summarized in Table 3.

$\partial_{\underline{z}^\dagger} \wedge$	$\frac{1}{2}(\partial^c \wedge)$
$\partial_{\underline{z}^\dagger} \bullet$	$\frac{1}{2}(\partial^* \bullet)$
$\partial_{\underline{z}} \wedge$	$-\frac{1}{2}(\partial \wedge)$
$\partial_{\underline{z}} \bullet$	$-\frac{1}{2}(\partial^{*c} \bullet)$

Table 3: Identification of the Hermitean Dirac operators

Through these identifications it becomes clear that the properties of the Hermitean Dirac operators on multivector functions listed in Section 2 and those of the Kählerian differential operators on complex differential forms listed in Section 3 are two faces of the same coin. This also implies that it suffices to prove a property in only one of these two worlds, automatically gaining the similar property in the other. To give an example, Proposition 2.12 is transposed as follows.

Proposition 4.1. A (p, q)-form $\omega^{p,q} \in \bigwedge^{p,q}(G)$ is harmonic in an open region G of $\mathbb{C}^n$, i.e. it satisfies the Hodge-de Rham system $\{d\omega^{p,q} = 0, d^*\omega^{p,q} = 0\}$, if and only if in G it is Hermitean monogenic, i.e. it satisfies the system $\{\partial\omega^{p,q} = 0, \partial^c\omega^{p,q} = 0, \partial^*\omega^{p,q} = 0, \partial^{*c}\omega^{p,q} = 0\}$, which implies that for a (p, q)-form $\omega^{p,q} \in \bigwedge^{p,q}(G)$ the notions *harmonic*, *twisted harmonic* and *Hermitean monogenic* coincide.

Another nice illustration of this identification is procured by the Euler operators. The Hermitean Euler operators

$$E_{\underline{z}} = \sum_{j=1}^{n} z_j \, \partial_{z_j} = 2 \, \underline{z} \bullet \partial_{\underline{z}}$$

$$E_{\underline{z}}^\dagger = \sum_{j=1}^{n} z_j^c \, \partial_{z_j^c} = 2 \, \underline{z}^\dagger \bullet \partial_{\underline{z}^\dagger}$$

have shown their importance in constructing the Fischer decomposition of homogeneous polynomials in terms of Hermitean monogenic polynomials and the corresponding Howe dual pair (see [16, 7]). They have a natural close connection with the traditional Euclidean Euler operators, since

$$\begin{aligned} E_{\underline{z}} &= \frac{1}{2}\,E_{\underline{X}} + \frac{i}{2}\,\underline{X} \bullet \partial_{\underline{X}|} \\ E_{\underline{z}}^{\dagger} &= \frac{1}{2}E_{\underline{X}} + \frac{i}{2}\,\underline{X}| \bullet \partial_{\underline{X}} \end{aligned}$$

whence

$$\begin{aligned} E_{\underline{z}} + E_{\underline{z}}^{\dagger} &= \sum_{j=1}^{n} \left(x_j\,\partial_{x_j} + y_j\,\partial_{y_j}\right) = E_{\underline{X}} = E_{\underline{X}|} = -\underline{X} \bullet \partial_{\underline{X}} = -\underline{X}| \bullet \partial_{\underline{X}|} \\ E_{\underline{z}} - E_{\underline{z}}^{\dagger} &= i\sum_{j=1}^{n} \left(-x_j\,\partial_{y_j} + y_j\,\partial_{x_j}\right) = i\,\underline{X} \bullet \partial_{\underline{X}|} = -i\,\underline{X}| \bullet \partial_{\underline{X}} \end{aligned}$$

It thus becomes clear that the Hermitean Euler operators are mutually complex conjugated scalar operators; note that they have the same expression in both worlds. In the world of differential forms we now focus on the contraction operators associated to the Hermitean Euler operators. To that end recall that we tend to denote contraction of a differential form by means of a "dot", more specifically $\partial_{X_\alpha} \rfloor = -dX_\alpha \bullet$, yielding

$$\partial_{\underline{X}} \rfloor = \sum_{\alpha=1}^{m} e_\alpha\,\partial_{X_\alpha} \rfloor = -\sum_{\alpha=1}^{m} e_\alpha\,dX_\alpha \bullet = -d\underline{X} \bullet$$

and also

$$\begin{aligned} \partial_{\underline{z}} \rfloor &= \sum_{j=1}^{n} \mathfrak{f}_j^{\dagger}\,\partial_{z_j} \rfloor = \sum_{j=1}^{n} \mathfrak{f}_j^{\dagger}\,(-\frac{1}{2}\,dz_j^c \bullet) = -\frac{1}{2}\,d\underline{z}^{\dagger} \bullet \\ \partial_{\underline{z}^{\dagger}} \rfloor &= \sum_{j=1}^{n} \mathfrak{f}_j\,\partial_{z_j^c} \rfloor = \sum_{j=1}^{n} \mathfrak{f}_j\,(-\frac{1}{2}\,dz_j \bullet) = -\frac{1}{2}\,d\underline{z} \bullet \end{aligned}$$

For the contracted Hermitean Euler operators we then obtain

$$\begin{aligned} E_{\underline{z}} \rfloor &= \sum_{j=1}^{n} z_j\,\partial_{z_j} \rfloor = (-\frac{1}{2}) \sum_{j=1}^{n} z_j\,(dz_j^c \bullet) \quad \text{or} \quad E_{\underline{z}} \rfloor = 2\underline{z} \bullet \partial_{\underline{z}} \rfloor = -\underline{z} \bullet d\underline{z}^{\dagger} \bullet \\ E_{\underline{z}}^{\dagger} \rfloor &= \sum_{j=1}^{n} z_j^c\,\partial_{z_j^c} \rfloor = (-\frac{1}{2}) \sum_{j=1}^{n} z_j^c\,(dz_j \bullet) \quad \text{or} \quad E_{\underline{z}}^{\dagger} \rfloor = 2\underline{z}^{\dagger} \bullet \partial_{\underline{z}^{\dagger}} \rfloor = -\underline{z}^{\dagger} \bullet d\underline{z} \bullet \end{aligned}$$

We could as well, for symmetry's sake, have introduced a so-called inflation operator (see [6]), denoted by a "wedge", i.e. $\partial_{X_\alpha} \rceil = -dX_\alpha \wedge$, yielding

$$\partial_{\underline{X}} \rceil = \sum_{\alpha=1}^{m} e_\alpha\,\partial_{X_\alpha} \rceil = -\sum_{\alpha=1}^{m} e_\alpha\,dX_\alpha \wedge = -d\underline{X} \wedge$$

and

$$\partial_{\underline{z}} \rceil = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \, \partial_{z_j} \rceil = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \, (-\frac{1}{2} \, dz_j^c \wedge) = -\frac{1}{2} \, d\underline{z}^\dagger \wedge$$

$$\partial_{\underline{z}^\dagger} \rceil = \sum_{j=1}^{n} \mathfrak{f}_j \, \partial_{z_j^c} \rceil = \sum_{j=1}^{n} \mathfrak{f}_j \, (-\frac{1}{2} \, dz_j \wedge) = -\frac{1}{2} \, d\underline{z} \wedge$$

Note that the above notations $\partial_{z_j} \rceil$ and $\partial_{z_j^c} \rceil$ where already used in Table 2. This leads to

$$E_{\underline{z}} \rceil = \sum_{j=1}^{n} z_j \, \partial_{z_j} \rceil = (-\frac{1}{2}) \sum_{j=1}^{n} z_j \, (dz_j^c \wedge) \quad \text{or} \quad E_{\underline{z}} \rceil = 2\underline{z} \bullet \partial_{\underline{z}} \rceil = -\underline{z} \bullet d\underline{z}^\dagger \wedge$$

$$E_{\underline{z}}^\dagger \rceil = \sum_{j=1}^{n} z_j^c \, \partial_{z_j^c} \rceil = (-\frac{1}{2}) \sum_{j=1}^{n} z_j^c \, (dz_j \wedge) \quad \text{or} \quad E_{\underline{z}}^\dagger \rceil = 2\underline{z}^\dagger \bullet \partial_{\underline{z}^\dagger} \rceil = -\underline{z}^\dagger \bullet d\underline{z} \wedge$$

These four contracted and inflated Hermitean Euler operators enjoy the properties summarized in the following two propositions.

Proposition 4.2. One has

(i) $E_{\underline{z}} \rfloor + E_{\underline{z}}^\dagger \rfloor = E_{\underline{X}} \rfloor = \underline{X} \bullet d\underline{X} \bullet$

(ii) $E_{\underline{z}} \rceil + E_{\underline{z}}^\dagger \rceil = E_{\underline{X}} \rceil = \underline{X} \bullet d\underline{X} \wedge$

(iii) $E_{\underline{z}} \rfloor - E_{\underline{z}}^\dagger \rfloor = i\underline{X}| \bullet d\underline{X} \bullet = -i\underline{X} \bullet d\underline{X}| \bullet$

(iv) $E_{\underline{z}} \rceil - E_{\underline{z}}^\dagger \rceil = i\underline{X}| \bullet d\underline{X} \wedge = -i\underline{X} \bullet d\underline{X}| \wedge$

(v) $E_z \rfloor + E_z \rceil = (-\frac{1}{2}) \sum_{j=1}^{n} z_j \, dz_j^c = -\underline{z} \bullet d\underline{z}^\dagger$

(vi) $E_z^\dagger \rfloor + E_z^\dagger \rceil = (-\frac{1}{2}) \sum_{j=1}^{n} z_j^c \, dz_j = -\underline{z}^\dagger \bullet d\underline{z}$

Proposition 4.3. One has

(i) $\left(E_z \rfloor + E_z^\dagger \rfloor\right)^2 = \left(E_{\underline{X}} \rfloor\right)^2 = 0$

(ii) $\left(E_z \rceil + E_z^\dagger \rceil\right)^2 = \left(E_{\underline{X}} \rceil\right)^2 = 0$

(iii) $(E_z \rfloor + E_z \rceil)^2 = 0$

(iv) $\left(E_z^\dagger \rfloor + E_z^\dagger \rceil\right)^2 = 0$

(v) $(E_z \rfloor + E_z \rceil) \left(E_z^\dagger \rfloor + E_z^\dagger \rceil\right) + \left(E_z^\dagger \rfloor + E_z^\dagger \rceil\right) (E_z \rfloor + E_z \rceil) = -|\underline{z}^2|$

(vi) $\left(E_z\rfloor + E_z^\dagger\rfloor\right)\left(E_z\rceil + E_z^\dagger\rceil\right) + \left(E_z\rceil + E_z^\dagger\rceil\right)\left(E_z\rfloor + E_z^\dagger\rfloor\right) = -|\underline{z}^2|$

These properties may be proven by direct calculation, but things become more transparent after identification in the multivector setting; to that end we look at the analogues of the operators involved, given by

$$\begin{aligned}
E_{\underline{z}}\rfloor &= (-\frac{1}{2})\sum_{j=1}^{n} z_j\,(dz_j^c\bullet) \longleftrightarrow (-\frac{1}{2})\sum_{j=1}^{n} z_j\,(2\mathfrak{f}_j\bullet) = -\underline{z}\,\bullet \\
E_{\underline{z}}^\dagger\rfloor &= (-\frac{1}{2})\sum_{j=1}^{n} z_j^c\,(dz_j\bullet) \longleftrightarrow (-\frac{1}{2})\sum_{j=1}^{n} z_j^c\,(-2\mathfrak{f}_j^\dagger\bullet) = \underline{z}^\dagger\,\bullet \\
E_{\underline{z}}\rceil &= (-\frac{1}{2})\sum_{j=1}^{n} z_j\,(dz_j^c\wedge) \longleftrightarrow (-\frac{1}{2})\sum_{j=1}^{n} z_j\,(2\mathfrak{f}_j\wedge) = -\underline{z}\wedge \\
E_{\underline{z}}^\dagger\rceil &= (-\frac{1}{2})\sum_{j=1}^{n} z_j^c\,(dz_j\wedge) \longleftrightarrow (-\frac{1}{2})\sum_{j=1}^{n} z_j^c\,(-2\mathfrak{f}_j^\dagger\wedge) = \underline{z}^\dagger\wedge
\end{aligned}$$

Propositions 4.2 and 4.3 now take a rather trivial form and are easily proven (see also [6]), as may be observed from their reformulation in the propositions below.

Proposition 4.4. One has

(i) $(-\underline{z}\bullet) + (\underline{z}^\dagger\bullet) = -\underline{X}\bullet$

(ii) $(-\underline{z}\wedge) + (\underline{z}^\dagger\wedge) = -\underline{X}\wedge$

(iii) $(-\underline{z}\bullet) - (\underline{z}^\dagger\bullet) = -i\underline{X}|\bullet$

(iv) $(-\underline{z}\wedge) - (\underline{z}^\dagger\wedge) = -i\underline{X}|\wedge$

(v) $(-\underline{z}\bullet) + (-\underline{z}\wedge) = -\underline{z}$

(vi) $(\underline{z}^\dagger\bullet) + (\underline{z}^\dagger\wedge) = \underline{z}^\dagger$

Proposition 4.5. One has

(i) $(-\underline{X}\bullet)(\underline{X}\bullet) = 0$

(ii) $(-\underline{X}\wedge)(\underline{X}\wedge) = 0$

(iii) $(-\underline{z}\bullet - \underline{z}\wedge)^2 = (-\underline{z})^2 = 0$

(iv) $\left(\underline{z}^\dagger\bullet + \underline{z}^\dagger\wedge\right)^2 = \left(\underline{z}^\dagger\right)^2 = 0$

(v) $(-\underline{z})\left(\underline{z}^\dagger\right) + \left(\underline{z}^\dagger\right)(-\underline{z}) = -|\underline{z}^2|$

(vi) $\left(-\underline{z}\bullet + \underline{z}^\dagger\bullet\right)\left(-\underline{z}\wedge + \underline{z}^\dagger\wedge\right) + \left(-\underline{z}\wedge + \underline{z}^\dagger\wedge\right)\left(-\underline{z}\bullet + \underline{z}^\dagger\bullet\right) = -|\underline{z}^2|$

In the same order of ideas, starting from the operators d and d^*, we introduce the contraction and inflation operators

$$\begin{aligned} d\rfloor &= \sum_{j=1}^{m} (dX^j \wedge)\, \partial_{X_j}\rfloor = \sum_{j=1}^{m} (dX^j \wedge)(-dX^j\ \bullet) \\ d^*\rceil &= \sum_{j=1}^{m} (dX^j\ \bullet)\, \partial_{X_j}\rceil = \sum_{j=1}^{m} (dX^j\ \bullet)(-dX^j \wedge) \end{aligned}$$

The operators $d\rfloor$ and $d^*\rceil$ have $\mathcal{E}^r(\Omega)$ as an eigenspace since

$$d\rfloor \omega^r = r\, \omega^r \quad \text{and} \quad d^*\rceil \omega^r = (m - r)\, \omega^r$$

In other words: they measure the order of a differential form. They are sometimes called *fermionic Euler operators*. In the Clifford analysis setting they read

$$\partial_{\underline{X}} \wedge \rfloor = \sum_{j=1}^{m} (e_j \wedge)(-e_j\ \bullet) \quad \text{and} \quad \partial_{\underline{X}} \bullet \rceil = \sum_{j=1}^{m} (e_j\ \bullet)(-e_j \wedge)$$

for which it indeed holds that

$$\partial_{\underline{X}} \wedge \rfloor F_r = r\, F_r \quad \text{and} \quad \partial_{\underline{X}} \bullet \rceil F_r = (m - r)\, F_r$$

Note that $d\rceil$, $d^*\rfloor, \partial_{\underline{X}} \wedge \rceil$ and $\partial_{\underline{X}} \bullet \rfloor$ are zero operators. The same can be done now with the Kählerian derivatives, leading to

$$\begin{aligned} \partial\rfloor &= (-\frac{1}{2}) \sum_{j=1}^{n} dz_j \wedge dz_j^c\ \bullet \\ \partial^c\rfloor &= (-\frac{1}{2}) \sum_{j=1}^{n} dz_j^c \wedge dz_j\ \bullet \\ \partial^*\rfloor &= (-\frac{1}{2}) \sum_{j=1}^{n} dz_j^c \bullet dz_j\ \bullet \\ \partial^{*c}\rfloor &= (-\frac{1}{2}) \sum_{j=1}^{n} dz_j \bullet dz_j^c\ \bullet \end{aligned}$$

and

$$\begin{aligned} \partial\rceil &= (-\frac{1}{2}) \sum_{j=1}^{n} dz_j \wedge dz_j^c \wedge \\ \partial^c\rceil &= (-\frac{1}{2}) \sum_{j=1}^{n} dz_j^c \wedge dz_j \wedge \\ \partial^*\rceil &= (-\frac{1}{2}) \sum_{j=1}^{n} dz_j^c \bullet dz_j \wedge \\ \partial^{*c}\rceil &= (-\frac{1}{2}) \sum_{j=1}^{n} dz_j \bullet dz_j^c \wedge \end{aligned}$$

while their Hermitean multivector analogues are given by

$$\begin{aligned}
\partial_{\underline{z}} \wedge \rfloor &= -\sum_{j=1}^{n} \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_j \bullet \\
\partial_{\underline{z}^\dagger} \wedge \rfloor &= \sum_{j=1}^{n} \mathfrak{f}_j \wedge \mathfrak{f}_j^\dagger \bullet \\
\partial_{\underline{z}^\dagger} \bullet \rfloor &= \sum_{j=1}^{n} \mathfrak{f}_j \bullet \mathfrak{f}_j^\dagger \bullet \\
\partial_{\underline{z}} \bullet \rfloor &= -\sum_{j=1}^{n} \mathfrak{f}_j^\dagger \bullet \mathfrak{f}_j \bullet
\end{aligned}$$

and by

$$\begin{aligned}
\partial_{\underline{z}} \wedge \rceil &= -\sum_{j=1}^{n} \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_j \wedge \\
\partial_{\underline{z}^\dagger} \wedge \rceil &= \sum_{j=1}^{n} \mathfrak{f}_j \wedge \mathfrak{f}_j^\dagger \wedge \\
\partial_{\underline{z}^\dagger} \bullet \rceil &= \sum_{j=1}^{n} \mathfrak{f}_j \bullet \mathfrak{f}_j^\dagger \wedge \\
\partial_{\underline{z}} \bullet \rceil &= -\sum_{j=1}^{n} \mathfrak{f}_j^\dagger \bullet \mathfrak{f}_j \wedge
\end{aligned}$$

The spaces $\mathcal{E}^{p,q}$ of smooth vector functions of bidegree (p, q) are eigenspaces of the operators $\partial_{\underline{z}} \wedge \rfloor, \partial_{\underline{z}^\dagger} \wedge \rfloor, \partial_{\underline{z}^\dagger} \bullet \rfloor$ and $\partial_{\underline{z}} \bullet \rfloor$. More precisely we have the following.

Proposition 4.6. For $F^{p,q} \in \mathcal{E}^{p,q}$, one has

(i) $\left(\partial_{\underline{z}} \wedge \rfloor\right) F^{p,q} = \left(-\frac{p}{2}\right) F^{p,q}$

(ii) $\left(\partial_{\underline{z}^\dagger} \wedge \rfloor\right) F^{p,q} = \left(\frac{q}{2}\right) F^{p,q}$

(iii) $\left(\partial_{\underline{z}^\dagger} \bullet \rceil\right) F^{p,q} = \left(\frac{n-p}{2}\right) F^{p,q}$

(iv) $\left(\partial_{\underline{z}} \bullet \rceil\right) F^{p,q} = \left(\frac{-n+q}{2}\right) F^{p,q}$

Note that, by similarity, the same eigenvalue equations hold for the operators $\partial \rfloor, \partial^c \rfloor, \partial^* \rceil$ and $\partial^{*c} \rceil$. Moreover observe that the eigenvalue equations for the operators $d \rfloor$ or $\partial_{\underline{X}} \wedge \rfloor$ and $d^* \rceil$ or

$\partial_{\underline{X}}\bullet\rceil$ are refined by the ones of Proposition 4.6, and may be recovered from them:

$$\begin{aligned}
\left(\partial_{\underline{X}}\wedge\rfloor\right) F^{p,q} &= 2\left(\partial_{\underline{z}^\dagger}\wedge\rfloor - \partial_{\underline{z}}\wedge\rfloor\right) F^{p,q} \;=\; 2\left(\frac{q}{2}+\frac{p}{2}\right) F^{p,q} \\
&= (p+q)F^{p,q} \\
\left(\partial_{\underline{X}}\bullet\rceil\right) F^{p,q} &= 2\left(\partial_{\underline{z}^\dagger}\bullet\rceil - \partial_{\underline{z}}\bullet\rceil\right) F^{p,q} \;=\; 2\left(\frac{n-p}{2}-\frac{-n+q}{2}\right) F^{p,q} \\
&= (2n-(p+q))\, F^{p,q}
\end{aligned}$$

Furthermore, it may be verified that $\partial_{\underline{X}}\wedge\rceil$ and $\partial_{\underline{X}}\bullet\rfloor$ indeed are zero operators:

$$\begin{aligned}
\partial_{\underline{X}}\wedge\rceil &= 2\left(\partial_{\underline{z}^\dagger}\wedge\rceil - \partial_{\underline{z}}\wedge\rceil\right) \;=\; 2\left(\sum_{j=1}^{n} \mathfrak{f}_j \wedge \mathfrak{f}_j^\dagger \wedge + \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_j \wedge\right) = 0 \\
\partial_{\underline{X}}\bullet\rfloor &= 2\left(\partial_{\underline{z}^\dagger}\bullet\rfloor - \partial_{\underline{z}}\bullet\rfloor\right) \;=\; 2\left(\sum_{j=1}^{n} \mathfrak{f}_j \bullet \mathfrak{f}_j^\dagger \bullet \;+ \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \bullet \mathfrak{f}_j \bullet\right) = 0
\end{aligned}$$

Finally, also the original expressions for $\partial_{\underline{X}}\wedge\rfloor$ and $\partial_{\underline{X}}\bullet\rceil$ as obtained in [6] may be recovered:

$$\begin{aligned}
\partial_{\underline{X}}\wedge\rfloor &= 2\left(\partial_{\underline{z}^\dagger}\wedge\rfloor - \partial_{\underline{z}}\wedge\rfloor\right) \;=\; 2\left(\sum_{j=1}^{n} \mathfrak{f}_j \wedge \mathfrak{f}_j^\dagger \bullet + \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \wedge \mathfrak{f}_j \bullet\right) \\
&= -\sum_{j=1}^{n} e_j \wedge e_j \bullet + e_{n+j} \wedge e_{n+j} \bullet \;=\; -\sum_{\alpha=1}^{2n} e_\alpha \wedge e_\alpha \bullet \\
\partial_{\underline{X}}\bullet\rceil &= 2\left(\partial_{\underline{z}^\dagger}\bullet\rceil - \partial_{\underline{z}}\bullet\rceil\right) \;=\; 2\left(\sum_{j=1}^{n} \mathfrak{f}_j \bullet \mathfrak{f}_j^\dagger \wedge + \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \bullet \mathfrak{f}_j \wedge\right) \\
&= -\sum_{j=1}^{n} e_j \bullet e_j \wedge + e_{n+j} \bullet e_{n+j} \wedge \;=\; -\sum_{\alpha=1}^{2n} e_\alpha \bullet e_\alpha \wedge
\end{aligned}$$

We shall encounter the operators $\partial_{\underline{z}}\wedge\rceil$ and $\partial_{\underline{z}}\bullet\rfloor$ again in the next section in a different context.

5 The Kählerian metric

We will now use known results from Kählerian geometry, however restricted to the flat Kählerian manifold $\mathbb{C}^n$, and transpose them to obtain results, not yet known in the Hermitean Clifford analysis setting. Our guides are [21, 20]. Each Kählerian metric induces a *fundamental form* Ω, which is a 2-form derived from the corresponding Kähler potential U, i.e.

$$\Omega \wedge = \frac{i}{2} \partial \partial^c U$$

The potential of the flat metric or the canonical Hermitean metric is given by

$$U = \frac{1}{2}|\underline{z}|^2 = \frac{1}{2}|\underline{z}^\dagger|^2 = \frac{1}{2}\sum_{i=1}^{n} z_i z_i^c = \underline{z} \bullet \underline{z}^\dagger = \frac{1}{2}(\underline{z}\underline{z}^\dagger + \underline{z}^\dagger \underline{z}) = \frac{1}{2}|\underline{X}|^2 = \frac{1}{2}|\underline{X}||^2$$

yielding the flat fundamental form

$$\begin{aligned}\Omega\wedge &= \frac{i}{2}(-2\partial_{\underline{z}}\wedge)(2\partial_{\underline{z}^\dagger}\wedge)|\underline{z}|^2 = (-2i)\left(\sum_{j=1}^{n}\partial_{z_j}\mathfrak{f}_j^\dagger\wedge\right)\left(\sum_{k=1}^{n}\partial_{z_k^c}\mathfrak{f}_k\wedge\right)|\underline{z}|^2\\ &= (-2i)\sum_{j=1}^{n}\mathfrak{f}_j^\dagger\wedge\mathfrak{f}_j\wedge = 2i\partial_{\underline{z}}\wedge\rceil\end{aligned}$$

or, in terms of the original basis vectors,

$$\Omega\wedge = \sum_{j=1}^{n} e_j \wedge e_{n+j}\wedge$$

Introducing the so-called spin-Euler operator, which is a parabivector valued multiplicative constant, i.e. the sum of a scalar and a bivector,

$$\beta = \sum_{j=1}^{n}\mathfrak{f}_j^\dagger\mathfrak{f}_j = \sum_{j=1}^{n}\left(\mathfrak{f}_j^\dagger\bullet\mathfrak{f}_j + \mathfrak{f}_j^\dagger\wedge\mathfrak{f}_j\right) = \frac{n}{2} + \sum_{j=1}^{n}\mathfrak{f}_j^\dagger\wedge\mathfrak{f}_j$$

we find that the fundamental form appears as the bivector part of that spin-Euler operator, meaning that we may write

$$\beta = \frac{n}{2} + \frac{i}{2}\Omega$$

Its complex conjugate is then given by

$$\beta^c = \sum_{j=1}^{n}\mathfrak{f}_j\mathfrak{f}_j^\dagger = \sum_{j=1}^{n}\left(\mathfrak{f}_j\bullet\mathfrak{f}_j^\dagger + \mathfrak{f}_j\wedge\mathfrak{f}_j^\dagger\right) = \frac{n}{2} - \sum_{j=1}^{n}\mathfrak{f}_j^\dagger\wedge\mathfrak{f}_j = \frac{n}{2} - \frac{i}{2}\Omega$$

Usually, one also introduces the associated fundamental form

$$\omega = \frac{1}{2i}\Omega = \frac{n}{2} - \beta = \beta^c - \frac{n}{2} = \sum_{j=1}^{n}\mathfrak{f}_j\wedge\mathfrak{f}_j^\dagger$$

For the sake of completeness we recall the following intertwining relations of the spin-Euler operator and its complex conjugate with the Witt basis vectors; for more of these intertwining relations we refer to [9].

Proposition 5.1. One has

(i) $[\mathfrak{f}_k, \beta] = \mathfrak{f}_k$, $[\mathfrak{f}_k^\dagger, \beta] = -\mathfrak{f}_k^\dagger$

(ii) $[\mathfrak{f}_k^\dagger, \beta^c] = \mathfrak{f}_k^\dagger$, $[\mathfrak{f}_k, \beta^c] = -\mathfrak{f}_k$

An important operator in Kähler geometry is the so-called L-operator, which is defined by means of the fundamental form.

Definition 5.2. The L-operator is defined as $L : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p+1,q+1} : F^{p,q} \mapsto \Omega \wedge F^{p,q}$, where, explicitly

$$\Omega \wedge F^{p,q} = \sum_{j=1}^{n} e_j \wedge e_{n+j} \wedge F^{p,q} = (-2i) \sum_{j=1}^{n} \mathfrak{f}_j^{\dagger} \wedge \mathfrak{f}_j \wedge F^{p,q} = 2i\partial_{\underline{z}} \wedge \rfloor F^{p,q}$$

The L-operator enjoys the properties listed in the proposition below (see also [21]).

Proposition 5.3. One has

(i) $[L, \partial_{\underline{X}} \wedge] = 0$, $[L, \partial_{\underline{X}} \bullet] = -\partial_{\underline{X}|} \wedge$

and also

(ii) $[L, \partial_{\underline{z}^{\dagger}} \wedge] = 0$, $[L, \partial_{\underline{z}} \wedge] = 0$

(iii) $[L, \partial_{\underline{z}^{\dagger}} \bullet] = i\partial_{\underline{z}^{\dagger}} \wedge$, $[L, \partial_{\underline{z}} \bullet] = -i\partial_{\underline{z}} \wedge$

The counterpart of the L-operator is the Λ-operator.

Definition 5.4. The Λ-operator is defined as $\Lambda : \mathcal{E}^{p,q} \longrightarrow \mathcal{E}^{p-1,q-1} : F^{p,q} \mapsto \Lambda F^{p,q}$, where, explicitly

$$\Lambda F^{p,q} = \sum_{j=1}^{n} e_j \bullet e_{n+j} \bullet F^{p,q} = (-2i) \sum_{j=1}^{n} \mathfrak{f}_j^{\dagger} \bullet \mathfrak{f}_j \bullet F^{p,q} = 2i\partial_{\underline{z}} \bullet \rfloor F^{p,q}$$

It shows the following properties (see also [21]).

Proposition 5.5. One has

(i) $[\Lambda, \partial_{\underline{X}} \wedge] = -\partial_{\underline{X}|} \bullet$, $[\Lambda, \partial_{\underline{X}} \bullet] = 0$

and also

(ii) $[\Lambda, \partial_{\underline{z}^{\dagger}} \wedge] = i\partial_{\underline{z}^{\dagger}} \bullet$, $[\Lambda, \partial_{\underline{z}} \wedge] = -i\partial_{\underline{z}} \bullet$

(iii) $[\Lambda, \partial_{\underline{z}^{\dagger}} \bullet] = 0$, $[\Lambda, \partial_{\underline{z}} \bullet] = 0$

A rather tedious computation leads to the commutator of the L and Λ operators.

Proposition 5.6. One has

$$[L\,,\,\Lambda]\; F^{p,q} = (n-p-q)F^{p,q}$$

Finally, putting for an arbitrary multivector function $F \equiv \sum_{p=0}^{n} \sum_{q=0}^{n} F^{p,q}$:

$$H\,[F] = \sum_{p=0}^{n} \sum_{q=0}^{n} (n-p-q)F^{p,q} = \left(n - 2\left(\partial_{\underline{z}^{\dagger}} \wedge \rfloor - \partial_{\underline{z}} \wedge \rfloor\right)\right) F$$

we obtain the following relations (see also [20]).

Proposition 5.7. One has

(i) $[L\,,\,\Lambda] = H$

(ii) $[H\,,\,\Lambda] = 2\Lambda$

(iii) $[H\,,\,L] = -2L$

meaning that the operators (L, Λ, H) generate the Lie algebra $\mathfrak{sl}_{\mathbb{C}}(2)$

6 The Hodge "star"-operator

The Hodge $*$-operator for smooth real differential forms in $\mathbb{R}^m$ may be defined as follows (see e.g. [19, 24]).

Definition 6.1. Let $\{j_1, \ldots, j_r\} \cup \{j_{r+1}, \ldots, j_m\} = \{1, \ldots, m\}$ and $\{j_1, \ldots, j_r\} \cap \{j_{r+1}, \ldots, j_m\} = \emptyset$, with $j_1 < \ldots < j_r$. Then

$$*\,(dX_{j_1} \wedge \cdots \wedge dX_{j_r}) = \sigma\, dX_{j_{r+1}} \wedge \cdots dX_{j_m}$$

where σ is the signature of the permutation $(j_{r+1}, \ldots, j_m, j_1, \ldots, j_r)$.

It constitutes an isomorphism $* : \bigwedge^r \longrightarrow \bigwedge^{m-r}$, its inverse being given by

$$*^{-1} = (-1)^{r(m-r)} *$$

which implies that

$$*^2 = (-1)^{r(m-r)}$$

By means of this $*$-operator the Hodge co-derivative d^* may be expressed in terms of the derivative d as

$$d^*\,\omega^r \;=\; (-1)^r\, *\, d\, *^{-1}\, \omega^r \;=\; (-1)^{r(m+1-r)}\, *\, d *\, \omega^r$$

In the actual case of even dimension ($m = 2n$) we find that the Hodge star-operator is an isomorphism $* : \bigwedge^{p,q} \longrightarrow \bigwedge^{n-q,n-p}$ for which $*^{-1} = (-1)^{(p+q)^2} *$ and thus $*^2 = (-1)^{(p+q)^2}$. For the Hodge derivative and co-derivative we then obtain

$$d^* = *\,d\,* \quad \text{and} \quad d = *\,d^*\,*$$

and similarly for the twisted versions

$$d|^* = *\,d|\,* \quad \text{and} \quad d| = *\,d^*|\,*$$

Simply applying the conversion rules of the foregoing section we obtain the counterparts of these relations in the setting of multivector functions, involving then the Dirac operators.

Proposition 6.2. One has

(i) $\partial_{\underline{X}} \bullet = * \left(\partial_{\underline{X}} \wedge\right) *$ and $\partial_{\underline{X}} \wedge = * \left(\partial_{\underline{X}} \bullet\right) *$

(ii) $\partial_{\underline{X}|} \bullet = * \left(\partial_{\underline{X}|} \wedge\right) *$ and $\partial_{\underline{X}|} \wedge = * \left(\partial_{\underline{X}|} \bullet\right) *$

and also

(iii) $\partial_{\underline{z}} \bullet = * \left(\partial_{\underline{z}} \wedge\right) *$ and $\partial_{\underline{z}} \wedge = * \left(\partial_{\underline{z}} \bullet\right) *$

(iv) $\partial_{\underline{z}^\dagger} \bullet = * \left(\partial_{\underline{z}^\dagger} \wedge\right) *$ and $\partial_{\underline{z}^\dagger} \wedge = * \left(\partial_{\underline{z}^\dagger} \bullet\right) *$

Clearly, we may convert Proposition 6.2 back to the differential form setting.

Proposition 6.3. One has

(i) $\partial^{*c} \bullet = * \left(\partial \wedge\right) *$ and $\partial \wedge = * \left(\partial^{*c} \bullet\right) *$

(ii) $\partial^{*} \bullet = * \left(\partial^{c} \wedge\right) *$ and $\partial^{c} \wedge = * \left(\partial^{*} \bullet\right) *$

Of course, it is also possible to express the Hodge star operator in $\mathbb{R}^m$ directly in the Clifford algebra setting; Definition 6.1 is then converted as follows (see [6]):

$$*(e_{j_1} \cdots e_{j_r}) = (-1)^{\frac{r(r+1)}{2}} \, e_M \, e_{j_1} \cdots e_{j_r}$$

where e_M is the so-called pseudoscalar given by $e_M = e_1 \cdots e_m$, of which the square equals $e_M^2 = (-1)^{\frac{m(m+1)}{2}}$. It follows that for 1-vectors the $*$-operation reduces to a multiplication from the left by $-e_M$. Also $*1 = e_M$ and $*e_M = 1$.

In the Hermitean case with even dimension $m = 2n$, let us compute e_M in terms of the Witt basis vectors. We consecutively obtain

$$\begin{aligned} e_M &= \prod_{\alpha=1}^{2n} e_\alpha = \prod_{j=1}^{n} (\mathfrak{f}_j - \mathfrak{f}_j^\dagger) \prod_{j=1}^{n} i(\mathfrak{f}_j + \mathfrak{f}_j^\dagger) \\ &= i^n \, (-1)^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} (\mathfrak{f}_j - \mathfrak{f}_j^\dagger)(\mathfrak{f}_j + \mathfrak{f}_j^\dagger) = i^n \, (-1)^{\frac{n(n-1)}{2}} \prod_{j=1}^{n} (\mathfrak{f}_j \mathfrak{f}_j^\dagger - \mathfrak{f}_j^\dagger \mathfrak{f}_j) \\ &= 2^n \, i^n \, (-1)^{\frac{n(n-1)}{2}} \, \mathfrak{f}_1 \wedge \mathfrak{f}_1^\dagger \wedge \mathfrak{f}_2 \wedge \mathfrak{f}_2^\dagger \wedge \cdots \wedge \mathfrak{f}_n \wedge \mathfrak{f}_n^\dagger \end{aligned}$$

showing that the pseudoscalar has bidegree (n, n).

As an example we have, for $m = 4, n = 2$, that the images under the $*$-operation of the Euclidean basis vectors are 3-vectors given by

$$*\,e_1 = -e_2 e_3 e_4, \quad *\,e_2 = e_1 e_3 e_4, \quad *\,e_3 = -e_1 e_2 e_4, \quad *\,e_4 = e_1 e_2 e_3$$

The Witt basis vectors $\mathfrak{f}_1$ and $\mathfrak{f}_2$, of bidegree $(0,1)$, transform into $(1,2)$-multivectors:

$$\begin{aligned} *\mathfrak{f}_1 &= -2\,\mathfrak{f}_1 \wedge \mathfrak{f}_2 \wedge \mathfrak{f}_2^\dagger \\ *\mathfrak{f}_2 &= -2\,\mathfrak{f}_1 \wedge \mathfrak{f}_1^\dagger \wedge \mathfrak{f}_2 \end{aligned}$$

while $\mathfrak{f}_1^\dagger$ and $\mathfrak{f}_2^\dagger$, of bidegree $(1,0)$, transform into $(2,1)$-multivectors:

$$\begin{aligned} *\mathfrak{f}_1^\dagger &= 2\,\mathfrak{f}_1^\dagger \wedge \mathfrak{f}_2 \wedge \mathfrak{f}_2^\dagger \\ *\mathfrak{f}_2^\dagger &= 2\,\mathfrak{f}_1 \wedge \mathfrak{f}_2 \wedge \mathfrak{f}_2^\dagger \end{aligned}$$

7 Afterword

In the previous sections we established and illustrated a "natural" isomorphism between on the one hand the algebra of complex differential forms (extended with the Hodge star operator and the inner product or dot product) with the underlying structure of a Grassmann algebra, and on the other hand the algebra of multi-vector functions in Hermitean Clifford analysis with the underlying structure of a complex Clifford algebra. The Hermitean Dirac operators, underlying the notion of Hermitean monogenicity, may well be identified with the Kählerian derivatives for complex differential forms, one of which is the famous $\overline{\partial}$ operator from several complex variables theory. It should be emphasized, as was done from the beginning, that only differential forms in $\mathbb{C}^n$ or in open regions thereof were considered, and that actually Hermitean Clifford analysis was developed only in flat space $\mathbb{C}^n$. As was also mentioned Hermitean Clifford analysis on curved Kählerian manifolds is the subject of the forthcoming paper [11]. Finally this paper is by no means a plea for substituting Hermitean multivector functions for complex differential forms. Both worlds, how convincing the similarities might be, have their own interest en properties; this paper intended to illustrate the very close connections between Hermitean Clifford analysis and complex analysis and the benefits obtained from exchanging knowledge between both.

Received: December 2009. Revised: April 2010.

References

[1] F. Brackx , J. Bureš, H. De Schepper, D. Eelbode, F. Sommen and V. Souček, *Fundaments of Hermitean Clifford Analysis*. Part I: Complex structure, Compl. Anal. Oper. Theory 1(3), 2007, 341–365.

[2] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen and V. Souček, *Fundaments of Hermitean Clifford Analysis*. Part II: Splitting of h-monogenic equations, Complex Var. Elliptic Eq. 52(10-11), 2007, 1063–1079.

[3] F. Brackx, B. De Knock and H. De Schepper, *A matrix Hilbert transform in Hermitean Clifford analysis*, J. Math. Anal. Appl. 344 (2008), 1068–1078.

[4] F. Brackx, B. De Knock, H. De Schepper and F. Sommen, *On Cauchy and Martinelli–Bochner integral formulae in Hermitean Clifford analysis*, Bull. Braz. Math. Soc. New Series 40(3), 2009, 395-416.

[5] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Pitman Publishers, Boston-London-Melbourne, 1982.

[6] F. Brackx, R. Delanghe and F. Sommen, *Differential forms and/or multivector functions*, Cubo 7(2), 2005, 139-169.

[7] F. Brackx, H. De Schepper, D. Eelbode and V. Souček, *The Howe dual pair in Hermitean Clifford analysis*, accepted for publication in Rev. Mat. Iberoam. 26(2), 2010, 449-479.

[8] F. Brackx, H. De Schepper and F. Sommen, *A theoretical framework for wavelet analysis in a Hermitean Clifford setting*, Comm. Pure Appl. Anal. 6(3), 2007, 549-567.

[9] F. Brackx, H. De Schepper and F. Sommen, *The Hermitian Clifford analysis toolbox*, Appl. Clifford Algebras 18(3-4), 2008, 451–487.

[10] F. Brackx, H. De Schepper and V. Souček, *On the Structure of Complex Clifford Algebra* Adv. Appl. Clifford Alg. DOI: 10.1007/s0006-010-0270-4.

[11] F. Brackx, H. De Schepper and V. Souček, *Hermitean Clifford Analysis on Kählerian manifolds* (in preparation).

[12] A. Damiano, D. Eelbode and I. Sabadini, *Invariant syzygies for the Hermitian Dirac operator*, Math. Zeitschrift 262, 2009, 929-945.

[13] R. Delanghe, R. Lávička and V. Souček, *On polynomial solutions of generalized Moisil-Théodoresco systems and Hodge-de Rham systems* (arXiv: 0908.0842).

[14] R. Delanghe, R. Lávička and V. Souček, *The Fischer decomposition for Hodge-de Rham systems in Euclidean spaces* (arXiv: 1012.4994).

[15] R. Delanghe, F. Sommen and V. Souček, *Clifford Algebra and Spinor–Valued Functions*, Kluwer Academic Publishers, Dordrecht, 1992.

[16] D. Eelbode, *Stirling numbers and spin-Euler polynomials*, Exp. Math. 16(1) (2007), 55-66.

[17] J. Gilbert and M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge University Press, Cambridge, 1991.

[18] K. Gürlebeck, K. Habetha and W. Sprössig, *Holomorphic Functions in the Plane and n-dimensional Space*, Birkhäuser Verlag, Basel, 2008.

[19] K. Maurin, *Analysis, part II*, D. Reidel Publishing Company, Dordrecht - Boston - London, PWN-Polish Scientific Publishers, Warszawa, 1980.

[20] M. L. Michelsohn, Clifford and Spinor *Cohomology of Kähler Manifolds*, American Journal of Mathematics 102(6) (1980), 1083-1146.

[21] A. Moroianu, *Lectures on Kähler geometry*, London Mathematical Society Student Texts 69, Cambridge University Press (Cambridge, 2007).

[22] I. R. Porteous, *Topological Geometry*, Van Nostrand Reinhold Company, London - New York - Toronto - Melbourne, 1969.

[23] I. Sabadini and F. Sommen, *Hermitian Clifford analysis and resolutions*, Math. Meth. Appl. Sci. 25 (16–18) (2002), 1395–1414.

[24] C. von Westenholz, *Differential Forms in Mathematical Physics*, Stud. Math. Appl., vol 3, North-Holland, Amsterdam, 1978.

CUBO A Mathematical Journal
Vol.13, Nº02, (119–126). June 2011

Degree theory for the sum of VMO maps and maximal monotone maps

Yuqing Chen
Faculty of Applied Mathematics,
Guangdong University of Technology,
Guangdong 510006, P. R. China,
email: ychen64@163.com
Donal O'Regan
Department of Mathematics,
National University of Ireland,
Galway, Ireland,
email: donal.oregan@nuigalway.ie
and
Ravi P. Agarwal
Department of Mathematical Science,
Florida Institute of Technology
Melbourne, FL, 32901, USA,
email: agarwal@fit.edu

ABSTRACT

Let $\Omega \subset \mathsf{R}^n$ be an open bounded domain, $\mathsf{f} : \Omega \to \mathsf{R}^n$ a VMO map, and $\mathsf{T} : \mathsf{D}(\mathsf{T}) \subseteq \mathsf{R}^n \to \mathsf{R}^n$ a maximal monotone map with $\mathsf{D}(\mathsf{T}) \cap \Omega \neq \emptyset$. We construct a degree for the sum of $\mathsf{f} + \mathsf{T}$, which can be viewed as a generalization of the degree both for VMO maps and maximal monotone maps.

RESUMEN

Sea $\Omega \subset \mathbb{R}^n$ un dominio abierto, $f : \Omega \to \mathbb{R}^n$ un mapa VMO, y $T : D(T) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ un mapa monotono maximal con $D(T) \cap \Omega \neq \emptyset$. Construimos un grado por la suma de $f + T$, que se puede ver como una generalización de la medida, tanto para los mapas de VMO y para los mapas monotono maximal.

Keywords and phrases: Degree theory, Maximal monotone map.
Mathematics Subject Classification: 47H11, 47H05

1. Introduction

Degree theory for continuous maps in finite dimensional spaces has a long history and has been extensively studied. In the early 80's of the last century a degree for some classes of non-continuous maps was established (see [8,1,17,18] and the references therein). In 1995 and 1996, H. Brezis and L. Nirenberg [12], [13] invented a degree theory for VMO maps; see [2-6,9-11,19,21,22]. Generally, VMO functions need not be continuous. Another important class of non-continuous maps is the class of maximal monotone maps, and there is no relation between the VMO maps and the maximal monotone maps. In this paper, we consider the sum of a VMO map and a maximal monotone map, and we will define a degree theory for such a map. First we recall some definitions. Let Ω be an open bounded domain in $\mathbb{R}^n$. The class of bounded mean oscillation functions (see [20]) are defined as

$$BMO(\Omega) = \{f : \Omega \to \mathbb{R}^n \text{ is locally integrable, and } |f|_{BMO} < \infty\},$$

where $|f|_{BMO} = \sup_{B \subset \Omega} \frac{1}{m(B)} \int_B |f(x) - \bar{f}| dx$, $\bar{f} = \frac{1}{m(B)} \int_B f(x) dx$ (here $m(\cdot)$ represents the Lebesgue measure), and the class of vanishing mean oscillation functions (see [23]) are defined as

$$VMO(\Omega) = \{f : \Omega \to \mathbb{R}^n \text{ is locally integrable, and } \lim_{m(B) \to 0} \frac{1}{m(B)} \int_B |f(x) - \bar{f}| dx = 0\},$$

where $B \subset \mathbb{R}^n$ is an open ball with its closure contained in Ω. It is well known that if $f \in VMO$, then $f_\epsilon(x) = \frac{1}{m(B_\epsilon(x))} \int_{B_\epsilon(x)} f(y) dy$ is continuous in ϵ and x where it is defined. Let $T : D(T) \subset \mathbb{R}^n \to \mathbb{R}^n$ be a function. If $(h - g, x - y) \geq 0$ for all $x, y \in D(T)$ and $h \in Tx, g \in Ty$, then T is said to be monotone. If T is monotone and T has no monotone extension in $\mathbb{R}^n$, then T is said to be maximal monotone. It is well known that T is maximal monotone iff T is monotone and $T + \epsilon I$ is surjective for all $\epsilon > 0$. If T is maximal monotone, we use $T_\epsilon = (T^{-1} + \epsilon I)^{-1}$ to represent the Yosida approximation, and $R_\epsilon = I - \epsilon T_\epsilon$, the resolvent with respect to T_ϵ. For maximal monotone maps we refer the reader to [7]. Let $f : \Omega \to \mathbb{R}^n$ be a VMO map, $T : D(T) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ a maximal monotone map, $p \in \mathbb{R}^n$, and $D(T) \cap \Omega \neq \emptyset$. Under appropriate assumptions, see (2.1) below, we define the degree $\deg(f + T, \Omega \cap D(T), p)$. If $T = 0$, this degree coincides with the degree for VMO

maps in [13], and if $f = 0$, then it coincides with the degree for maximal monotone maps (see [14-16]).

2. Results

In this section, $\Omega \subset R^n$ is an bounded open domain, $f \in VMO(\Omega)$, $T : D(T) \subseteq R^n \to R^n$ is a maximal monotone map, $p \in R^n$, and $\Omega \cap D(T) \neq \emptyset$. Suppose there exists an open neighborhood U of $\partial\Omega$ in Ω and a constant $\beta > 0$ such that

$$\frac{1}{m(B_\epsilon(y))}\int_{B_\epsilon(y)} |f(x) + g - p|dx \geq \beta \tag{2.1}$$

for all $0 < \epsilon < \frac{1}{2}d(y, \partial\Omega)$, $g \in Tz$, $z \in D(T) \cap B_\epsilon(y)$, where $B_\epsilon(y)$ is an open ball centered at y with radius ϵ such that $\overline{B_\epsilon(y)} \subset U$, and $d(y, \partial\Omega)$ is the distance between y and $\partial\Omega$.

We remark that if $T = 0$, then (2.1) was first used in [13]. If $f = 0$, then (2.1) is equivalent to $|g - p| \geq \beta$ for all $z \in D(T) \cap U$ and $g \in Tz$, and in this case Proposition 2.1 below shows that the assumption $p \notin T(\partial\Omega \cap D(T))$ will guarantee (2.1) holds.

Proposition 2.1. If $p \notin T(\partial\Omega \cap D(T))$, then there exists $d_0 > 0, \alpha_0 > 0$ such that $d(p, Tx) \geq d_0$ for all $x \in \Omega \cap D(T)$ with $d(x, \partial\Omega) < \alpha_0$.

Proof. Suppose the conclusion is not true. There exist $x_n \in \Omega \cap D(T)$, $g_n \in Tx_n$ such that $d(x_n, \partial\Omega) \to 0$, and $g_n - p \to 0$. Without loss of generality, we may assume that $x_n \to x_0 \in \partial\Omega$.

Since $(g_n - g, x_n - x) \geq 0$ for all $x \in D(T)$, $g \in Tx$, we have

$$(p - g, x_0 - x) \geq 0, \text{ for all } x \in D(T), g \in Tx.$$

Therefore $x_0 \in \partial\Omega \cap D(T)$, $p \in Tx_0$, which is a contradiction.

As in [13], we define $\Omega_\epsilon = \{x \in \Omega : d(x, \partial\Omega) > 2\epsilon\}$ for each $\epsilon > 0$. By definition of VMO functions, there exists $\epsilon_0 > 0$ such that

$$\frac{1}{m(B_\epsilon(x))}\int_{B_\epsilon(x)} |f(y) - \bar{f}|dy < \frac{\beta}{2} \tag{2.2}$$

for all $\epsilon < \epsilon_0$, $x \in \Omega$ and $\epsilon < \frac{d(x,\partial\Omega)}{2}$. We may also take ϵ_0 such that $\{x \in \Omega : d(x, \partial\Omega) \leq 3\epsilon_0\} \subset U$, where U is the same as in (2.1). Now for $0 < \epsilon < \epsilon_0$, and $x \in \partial\Omega_\epsilon \cap D(T)$, $g \in Tx$, by (2.1) and (2.2), we obtain

$$|f_\epsilon(x) + g - p| \geq \frac{\beta}{2}, \tag{2.3}$$

where $f_\epsilon(x) = \frac{1}{m(B_\epsilon(x))}\int_{B_\epsilon(x))} f(y)dy$.

Lemma 2.2. Suppose $|f_\varepsilon(x) + g - p| \geq \frac{\beta}{2}$, for $x \in \partial\Omega_\varepsilon \cap D(T)$, $g \in Tx$. Then there exists $\lambda_0(\varepsilon) > 0$ such that

$$p \neq f_\varepsilon(x) + T_\lambda(x), \text{ for all } x \in \partial\Omega_\varepsilon, \lambda \in (0, \lambda_0(\varepsilon)).$$

Proof. If this is not true, there exist $\lambda_n \to 0^+$, $x_n \in \partial\Omega_\varepsilon$ with $x_n \to x_0 \in \partial\Omega_\varepsilon$, such that

$$f_\varepsilon x_n + T_{\lambda_n} x_n = p, \ \ n \in \{1, 2, \cdots\}.$$

Since $f_\varepsilon x_n \to f_\varepsilon x_0$, $R_{\lambda_n} x_n = x_n - \lambda_n T_{\lambda_n} x_n \to x_0$, the maximal monotonicity of T implies that $x_0 \in D(T)$, and $p - f_\varepsilon x_0 \in Tx_0$, which is a contradiction.

Now, assume that (2.1) holds. In view of (2.3) and Lemma 2.2, we define the degree $\deg(f + T, \Omega \cap D(T), p)$ by

$$\deg(f + T, \Omega \cap D(T), p) = \lim_{\varepsilon \to 0^+} \lim_{\lambda \to 0^+} \deg(f_\varepsilon + T_\lambda, \Omega_\varepsilon, p). \tag{2.4}$$

We claim this definition is reasonable. First, for each $\varepsilon < \varepsilon_0$, and $\lambda_1, \lambda_2 \in (0, \lambda_0(\varepsilon))$, since $T_{t\lambda_1 + (1-t)\lambda_2} x$ is continuous in (t, x) (see Corollary 2.8 in [15]) we know that $\{f_\varepsilon + T_{t\lambda_1+(1-t)\lambda_2}\}_{t \in [0,1]}$ is a homotopy, so

$$\deg(f_\varepsilon + T_{\lambda_1}, \Omega_\varepsilon, p) = \deg(f_\varepsilon + T_{\lambda_2}, \Omega_\varepsilon, p).$$

Now, for any $\varepsilon \in (0, \varepsilon_0)$, by the continuity of $f_t(x)$ in (t, x) and (2.3), there exists $\delta > 0$ such that

$$|f_t(x) + g - p| > \frac{\beta}{4},$$

for $|t - \varepsilon| \leq \delta$ and $x \in \partial\Omega_\varepsilon$ and $g \in Tx$. The same proof as in Lemma 2.2 guarantees that there exists $\lambda_1 > 0$ such that

$$p \neq f_t(x) + T_\lambda(x), \text{ for all } x \in \partial\Omega_\varepsilon, |t - \varepsilon| \leq \delta, \lambda \in (0, \lambda_1),$$

so $\deg(f_t + T_\lambda, \Omega_\varepsilon, p)$ is well defined for $\lambda \in (0, \lambda_1)$, and $|t - \varepsilon| \leq \delta$. By homotopy invariance, we have

$$\deg(f_t + T_\lambda, \Omega_\varepsilon, p) = \deg(f_\varepsilon + T_\lambda, \Omega_\varepsilon, p),$$

so the degree in (2.4) is well defined.

For a measurable function $f : \Omega \to R^n$, we recall that the essential range of f is defined as the smallest closed subset $essR(f)$ such that $f(x) \in essR(f)$ a. e. $x \in \Omega$ (see [12]).

Proposition 2.3. If $\deg(f + T, \Omega \cap D(T), p) \neq 0$, then $p \in \overline{essR(f) + T(\overline{\Omega} \cap D(T))}$.

Proof. Suppose the conclusion is not true. Then exists $r > 0$ such that $B(p, r) \cap essR(f) + T(\overline{\Omega} \cap D(T)) = \emptyset$. Set $\Sigma = R^n \setminus (B(p, r) - T(\overline{\Omega} \cap D(T)))$. Clearly, $essR(f) \subset \Sigma$. Also $f(x) \in essR(f)$,

a. e. $x \in \Omega$, and $f \in VMO(\Omega)$, so we deduce that $\lim_{\epsilon\to 0^+} d(f_\epsilon(x), \Sigma) = 0$ uniformly. Therefore, there exists $\epsilon_1 \in (0, \epsilon_0)$ such that

$$|f_\epsilon(x) - p + g| \geq \frac{r}{2},$$

for all $x \in \Omega$, $z \in D(T) \cap \overline{\Omega}$, $g \in Tz$, $\epsilon \in (0, \epsilon_1)$.

Thus $\deg(f_\epsilon + T_\lambda, \Omega, p) = 0$ for all $\lambda \in (0, \lambda_0(\epsilon))$, and $\epsilon \in (0, \epsilon_1)$. Consequently, it follows from the definition that $\deg(f + T, \Omega \cap D(T), p) = 0$, which is a contradiction.

Proposition 2.4. Let $\{h_t(\cdot)\}_{t\in[0,1]}$ be a family of functions in $VMO(\Omega)$, and $h_t(\cdot)$ depends continuously on the parameter t in the topology of $BMO \cap L^1_{loc}(\Omega)$. Assume that there exists an open neighborhood U of $\partial\Omega$ in Ω and a constant $\beta > 0$ such that

$$\frac{1}{m(B_\epsilon(y))} \int_{B_\epsilon(y)} |h_t(x) + g - p| dx \geq \beta \tag{2.5}$$

for all $0 < \epsilon < \frac{1}{2} d(y, \partial\Omega)$, $g \in Tz$, $z \in D(T) \cap B_\epsilon(y)$, $t \in [0, 1]$, where $B_\epsilon(y)$ is an open ball centered at y with radius ϵ such that $\overline{B_\epsilon(y)} \subset U$. Then $\deg(h_t + T, \Omega \cap D(T), p)$ does not depend on $t \in [0, 1]$.

Proof. Since $h_t(\cdot)$ depends continuously on the parameter t in the topology of $BMO \cap L^1_{loc}(\Omega)$, we have

$$\lim_{m(B)\to 0} \frac{1}{m(B)} \int_B |h_t(x) - \overline{h_t}| = 0, \tag{2.6}$$

uniformly in t. From (2.5), (2.6), and using the same proof as in (2.3), we know that there exists $\epsilon_0 > 0$, such that

$$|h_{t,\epsilon}(x) + g - p| \geq \frac{\beta}{2}, \tag{2.7}$$

for all $x \in \partial\Omega_\epsilon \cap D(T)$, $g \in Tx$, $t \in [0, 1]$, $\epsilon \in (0, \epsilon_0)$. By using the same proof as in Lemma 2.2, we know that there exists $\lambda(\epsilon) > 0$, such that

$$p \neq h_{t,\epsilon}(x) + T_\lambda x,$$

for all $x \in \partial\Omega_\epsilon$, $t \in [0, 1]$, $\lambda \in (0, \lambda(\epsilon))$. Thus $\deg(h_{t,\epsilon} + T_\lambda, \Omega_\epsilon, p)$ does not depend on t for each $\epsilon \in (0, \epsilon_0)$, $\lambda \in (0, \lambda(\epsilon))$. Thus $\deg(h_t + T, \Omega \cap D(T), p)$ does not depend on $t \in [0, 1]$.

Corollary 2.5. Let $f_1, f_2 \in VMO(\Omega)$ satisfying (2.1). Suppose there exists $0 < \beta_0 < \beta$ such that

$$\frac{1}{m(B)} \int_B |f_1(x) - f_2(x)| dx < \beta_0,$$

for all $B \subset U$. Then $\deg(f_1 + T, \Omega \cap D(T), p) = \deg(f_2 + T, \Omega \cap D(T), p)$.

Proof. Set $h_t(x) = tf_1(x) + (1 - t)f_2(x)$ for $t \in [0, 1]$, $x \in \Omega$. Then it is easy to see that h_t depends continuous on t in the topology of $BMO \cap L^1_{loc}(\Omega)$. Also we have

$$\frac{1}{m(B_\epsilon(y))} \int_{B_\epsilon(y)} |h_t(x) + g - p| dx \geq \beta - \beta_0$$

for all $0 < \epsilon < \frac{1}{2}d(y, \partial\Omega)$, $g \in Tz$, $z \in D(T) \cap B_\epsilon(y)$, $t \in [0,1]$, where $B_\epsilon(y)$ is an open ball centered at y with radius ϵ such that $\overline{B_\epsilon(y)} \subset U$. Therefore the conclusion follows from Proposition 2.4.

Proposition 2.6. Let $T_i : D \subseteq R^n$, $i = 1, 2$, be two maximal monotone maps. If $tT_1 + (1-t)T_2$ is maximal monotone for each $t \in [0,1]$, and there exist an open neighborhood U of $\partial\Omega$ in Ω and a constant $\beta > 0$ such that

$$\frac{1}{m(B_\epsilon(y))}\int_{B_\epsilon(y)} |f(x) + g_t - p| dx \geq \beta \tag{2.8}$$

for all $0 < \epsilon < \frac{1}{2}d(y, \partial\Omega)$, $g_t \in [tT_1 + (1-t)T_2]z$, $z \in D \cap B_\epsilon(y))$, $t \in [0,1]$, where $B_\epsilon(y)$ is an open ball centered at y with radius ϵ such that $\overline{B_\epsilon(y)} \subset U$. Then $deg(f + [tT_1 + (1-t)T_2], \Omega \cap D, p)$ does not depend on $t \in [0,1]$.

Proof. By (2.8), using the same proof as in (2.3), we know that there exists $\epsilon_0 > 0$, such that

$$|f_\epsilon(x) + g_t - p| \geq \frac{\beta}{2}, \tag{2.9}$$

for all $x \in \partial\Omega_\epsilon \cap D$, $g_t \in tT_1 x + 1 - t)T_2 x$, $t \in [0,1]$, $\epsilon \in (0, \epsilon_0)$. From (2.9), and using the same proof as in Lemma 2.2, we know that there exists $\lambda(\epsilon) > 0$, such that

$$p \neq f_\epsilon(x) + T^t_\lambda x,$$

for all $x \in \partial\Omega_\epsilon$, $t \in [0,1]$, $\lambda \in (0, \lambda(\epsilon))$, where T^t_λ is the Yosida approximation of $tT_1 + (1-t)T_2$. From Lemma 2.7 in [15], we know

$$deg(f_\epsilon + T^t_\lambda, \Omega_\epsilon, p)$$

does not depend on $t \in [0,1]$, $\lambda \in (0, \lambda(\epsilon))$. Therefore, $deg(f + [tT_1 + (1-t)T_2], \Omega \cap D, p)$ does not depend on $t \in [0,1]$.

Acknowledgement: The first author was supported by a NSFC grant, grant no. 10871052.

Received: April 2009. Revised: May 2010.

Referencias

[1] A. Boutet de Monvel-Berthier, V. Georgescu and R. Purice, *A boundary value problem related to the Ginzburg-Landau model*, Comm. Math. Phys 1991 pag. 1-23.

[2] J. Bourgain, H. Brezis and P. Mironescu, *Lifting in Sobolev spaces*, J. Analyse Math. 2000 pag. 37-86.

[3] J. Bourgain, H. Brezis and P. Mironescu, *Another look at Sobolev spaces, in Optimal Control and Partial Differential Equations*, (J.L. Menaldi, E. Rofman et A. Sulem, eds), a volume in honour of A. Bensoussan¡¯s 60th birthday IOS Press 2001 pag. 439-455.

[4] J. Bourgain, H. Brezis and P. Mironescu, *H1/2 maps into the circle: minimal connections, lifting, and the Ginzburg-Landau equation*, Publications math¡äematiques de l¡¯ IHES 2004 pag. 1-115.

[5] J. Bourgain, H. Brezis and P. Mironescu, *Lifting, Degree and Distributional Jacobian Revisited*, Comm. Pure Appl. Math. 2005 pag. 529-551.

[6] J. Bourgain, H. Brezis and H.-M. Nguyen, *A new estimate for the topological degree*, C. R. Acad. Sc. Paris 2005 pag. 787-791.

[7] H. Brezis, *Operateurs Maximaux monotones* North-Holland 1973.

[8] H. Brezis and J. M. Coron, *Large solutions for harmonic maps in two dimensions*, Comm. Math. Phys. 1983 pag. 203-215.

[9] H. Brezis, *Degree theory: old and new, in Topological Nonlinear Analysis II: Degree, Singularity and Variations*, (M. Matzeu and A. Vignoli ed.) Birkhauser 1997 pag. 87-108.

[10] H. Brezis and Y. Li, *Topology and Sobolev spaces*, J. Funct. Anal. 2001 pag. 321-369.

[11] H. Brezis, Y. Li, P. Mironescu and L. Nirenberg, *Degree and Sobolev spaces Topological methods in Nonlinear Analysis* 1999 pag. 181-190.

[12] H. Brezis and L. Nirenberg, *Degree theory and BMO*, Part I : compact manifolds without boundaries Selecta Math. 1995 pag. 197-263.

[13] H. Brezis and L. Nirenberg , *Degree Theory and BMO*, Part II: Compact Manifolds with Boundaries Selecta Math. 1996 pag. 1-60.

[14] D. O'Regan, Y. J. Cho, Y. Q. Chen, *Topological Degree Theory and Applications*. Chapman and Hall/CRC Press 2006.

[15] Y.Q.Chen, D.O'Regan, *On the homotopy property of topological degree for maximal monotone mappings*. Appl. Math. Comput. 2009 pag. 373-377.

[16] Y.Q.Chen, D. O'Regan, *F.L.Wang, R. Agarwal A note on degree theory for maximal monotone mappings in finite dimensional spaces*, Appl. Math. Lett. 2009 pag. 1766-1769.

[17] M.J. Esteban and S. Miiller, *Sobolev maps with integer degree and applications to Skyrme's problem*, Proc. Roy. Soc. London A 1992 pag. 197-201.

[18] M. Giaquinta, G. Modica and J. Soucek, *Remarks on the degree theory*, J. Funct. Anal. 1994 pag. 172-200.

[19] F.B. Hang and F.H. Lin, *Topology of Sobolev mappings II*, Acta Math. 2003 pag. 55-107.

[20] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. 1961 pag. 415-426.

[21] J. Korevaar, *On a question of Brezis and Nirenberg concerning the degree of circle maps Selecta Math.* 1999 pag. 107-122.

[22] P. Mironescu and A. Pisante, *A variational problem with lack of compactness for H1/2(S1, S1) maps of prescribed degree.* J. Funct. Anal. 2004 pag. 249-279.

[23] D. Sarason, *Functions of vanishing mean oscillation Trans.* Amer. Math. Soc. 1975 pag. 391-405

CUBO A Mathematical Journal
Vol.13, Nº02, (127–137). June 2011

Module amenability for Banach modules

D. Ebrahimi Bagha
Department of Mathematics,
Islamic Azad University,
Central Tehran Branch, Tehran, Iran.
email: d.ebrahimibagha@iauctb.ac.ir
and
M. Amini
Department of Mathematics,
Tarbiat Modares University,
P.O.Box 14115-175, Tehran, Iran.
email: mamini@modares.ac.ir

ABSTRACT

We study the module amenability of Banach modules. This is a natural generalization of Johnson's amenability of Banach algebras. As an example we show that for a discrete abelian group G, $\ell^{\mathsf{p}}(\mathsf{G})$ is amenable as an $\ell^1(\mathsf{G})$-module if and only if G is amenable, where $\ell^1(\mathsf{G})$ is a Banach algebra with pointwise multiplication.

RESUMEN

Se estudia el módulo de receptividad de los módulos de Banach. Esta es una generalización natural de la receptividad de Johnson de las álgebras de Banach. Como ejemplo se muestra que para un grupo abeliano discreto G $\ell^{\mathsf{p}}(\mathsf{G})$ es receptivo como un G $\ell^{\mathsf{p}}(\mathsf{G})$-módulo, si y sólo si G es receptivo, donde $\ell^1(\mathsf{G})$ es un álgebra de Banach con producto punto.

Keywords and phrases: Banach modules, module amenability, weak module amenability, semigroup algebra, inverse semigroup.
Mathematics Subject Classification: 43A07, 46H25

1. Introduction

The concept of amenability for Banach algebras was introduced by B.E. Johnson in [J]. The main example in [J] asserts that the group algebra $L^1(G)$ of a locally compact group G is amenable if and only if G is amenable. This is far from true for semigroups. If S is a discrete inverse semigroup with the set of idempotents E_S, $\ell^1(S)$ is amenable if and only if E_S is finite and all the maximal subgroups of S are amenable [DN]. For an arbitrary discrete semigroup S, $\ell^1(S)$ is amenable if and only if the minimum ideal of S exists and is an amenable group and S has a principal series whose corresponding quotients are regular Rees matrix semigroups of special form [DLS, 10.12].This failure is partly due to the fact that $\ell^1(S)$ is equipped with two (related) algebraic structures. It is a Banach algebra and a Banach module over $\ell^1(E_S)$. This consideration was the motivation of the second named author to study the concept of *module amenability* for Banach algebras which have an extra Banach module structure (with compatible actions) in [A]. In particular it is shown in [A] that for an inverse semigroup S, $\ell^1(S)$ is module amenable as a Banach module over $\ell^1(E_S)$ if and only if S is amenable. The authors introduced the concept of *weak module amenability* in [AE] and showed that for a commutative inverse semigroup S, $\ell^1(S)$ is always weak module amenable as a Banach module over $\ell^1(E_S)$.

The present paper investigates module amenability from a different angle. There are many examples of Banach modules which do not have any natural algebra structure. One example is $L^p(G)$ which is a left Banach $L^1(G)$-module, for a locally compact group G [D, 3.3.19]. As another example of this sort, one may consider a Banach algebraic bundle over a locally compact group G [FD]. Then the fibers on elements of G are Banach modules over the fiber on the identity. Crossed products of Banach algebra by groups are special cases of Banach algebraic bundles. The theory of module amenability developed in [A] does not cover these examples. There is one thing in common in these examples and that is the existence of a module homomorphism from the Banach module to the underlying Banach algebra. For instance in the case of crossed products, X is a Banach algebra, G is a topological group, and $X_g = X \times \{g\}$, for $g \in G$, and $\{X_g\}$ is a Banach algebraic bundle over G. In this case we have a module homomorphism $\Delta_g : X_g \to X_e$ which sends (x, g) to (x, e), where e is the identity of G. Also if G is a compact group and $f \in L^q(G)$, then one has the module homomorphism $\Delta_f : L^p(G) \to L^1(G)$ which sends g to $f * g$.

In this paper, the concept of module amenability (more precisely Δ-amenability) is defined for a Banach module E over a Banach algebra A with a given module homomorphism $\Delta : E \to A$. The next section gives the basic properties of module amenability and in particular establishes the equivalence of this concept with the existence of module virtual (approximate) diagonals in an appropriate sense. Section 3 covers the weak Δ-amenability. A few examples are discussed in the last section.

2. Module Amenability

Let A be a Banach algebra and E be a Banach space with a left A-module structure such that, for some $M > 0$,

$$\|a.x\| \leq M\|a\|\,\|x\| \quad (a \in A, x \in E),$$

then E is called a left Banach A-module. Right and two-sided Banach A-modules are defined similarly. Throughout this section E is a Banach A-bimodule and $\Delta : E \to A$ is a bounded Banach A-bimodule homomorphism.

Definition 2.1. Let X be a Banach A-bimodule. A bounded linear map $D : A \to X$ is called a module derivation (or more specifically a Δ-derivation) if

$$D(\Delta(a.x)) = a.D(\Delta(x)) + D(a).\Delta(x),\ \ D(\Delta(x.a)) = D(\Delta(x)).a + \Delta(x).D(a),$$

For each $a \in A$ and $x \in E$. Also D is called inner (or Δ-inner) if there is $f \in X$ such that

$$D(\Delta(x)) = f.\Delta(x) - \Delta(x).f =: D_f(\Delta(x)) \quad (x \in E).$$

When Δ has a dense range, D_f extends uniquely to a Δ-derivation from A to X.

Definition 2.2. A bimodule E is called *module amenable* (or more specifically Δ-amenable as a A-bimodule) if for each Banach A-bimodule X, all Δ-derivations from A to X^* are Δ-inner.

It is clear that A is A-module amenable (with $\Delta = id$) if and only if it is amenable as a Banach algebra. A right *bounded approximate identity* of E is a bounded net $\{a_\alpha\}$ in A such that for each $x \in E$, $\|\Delta(x).a_\alpha - \Delta(x)\| \to 0$, as $\alpha \to \infty$. The left and two-sided approximate identities are defined similarly.

Proposition 2.3. If E is module amenable, then E has a bounded approximate identity.

Proof Consider the double conjugate space A^{**} as a Banach A-module with

$$\langle F.a, f\rangle = \langle F, a.f\rangle, \langle a.f, b\rangle = f(ba), a.F = 0 \quad (a, b \in A, f \in A^*, F \in A^{**}).$$

Then the canonical embedding $D : A \to A^{**}$ is a module derivation, hence $D = D_F$ on $\Delta(E)$, for some $F \in A^{**}$. Choose a net $\{a_\alpha\}$ in A which is w^*-convergent to F in A^{**}. Clearly $\{a_\alpha\}$ is a left bounded approximate identity of E. Right and two sided approximate identities now could be constructed similar to the classical case [D]. □

Definition 2.4. A Banach A-module X is called right Δ-essential if for each $x \in X$ there is $a \in \Delta(E)$ and $y \in X$ such that $x = y.a$. The left Δ-essential and (two sided) Δ-essential modules are defined similarly.

The following two results are proved as in the classical case [J]. We just include the proof of Lemma 2.5(i), as it involves a variation of the Cohen factorization theorem.

Lemma 2.5. (i) If Δ has a dense range and E has a (right) bounded approximate identity, then E is module amenable iff for each (right) Δ-essential Banach A-bimodule X, all Δ-derivations from A to X^* are Δ-inner.

(ii) If E and $E^{'}$ are Banach A-modules with module homomorphisms Δ and $\Delta^{'}$ and $\theta : E \to E^{'}$ is a bounded module map with dense range such that $\Delta^{'} \circ \theta = \Delta$, then Δ-amenability of E implies $\Delta^{'}$-amenability of $E^{'}$.

(iii) If J is a closed submodule of E and J_Δ is the closed ideal of A generated by $\Delta(J)$, and $q : A \to A/J_\Delta$ and $\tilde{q} : E \to E/J$ are the corresponding quotient maps, then E is Δ amenable whenever J is $\Delta_{|J}$-amenable and E/J is $\tilde{\Delta}$-amenable, where $\tilde{\Delta} : E/J \to A/J_\Delta$ is the unique A/J_Δ-module map with $\tilde{\Delta} \circ \tilde{q} = q \circ \Delta$.

Proof We prove part (i) as promised. We just need to check the necessity. Let $\{a_\alpha\} \subseteq A$ be a right bounded approximate identity for E. Let X be a Banach A-bimodule. Consider $T_\alpha : X^* \to X^*$ defined by $T(f) = a_\alpha.f$, for $f \in X^*$, where $a.f(x) = f(x.a)$, for $a \in A, x \in X$. Since $\{a_\alpha\}$ is bounded in A, $\{T_\alpha\}$ is bounded in $\mathcal{B}(X^*)$. Hence it has a w^*-cluster point T. We may assume that $T_\alpha \to T$ in w^*-topology.

For each $e \in E, x \in X, f \in X^*$, we have

$$\begin{aligned}\langle x, \Delta(e), Tf\rangle &= \lim_\alpha \langle x, \Delta(e), T_\alpha f\rangle = \lim_\alpha \langle x, \Delta(e), a_\alpha.f\rangle \\ &= \lim_\alpha \langle x, \Delta(e)a_\alpha, f\rangle = \langle x, \Delta(e), f\rangle.\end{aligned}$$

Hence $T - I : X^* \to (X.\Delta(E))^\perp$ is a bounded projection and we have the admissible short exact sequence

$$0 \to (X.\Delta(E))^\perp \to X^* \to (X.\Delta(E))^* \to 0$$

of Banach A-bimodules. But $\Delta(E).(X/\overline{(X.\Delta(E))}) = 0$ and Δ has a dense range, hence each bounded Δ-derivation $D_1 : A \to (X.\Delta(E))^\perp = (X/\overline{(X.\Delta(E))})^*$ is zero. On the other hand, each bounded Δ-derivation $D_2 : A \to (X.\Delta(E))^*$ is Δ-inner, by assumption. Therefore each bounded Δ-derivation $D : A \to (X.\Delta(E))^*$ is Δ-inner, and we are done. □

Lemma 2.6. Assume that A and B are Banach algebras, J is a closed ideal of A, E is a Banach A-module, and $\Delta : E \to A$ is an A-module homomorphism.

(i) If F is a Banach A-module and $\Phi : E \to F$ is an A-module homomorphism with dense range, then Δ-amenability of E implies $\Delta \circ \Phi$-amenability of F.

(ii) If $\Psi : A \to B$ is a Banach algebra epimorphism with

$$E.\mathrm{Ker}(\Psi) = \mathrm{Ker}(\Psi).E = \{0\},$$

and E is considered as a B-module via

$$b.x := a.x,\ x.b := x.a \quad (b \in B, x \in E),$$

where $a \in A$ on the right hand side is any element with $b = \Psi(a)$. Then Δ-amenability of E, as an A-module, implies $\Psi \circ \Delta$-amenability of E as a B-module.

(iii) In (ii), if $B = A/J$, $\Psi : A \to A/J$ is the quotient map, and $E.J = J.E = \{0\}$, then Δ-amenability of E, as an A-module, implies $\Psi \circ \Delta$-amenability of E as a A/J-module.

(iv) If I is a closed ideal of A, E' is the closed submodule of E generated by IE, and $\Delta' : E' \to I$ is the restriction of $\Delta : E \to A$, then E' is Δ'-amenable whenever E is Δ-amenable and E' has a bounded approximate identity.

Proposition 2.7. If I is a closed ideal of A which contains a bounded approximate identity (of itself), E is a Banach A-bimodule with module homomorphism $\Delta : E \to A$, and X is an essential Banach I-module, then X is (canonically) a Banach A-module and each $\Delta_{|I}$-derivation $D : I \to X^*$ uniquely extends to a Δ-derivation $\tilde{D} : A \to X^*$ which is continuous with respect to the strict topology of A (induced by I) and w^*-topology of X^*.

Proof Each $x \in X$ decomposes (not uniquely) as $x = a.y$, for some $a \in I$ and $y \in X$. It is easy to see that X is a left Banach A-module under the action

$$b.x = ba.y \quad (a \in I, b \in A, x, y \in X, x = a.y).$$

This is well defined, as I has a bounded approximate identity. Define $\tilde{D} : A \to X^*$ by

$$\tilde{D}(b) = w^*\text{-}\lim_{\alpha}(D(be_\alpha) - b.D(e_\alpha)),$$

where $\{e_\alpha\}$ is a bounded approximate identity of I. Now $be_\alpha \to b$ strictly, for each $b \in A$. Hence, given $b \in A$ and $e \in E$, we have

$$\begin{aligned}\tilde{D}(\Delta(b.e)) = \tilde{D}(b\Delta(e)) &= w^*\text{-}\lim_{\alpha} w^*\text{-}\lim_{\beta} D(be_\alpha \Delta(e) e_\beta) \\ &= w^*\text{-}\lim_{\alpha} w^*\text{-}\lim_{\beta}[be_\alpha D(\Delta(e)e_\beta) + D(be_\alpha).\Delta(e)e_\beta] \\ &= b\tilde{D}(\Delta(e)) + \tilde{D}(b).\Delta(e).\end{aligned}$$

Hence $\tilde{D}$ is a Δ-derivation. The rest of the proof is similar to [Ru, 2.1.6]. □

Proposition 2.8. If $\Delta : E \to A$ has a dense range, then Δ-amenability of E is equivalent to amenability of A.

Proof If E is Δ-amenable, then each derivation $D : A \to X^*$, where X is a Banach A-module, is a module derivation and so inner on $\Delta(E)$. By continuity, D is inner on A. Conversely each module derivation $D : A \to X^*$ is a derivation. Indeed, given $b \in A$, there is a sequence $\{x_n\} \subseteq E$ such that $\Delta(x_n) \to b$, and so

$$D(ab) = \lim_{n} D(a\Delta(x_n)) = \lim_{n}(D(a).\Delta(x_n) + a.D(\Delta(x_n))) = D(a).b + a.D(b),$$

for each $a \in A$. Hence, if A is amenable, then E is Δ-amenable. □

Definition 2.9. Let $\pi : A\hat{\otimes}A \to A$ be the continuous lift of the multiplication map of A to the projective tensor product $A\hat{\otimes}A$. A *module approximate diagonal* of E is a bounded net $\{e_\alpha\}$ in $A\hat{\otimes}A$ such that

$$\|e_\alpha.\Delta(x) - \Delta(x).e_\alpha\| \to 0, \ \|\pi(e_\alpha).\Delta(x) - \Delta(x)\| \to 0 \quad (x \in E),$$

as $\alpha \to \infty$. A *module virtual diagonal* of E is an element M in $(A\hat{\otimes}A)^{**}$ such that

$$M.\Delta(x) - \Delta(x).M = 0, \ \pi^{**}(M).\Delta(x) - \Delta(x) = 0 \quad (x \in E).$$

It is clear that if E has a module virtual diagonal, then A contains a bounded approximate identity.

Theorem 2.10. *Consider the following assertions.*

(i) E *is module amenable,*

(ii) E *has a module virtual diagonal,*

(iii) E *has a module approximate diagonal.*

We have $(i) \to (ii) \leftrightarrow (iii)$. *If moreover* Δ *has a dense range, all the assertions are equivalent.*

Proof $(i) \to (ii)$. By Proposition 2.3, we may choose a bounded approximate identity $\{e_\alpha\}$ for E. We may assume that $\{e_\alpha \otimes e_\alpha\}$ is w^*-convergent to a point $P \in (A\hat{\otimes}A)^{**}$. Then for each $x \in E$ and $f \in A^*$,

$$\begin{aligned}\langle \pi^{**}((D_P \circ \Delta)(x), f\rangle &= w^*\text{-}\lim_\alpha \langle \pi^*(f).\Delta(x) - \Delta(x).\pi^*(f), e_\alpha \otimes e_\alpha\rangle \\ &= w^*\text{-}\lim_\alpha f(\Delta(x.e_\alpha)e_\alpha - e_\alpha\Delta(e_\alpha.x)) = 0.\end{aligned}$$

Hence $Im(D_P \circ \Delta) \subseteq Ker(\pi^{**})$. Now $Ker(\pi^{**})$ is isometrically isomorphic to X^*, where $X = (\Delta(E)\hat{\otimes}A)^{**}/Im(\pi^*)^\perp$, so by assumption, there is $Q \in Ker(\pi^{**})$ with $D_P \circ \Delta = D_Q \circ \Delta$. It is easy to see that $M := P - Q$ is a module virtual diagonal for E.

$(ii) \to (iii)$. Let M be a module virtual diagonal and let $\{e_\alpha\}$ be a net in $A\hat{\otimes}A$ which w^*-clusters to M. Then clearly $e_\alpha.\Delta(x) - \Delta(x).e_\alpha \to 0$, $\Delta \circ \pi(e_\alpha).\Delta(x) - \Delta(x) \to 0$ as $\alpha \to \infty$ for each $x \in E$ in the w^*-topology of $(A\hat{\otimes}A)^{**}$. A standard argument based on Mazur's theorem shows that the same holds in the norm topology for a net consisting of appropriate convex combinations of elements of $\{e_\alpha\}$.

$(iii) \to (ii)$. Just take any w^*-cluster point.

$(iii) \to (i)$. Now assume that Δ has a dense range. Let $\{m_\alpha\}$ be a module approximate diagonal for E with w^*-cluster point M, then $\{\pi(m_\alpha)\}$ is a bounded approximate identity for E. By Lemma 2.5(i), it is enough to show that for each essential A-module Y, all module derivation D from A to Y^* are inner. Each $y \in Y$ could be regarded as a bounded linear functional $\hat{y}$ on $A\hat{\otimes}A$ via

$$\langle \hat{y}, b \otimes a\rangle := \langle b.D(a), y\rangle \quad (a, b \in A).$$

Then for each $x, x' \in E$, $a \in A$, and $y \in Y$

$$\langle (y.\Delta(x) - \Delta(x).y)\hat{}, \Delta(x') \otimes a\rangle = \langle \hat{y}.\Delta(x) - \Delta(x).\hat{y}, \Delta(x') \otimes a\rangle + \langle \Delta(x')a.D \circ \Delta(x), y\rangle.$$

It follows that

$$\langle (y.\Delta(x) - \Delta(x).y)\hat{}, m\rangle = \langle \hat{y}.\Delta(x) - \Delta(x).\hat{y}, m\rangle + \langle \pi(m).D \circ \Delta(x), y\rangle,$$

for each $m \in A\hat{\otimes}A$. If we identify M with an element of Y^* with $M(y) = \langle \hat{y}, M\rangle$, for $y \in Y$, then

$$\begin{aligned}\langle D_M \circ \Delta(x), y\rangle &= w^*\text{-}\lim_\alpha \langle y.\Delta(x) - \Delta(x).y, m_\alpha\rangle \\ &= \langle M, \hat{y}.\Delta(x) - \Delta(x).\hat{y}\rangle + w^*\text{-}\lim_\alpha \langle \pi(m_\alpha).D \circ \Delta(x), y\rangle.\end{aligned}$$

Now in the last equation, the first term is zero, as M is a module virtual diagonal, and the second term is easily seen to be equal to $\langle D \circ \Delta(x), y\rangle$, using the fact that $y = z.\Delta(x')$, for some $z \in Y$ and $x' \in E$. Therefore $D = D_M$ on $\Delta(E)$, as required. □

3. Weak Module Amenability

In this section we study weak module amenability of Banach modules. All over this section E is a commutative Banach A-module (that is $a.x = x.a$, for each $a \in A, x \in E$) and $\Delta : E \to A$ is a bounded Banach A-module homomorphism. A Banach A-module X is called Δ-*commutative* (or more specifically $\Delta(E)$-commutative) if

$$a.x = x.a \quad (a \in \Delta(E), x \in X).$$

Definition 3.1. E is called *weak module amenable* (or more specifically weak Δ-amenable as an A-module) if each Δ-derivation from A to $\Delta(E)^*$ is inner on $\Delta(E)$.

Clearly A is weak A-module amenable (with $\Delta = id$) if and only if it is weakly amenable as a Banach algebra. The following result could be proved as in the classical case.

Proposition 3.2. (i) If E' is a commutative A-module and $\Phi : E' \to E$ is a module homomorphism with dense range, and E is weak Δ-amenable then E' is weak $\Delta \circ \Phi$-amenable.

(ii) If I is a closed ideal of A with $IE = EI = \{0\}$ and $q : A \to A/I$ is the quotient map, then E is weak $q \circ \Delta$-amenable as an A/I-module if it is Δ-amenable as an A-bimodule.

Proposition 3.3. If E is weak Δ-amenable, then the closed linear span F of $A\Delta(E)$ is dense in $\Delta(E)$.

Proof If not, there is a nonzero bounded linear functional λ in $\Delta(E)^*$ which vanishes on F. By Hahn-Banach Theorem λ extends to an element of A^*, which we still denote by λ. Define $D : A \to \Delta(E)^*$ by $D(a) = \lambda(a)\lambda$. This is a module derivation which is not inner, a contradiction. □

Now if A is a (commutative) Banach algebra with maximal ideal space $\mathcal{M}_A$ and $\phi \in \mathcal{M}_A$, then $\mathbb{C}$ is a Banach A-module with respect to the module action

$$a.z = z.a = \phi(a)z \quad (a \in A, z \in \mathbb{C}),$$

which is denoted by $\mathbb{C}_\phi$. Each module derivation $D : A \to \mathbb{C}_\phi$ is called a *module point derivation* (at ϕ). Clearly when a commutative Banach A-module E is Δ-weak amenable, all module point derivations vanish on $\Delta(E)$. This holds in general.

Proposition 3.4. If E is weak Δ-amenable, there is no nonzero point derivation on A.

Proof Let $d : A \to \mathbb{C}_\phi$ be a nonzero module point derivation. Let ψ be the restriction of ϕ to $\Delta(E)$ and define $D : A \to \Delta(E)^*$ by $D(a) = d(a)\psi$. Then D is a Δ-derivation and so $D = D_\lambda$ on $\Delta(E)$, for some $\lambda \in \Delta(E)^*$. Choose $e, f \in E$ so that $\psi(\Delta(e)) = 1$, $\psi(\Delta(f)) = 0$, and $d(\Delta(f)) = 1$. Then for $a = \Delta(e) + (1 - d(\Delta(e)))\Delta(f)$, we have $\psi(a) = d(a) = 1$, hence $D(a)(a) = 1$, a contradiction. □

Theorem 3.5. *If E is a commutative A-module and there is a Δ-commutative A-module X and a module derivation $D_0 : A \to X$ which is not identically zero on $\Delta(E)$, then there is a nonzero module derivation $D : A \to \Delta(E)^*$.*

Proof We consider two cases. First assume that $A\Delta(E)$ is not dense in $\Delta(E)$. By Hahn-Banach Theorem, there is a nonzero functional $\lambda \in (\Delta(E))^*$ whose kernel contains $A\Delta(E)$. Extend λ to an element of A^* (still denoted by λ) and define $D : A \to \Delta(E)^*$ by

$$D(a) = \lambda(a)\lambda \quad (a \in A),$$

Next consider the case where $A\Delta(E)$ is dense in $\Delta(E)$. We know that there is a Δ-commutative A-module X and a module derivation $D_0 : A \to X$ which is not identically zero on $\Delta(E)$. Choose $a \in A$, $e \in E$, and $\lambda \in X^*$ such that $D_0(a\Delta(e)) \neq 0$ and $\lambda(D_0(a\Delta(e))) \neq 0$. Define $D : A \to \Delta(E)^*$ by

$$\langle D(a), \Delta(e)\rangle = \lambda(\Delta(e).D_0(a)) \quad (e \in E, a \in A).$$

In both cases D is a nonzero module derivation. □

4. Examples

In this sections we give three examples in which strong and weak module amenability of some Banach modules are demonstrated.

Example 4.1. Let S be an inverse semigroup and E_S be the commutative sub-semigroup of idempotents in S. Then $A = \ell^1(E_S)$ is a commutative Banach algebra and $E = \ell^1(S)$ is a commutative Banach A-bimodule with the module actions

$$\delta_e.\delta_x = \delta_x.\delta_e = \delta_{ex} \quad (e \in E_S, x \in S).$$

Also there is a surjective module homomorphism $\Delta : \ell^1(S) \to \ell^1(E_S)$ defined by

$$\Delta(\delta_x) = \delta_{xx^*} \quad (x \in S).$$

We show that $\ell^1(S)$ is always module weakly amenable. If $D : \ell^1(E_S) \to \ell^\infty(E_S)$ is a Δ-derivation, then for each $e \in E_S$,

$$\begin{aligned} D(\delta_e) &= D(\delta_{ee^*}) = D(\Delta(\delta_e)) \\ &= D(\Delta(\delta_e.\delta_e)) = \Delta(\delta_e).D(\delta_e) + \delta_e.D(\Delta(\delta_e)) = 2\delta_e D(\delta_e). \end{aligned}$$

Applying the same formula to the right hand side,

$$\delta_e.D(\delta_e) = 2\delta_e.(\delta_e.D(\delta_e)) = 2(\delta_e * \delta_e).D(\delta_e)) = 2\delta_e.D(\delta_e),$$

hence $D(\delta_e) = \delta_e.D(\delta_e) = 0$.

Example 4.2. In the above example, if $\ell^1(S)$ is Δ-amenable, then $\ell^1(E_S)$ is amenable (Proposition 2.8). Hence E_S is finite, $\ell^1(S)$ is weakly amenable, and it has a bounded approximate identity [DN].

Example 4.3. It is well known that the disk algebra $A(D)$ is non amenable [BD]. $A(D)$ is a $\mathbb{C}$-module with respect to the scalar product. Now evaluation at zero defines a module epimorphism $\Delta : A(D) \to \mathbb{C}$ and $A(D)$ is Δ-amenable.

Example 4.4. If A is an amenable Banach algebra, the canonical map $\pi : A\hat{\otimes}A \to A$ is an A-module epimorphism (it is surjective, since A has a bounded approximate identity) and $A\hat{\otimes}A$ is π-amenable.

Example 4.5. For a locally compact group G, $L^1(G)$ is a closed two sided ideal in $M(G)$, so we can consider it as a Banach $M(G)$ module. Now if G is a non discrete amenable group, $M(G)$ is not amenable [DGH] but $L^1(G)$ is i-amenable, where $i : L^1(G) \to M(G)$ is the canonical injection.

Example 4.6. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then ℓ^1 is a Banach algebra and ℓ^p is a Banach ℓ^1-bimodule, both with respect to pointwise multiplication. Also each $f \in \ell^q$ defines a module homomorphism $\Delta_f : \ell^p \to \ell^1$ by $\Delta_f(g) = g * f$. If $f = \sum_{k=-\infty}^{\infty} \frac{1}{k}\delta_k$, then Δ_f has dense range(as its range contains all finitely supported element) and ℓ^p is Δ_f-amenable, by Proposition 2.8. This example could also be stated for any discrete group G, where $\ell^p(G)$ is considered as a Banach $\ell^1(G)$-bimodule. Same is true for $L^p(G)$ with convolution, when G is a compact group [D, 3.3.19]. In this case we have the module homomorphism $\Delta_f : L^p(G) \to L^1(G)$ defined by $\Delta_f(g) = g * f$, where $f \in L^q(G)$. If Δ_f has a dense range, then $L^p(G)$ is Δ_f -amenable. This is always the case when G is an abelian compact group. We illustrate this for $G = \mathbb{T}$. The same proof basically works for arbitrary abelian compact groups as well. Take $f = \sum_{k=-\infty}^{\infty} \frac{1}{k} e^{2\pi i k t} \in L^q(\mathbb{T})$ (which is basically the Fourier transform of the above function f used in the discrete case). Then, for each $g \in L^p(G)$,

$$\Delta_f(g) = \sum_{k=-\infty}^{\infty} \frac{1}{k}\hat{g}(k)e^{2\pi i k t},$$

where $\hat{g} \in c_0$ is the Fourier transform of g. In particular, range of Δ_f includes all trigonometric functions which are dense in $L^1(G)$.

Example 4.7. If G is a discrete group (with identity e) which acts on a C^*-algebra A, then $A \times \{e\}$ could be identified with A and $A \times \{g\}$ is a Banach A-module under

$$a.(b, g) = ((g.a)b, g)\ ,\ (b, g).a = (ba, g) \quad (a, b \in A, g \in G),$$

and there is a natural surjective module homomorphism $\Delta_g : A \times \{g\} \to A$ which sends (a, g) to a. The crossed product C^*-algebra $A \rtimes G$ is nuclear iff A is nuclear and G is amenable [Ro] iff G is amenable and modules $A \times \{g\}$ are Δ_g-amenable, for each $g \in G$.

Example 4.8. If A is a Banach algebra such that $A^* \subseteq A$ and A^* is a dense subspace of A, then A^* is a Banach A-bimodule (with canonical Arens actions) and $\Delta = \mathrm{id} : A^* \to A$ has dense range. Therefore A^* is Δ-amenable as a Banach A-bimodule iff A is amenable as a Banach algebra. There are many examples of this type. If G is a compact group, then the Fourier algebra $A(G)$ is dense in the group C^*-algebra $C^*(G)$. Indeed

$$A(G) \subseteq C(G) \subseteq L^1(G) \subseteq C^*(G),$$

and each space in this chain is dense in the subsequent space (with respect to the norm of the bigger space). But the norms of the last three spaces satisfy $\|.\|_{C^*(G)} \leq \|.\|_1 \leq \|.\|_\infty$ [Ey]. Hence $A(G)$ is dense in $C^*(G)$. Also, since G is compact, $A(G) = B(G) \simeq C^*(G)^*$, where $B(G)$ is the Fourier-Stieltjes algebra [Ey]. But $C^*(G)$ is amenable when G is compact [Ru]. Hence $A(G)$ is id-amenable in this case. This becomes more interesting when we recall that there are compact groups for which the Fourier algebra $A(G)$ is not amenable [J2]. Another example is ℓ^1 which is dense in c_0. It follows that $\ell^1 \simeq c_0^*$ is id-amenable as a c_0-bimodule. Finally, for a compact group G, $L^1(G)$ is an amenable Banach algebra with convolution, and so $L^1(G)$ is id-amenable as a Banach $L^1(G)$-bimodule.

Received: December 2009. Revised: April 2010.

Referencias

[A] M. Amini, *Module amenability for semigroup algebras*, Semigroup Forum 69 (2004) 243-254.

[AE] M. Amini and D. Ebrahimi Bagha, *Weak module amenability for semigroup algebras*, Semigroup Forum 71 (2005) 18-26.

[BD] F. F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, New York, 1973.

[D] H.G. Dales, *Banach algebras and automatic continuity*, London Math. Soc. Monographs, Volume 24, Clarendon Press, Oxford, 2000.

[DGH] H.G. Dales, F. Ghahramani and A. Ya. Helemskii, *The amenability of measure algebras*, J. London Math. Soc. (2) 66 (2002), no. 1, 213-226.

[DLS] H.G. Dales, A.T.M. Lau and D. Strauss, *Banach algebras on semigroups and their compactification*, to appear in Memoirs Amer. Math. Soc.

[DN] J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras*, Proceedings of the Royal Society of Edinburgh 80A (1978), 309-321.

[Ey] P. Eymard, *L'algebre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), 181-236.

[FD] J. M. G. Fell and R. S. Doran, *Representations of *-algebras*, locally compact groups, and Banach *-algebraic bundles, Pure and Applied Mathematics, Vols. 125 & 126, Academic Press Inc., Boston, MA, 1988.

[J] B.E. Johnson, *Cohomology in Banach algebras.* Memoirs of the American Mathematical Society, No. 127, American Mathematical Society, Providence, 1972.

[J2] B.E. Johnson,*Non-amenability of the Fourier algebra of a compact group*, J. London Math. Soc. (2), 50 (1994), 361-374.

[Ro] J. Rosenberg, *Amenability of crossed products of* C^**-algebras*, Comm. Math. Phys. 57 (1977), no. 2, 187-191.

[Ru] V. Runde, *Lectures on amenability*, Lecture Notes in Mathematics 1774, Springer-Verlag, Berlin, 2002.

CUBO A Mathematical Journal
Vol.13, N^{o}02, (139–149). June 2011

Fractional Order Differential Inclusions via the Topological Transversality Method

Mouffak Benchohra[1] and Naima Hamidi[2]
Laboratoire de Mathématiques,
Université de Sidi Bel Abbès,
B.P. 89, Sidi Bel-Abbès, 22000.
email: benchohra@univ-sba.dz
email: hamidi.naima@yahoo.fr

ABSTRACT

The aim of this paper is to present new results on the existence of solutions for a class of boundary value problem for differential inclusions involving the Caputo fractional derivative. Our approach is based on the topological transversality method.

RESUMEN

El objetivo de este trabajo es presentar nuevos resultados sobre la existencia de soluciones para una clase de problemas de contorno para inclusiones diferenciales derivados de la participación de Caputo fraccionada. Nuestro enfoque se basa en el método de la transversalidad topológica.

Keywords and phrases: Fractional differential inclusions; fixed point, Caputo fractional derivative, existence, topological transversality theorem.

Mathematics Subject Classification: 26A33, 26A42, 34A60, 34B15.

1 Introduction

This paper deals with the existence of solutions for boundary value problems (BVP for short) for fractional order differential inclusion of the form

$$^{c}D^{\alpha}y(t) \in F(t, y(t)), \quad t \in J := [0, T], \quad 1 < \alpha \leq 2, \tag{1.1}$$

$$y(0) = y_0, \quad y(T) = y_T \tag{1.2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a Carathéodory multifunction, $y_0, y_T \in \mathbb{R}^n$. Here $\mathcal{P}(\mathbb{R}^n)$ denotes the family of all nonempty subsets of $\mathcal{P}(\mathbb{R}^n)$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [11, 16, 20, 24, 27, 28]). There has been a significant development in fractional differential equations and inclusions in recent years; see the monographs of Kilbas *et al* [21], Lakshmikantham *et al.* [22], Miller and Ross [25], Podlubny [28], Samko *et al* [29] and the survey by Agarwal *et al.* [1], Benchohra *et al.* [5, 6, 7], Chang and Nieto [10], Diethelm *et al* [11, 12], Ouahab [26], Yu and Gao [30] and Zhang [31] and the references therein. Very recently, in [4, 8] the authors studied the existence and uniqueness of solutions of some classes of functional differential equations with infinite delay and fractional order, and in [3] a class of perturbed functional differential equations involving the Caputo fractional derivative has been considered. These papers have relied on different methods such as Banach fixed point theorem, Schaefer's theorem, Leray-Schauder nonlinear alternative.

In this paper we use a powerful method due to Granas [17] to prove the existence of solution to BVP (1.1)-(1.2). Granas' method is commonly known as topological transversality and relies on the idea of an essential map. The method has been highly useful for proving existence of solutions for initial and boundary value problem for integer order differential equations, see for example [9, 14, 18, 19].

This paper is organized as follows: in Section 2 we introduce some backgrounds on fractional calculus and the topological transversality theorem. In Section 3 we present our main results and an illustrative example will be presented in Section 4. This paper initiates the application of the topological transversality method to boundary value problems for fractional order differential inclusions.

2 Preliminaries

We now introduce notations, definitions, and preliminary facts that will be used in the remainder of this paper.
We denote by $\|y\|$ the norm of any element $y \in \mathbb{R}^n$.

$C(J, \mathbb{R}^n)$ is the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the usual norm

$$|y|_\infty = \sup\{|y(t)| : 0 \le t \le T\}.$$

$AC^1(J, \mathbb{R}^n)$ is the space of differentiable functions $y : J \to \mathbb{R}^n$, whose first derivative, y' is absolutely continuous.
$L^1(J, \mathbb{R}^n)$ denote the Banach space of functions $y : J \longrightarrow \mathbb{R}^n$ that are Lebesgue integrable with the norm

$$\|y\|_{L^1} = \int_0^T \|y(t)\| dt.$$

2.1 Some Properties of Fractional Calculus

Definition 1. ([21, 28]). Given an interval $[a, b]$ of $\mathbb{R}$. The fractional (arbitrary) order integral of the function $h \in L^1([a, b], \mathbb{R})$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Definition 2. ([21]). For a given function h on the interval $[a, b]$, the Caputo fractional-order derivative of h, is defined by

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$.

Lemma 3. (Lemma 2.22 [21]). Let $\alpha > 0$, then the differential equation

$${}^cD^\alpha h(t) = 0$$

has solutions

$$h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}, c_i \in \mathbb{R}, \quad i = 0, 1, 2, \ldots, n-1, \quad n = [\alpha] + 1.$$

Lemma 4. (Lemma 2.22 [21]). Let $\alpha > 0$, then

$$I^\alpha \; {}^cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

for arbitrary $c_i \in \mathbb{R}$, $\quad i = 0, 1, 2, \ldots, n-1, \quad n = [\alpha] + 1$.

2.2 Set-valued maps.

Let X and Y be Banach spaces. A set-valued map $G : X \to \mathcal{P}(Y)$ is said to be compact if $G(X) = \overline{\bigcup\{G(y); y \in X\}}$ is compact. G has convex (closed, compact) values if $G(y)$ is convex(closed, compact) for every $y \in X$. G is bounded on bounded subsets of X if $G(B)$ is bounded in Y for every bounded subset B of X. A set-valued map G is upper semicontinuous (usc for short) at $z_0 \in X$ if for every open set O containing Gz_0, there exists a neighborhood V of z_0 such that $G(V) \subset O$. G is usc on X if it is usc at every point of X if G is nonempty and compact-valued then G is usc if and only if G has a closed graph. The set of all bounded closed convex and nonempty subsets of X is denoted by $\mathtt{bcc}(X)$. A closed valued set-valued map $G : J \to \mathcal{P}(X)$ is measurable if for each $y \in X$, the function $t \mapsto \mathtt{dist}(y, G(t))$ is measurable on J. If $X \subset Y$, G has a fixed point if there exists $y \in X$ such that $y \in Gy$. Also, $\|G(y)\|_{\mathcal{P}} = \sup\{|x|; x \in G(y)\}$.

Definition 5. A multivalued map $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is said to be L^1-Carathéodory if

(i) $t \longmapsto F(t, y)$ is measurable for each $x \in \mathbb{R}^n$;

(ii) $y \longmapsto F(t, y)$ is upper semicontinuous for almost each $t \in J$;

(iii) for each $q > 0$, there exists $\varphi_q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, y)\|_{\mathcal{P}} = \sup\{\|v\| : v \in F(t, y)\} \le \varphi_q(t) \text{ for all } \|y\| \le q \text{ and for } a.e.\ t \in [0, 1].$$

For each $y \in C(J, \mathbb{R}^n)$, define the set of selections of F by

$$S^1_{F,y} = \{v \in L^1(J, \mathbb{R}^n) : v(t) \in F(t, y(t))\ a.e.\ t \in J\},$$

denotes the set of selections of F.

Remark 6. Note that for an L^1-Carathéodory multifunction $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ the set $S^1_{F,y}$ is not empty (see [23]).

For more details on set-valued maps we refer to [2, 13].

2.3 Topological transversality theory.

Let X be a Banach space, C a convex subset of X and U an open subset of C. $K_{\partial U}(\bar{U}, \mathcal{P}(C))$ denotes the set of all set-valued maps $G : \overline{U} \to \mathcal{P}(C)$ which are compact, usc with closed convex values and have no fixed points on ∂U (i.e., $u \in Gu$ for all $u \in \partial U$). A compact homotopy is a set-valued map $H : [0, 1] \times \bar{U} \to \mathcal{P}(C)$ which is compact, usc with closed convex values.

If $u \in H(\lambda, u)$ for every $\lambda \in [0, 1]$, $u \in \partial U$, H is said to be fixed point free on ∂U.

Two set-valued maps $F, G \in K_{\partial U}(\bar{U}, \mathcal{P}(C))$ are called homotopic in $K_{\partial U}(\bar{U}, \mathcal{P}(C))$ if there exists a compact homotopy $H : [0, 1] \times \bar{U} \to \mathcal{P}(C)$ which is fixed point free on ∂U and such that $H(0, .) = F$ and $H(1, .) = G$. The function $G \in K_{\partial U}(\bar{U}, \mathcal{P}(C))$ is called essential if every $F \in K_{\partial U}(\bar{U}, \mathcal{P}(C))$ such that $G|_{\partial U} = F|_{\partial U}$, has a fixed point. Otherwise G is called inessential.

Theorem 7. [17] Let $G : \bar{U} \to \mathcal{P}(C)$ be the constant set-valued map $G(u) \equiv u_0$. Then, if $u_0 \in U$, G is essential.

Theorem 8. (Topological transversality theorem) [17]. Let F, G be two homotopic maps in $K_{\partial U}(\bar{U}, \mathcal{P}(C)))$. Then F is essential if and only if G is essential.

For further details of the Topological Transversality Theory we refer the reader to [18].

3 Main results

In this section, we are concerned with the existence of solutions for the problem (1.1)-(1.2). Consider the following spaces

$$AC^1_B(J, \mathbb{R}^n) = \{y \in AC^1(J, \mathbb{R}^n);\ y(0) = y_0,\ \ y(T) = y_T\},$$

$$AC^{1,\alpha}(J, \mathbb{R}^n) = \{y \in AC^1_B(J, \mathbb{R}^n);\ \int_0^T |^cD^\alpha y(t)|dt < \infty\}.$$

$AC^{1,\alpha}(J, \mathbb{R}^n)$ is a Banach space with norm

$$\|y\|_{AC^{1,\alpha}} = \max\{\|y\|_\infty, \|y'\|_\infty, \|^cD^\alpha y\|_{L^1}\}.$$

For the existence of solutions for the problem (1.1)-(1.2), we have the following result which is useful in what follows.

Definition 9. A function $y \in AC^{1,\alpha}(J, \mathbb{R}^n)$ is said a solution to BVP (1.1)-(1.2) if there exists a function $v \in L^1(J, \mathbb{R})$ with $v(t) \in F(t, y(t))$, for a.e. $t \in J$, such that
$^cD^\alpha y(t) = v(t)$, a.e $t \in J$, $1 < \alpha \leq 2$, and the function y satisfies condition (1.2).

Let $h : J \to \mathbb{R}^n$ be continuous, and consider the linear fractional order differential equation

$$^cD^\alpha y(t) = h(t), \quad t \in J, \quad 1 < \alpha \leq 2. \tag{3.1}$$

We shall refer to (3.1)-(1.2) as (LP). For the existence of solutions for the problem (1.1)-(1.2), we have the following result which is useful in what follows.

Lemma 10. Let $1 < \alpha \leq 2$ and let $h : J \to \mathbb{R}^n$ be continuous. A function y is a solution of the fractional integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^T G(t, s)h(s)ds + y_0 + \frac{(y_T - y_0)t}{T}, \tag{3.2}$$

if and only if y is a solution of (LP), where $G(t, s)$ is the Green's function defined by

$$G(t, s) = \begin{cases} (t-s)^{\alpha-1} - \frac{t(T-s)^{\alpha-1}}{T}, & 0 \leq s \leq t \leq T, \\ \frac{-t(T-s)^{\alpha-1}}{T}, & 0 \leq t \leq s \leq T. \end{cases} \tag{3.3}$$

Proof. Assume y satisfies (3.1), then Lemma 4 implies that

$$y(t) = c_0 + c_1 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds.$$

From (1.2), a simple calculation gives

$$c_0 = y_0,$$

and

$$c_1 = -\frac{1}{T\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} h(s) ds + \frac{y_T - y_0}{T}.$$

Hence we get equation (3.2). Inversely, it is clear that if y satisfies equation (3.2), then equations (3.1)-(1.2) hold.

Our main result is

Theorem 11. Assume the following hypotheses hold:

(A_1) The function $F : J \times \mathbb{R}^n \to bcc(\mathbb{R}^n)$ is a L^1-Carathéodory multi-valued map;

(A_2) There exist a function $p \in L^1(J, \mathbb{R}_+)$, and a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$, such that

$$\|F(t, y)\|_{\mathcal{P}} \leq p(t)\psi(\|y\|) \quad \text{for each } (t, y) \in J \times \mathbb{R}^n;$$

(A_3) $\limsup_{r \to +\infty} \frac{r}{\psi(r)} = +\infty.$

Then, the fractional BVP (1.1)-(1.2) has a least one solution on J.

Proof. This proof will be given in several steps.

Step 1: Consider the set-valued operator $\mathcal{F} : C(J, \mathbb{R}^n) \to \mathcal{P}(L^1(J, \mathbb{R}^n))$ defined by

$$(\mathcal{F}y)(t) = F(t, y(t)).$$

$\mathcal{F}$ is well defined, upper semicontinuous, with convex values and sends bounded subsets of $C(J, \mathbb{R}^n)$ into bounded subsets of $L^1(J, \mathbb{R}^n)$. In fact, we have

$$\mathcal{F}y := \{u : J \to \mathbb{R}^n, \quad \text{measurable } u(t) \in F(t, y(t)), \ a.e. \ t \in J\}.$$

Let $z \in C(J, \mathbb{R}^n)$. and $u \in \mathcal{F}z$. Then

$$\|u(t)\| \leq p(t)\psi(\|z(t)\|) \leq p(t)\psi(\|z\|_0).$$

Hence $\|u\|_{L^1} \leq k_0 := \|p\|_{L^1}\psi(\|z\|_0)$. This shows that $\mathcal{F}$ is well defined.
It is clear that $\mathcal{F}$ is convex valued.

Now, let B be a bounded subset of $C(J, \mathbb{R}^n)$. Then, there exists $k > 0$ such that $\|u\|_0 \leq k$ for $u \in B$. So, for $w \in \mathcal{F}u$ we have $\|w\|_{L^1} \leq k_1$, where $k_1 = \|p\|_{L^1}\psi(k)$.
Also, we can argue as in [15] to show that $\mathcal{F}$ is usc.

Step 2: *A priori bounds on solutions.*

We shall show that if y be a possible solution of (1.1)-(1.2), then there exists a positive constant R^*, independent of y, such that

$$\|y\|_{AC^{1,\alpha}} \leq R^*.$$

Let y be a possible solution of (1.1)-(1.2), by Lemma 10, there exits $v \in S^1_{F,y}$ such that, for each $t \in J$

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)v(s)ds - \left(\frac{t}{T} - 1\right) y_0 + \frac{1}{T} y_T, \tag{3.4}$$

where G is given by (3.3). Let

$$G_0 := \sup\{\|G(t,s)\|;\ (t,s) \in J \times J\},$$

$$p_0 = \sup\{p(t) : t \in J\}.$$

Hence for $t \in J$

$$\begin{aligned}
\|y(t)\| &\leq \frac{1}{\Gamma(\alpha)} \int_0^T G(t,s)\|v(s)\|ds - \left(\frac{t}{T} - 1\right) \|y_0\| + \frac{1}{T}\|y_T\| \\
&\leq \frac{G_0}{\Gamma(\alpha)} \int_0^T p(s)\psi(\|y(s)\|)ds + \left|\left(\frac{t}{T} - 1\right)\right| \|y_0\| + \frac{1}{T}\|y_T\| \\
&\leq \frac{G_0}{\Gamma(\alpha)} \int_0^T p(s)\psi(\|y(s)\|)ds + \|y_0\| + \frac{1}{T}\|y_T\|.
\end{aligned}$$

Since ψ is nondecreasing we have

$$\|y\|_\infty \leq \frac{G_0\psi(\|y\|_\infty)p_0T}{\Gamma(\alpha)} + \|y_0\| + \frac{1}{T}\|y_T\|.$$

Thus

$$\frac{\|y\|_\infty}{\psi(\|y\|_\infty)} \leq \frac{G_0p_0T}{\Gamma(\alpha)} + \frac{\|y_0\|}{\psi(\|y\|_\infty)} + \frac{\|y_T\|}{T\psi(\|y\|_\infty)} = \tilde{R}$$

So

$$\frac{\|y\|_\infty}{\psi(\|y\|_\infty)} \leq \tilde{R}. \tag{3.5}$$

Now, the condition ψ in (A_3) shows that there exists $R_1^* > 0$ such that for all $R > R_1^*$

$$\frac{R}{\psi(R)} > \tilde{R}. \tag{3.6}$$

Comparing these last two inequalities (3.5) and (3.6) we see that $R_0 \leq R_1^*$. Consequently, we obtain

$$\|y(t)\| \leq R_1^* \quad \text{for all } t \in J.$$

From (3.4) we have for each $t \in J$

$$y'(t) = \frac{1}{\Gamma(\alpha)} \int_0^T \frac{\partial G(t,s)}{\partial t} f(s, y(s))ds - \frac{y_0}{T}, \tag{3.7}$$

Using a similar argument as before we can show that there exists $R_2^* > 0$ such that

$$\|y'(t)\| \leq R_2^* \text{ for all } t \in J. \tag{3.8}$$

Now from (1.1) and (A_1) we have

$$\int_0^T \|{}^c D^\alpha y(t)\| dt \leq \psi(R_1^*) \int_0^T p(s)ds := R_3^*. \tag{3.9}$$

Hence

$$\|y\|_{AC^{1,\alpha}} \leq \max\{R_1^*, R_2^*, R_3^*\} := R^*.$$

Step 3: *Existence of solutions.*

For $0 \leq \lambda \leq 1$ consider the one-parameter family of problems

$$^c D^\alpha y(t) \in \lambda F(t, y(t)), \text{ a.e. } t \in J, \quad 1 < \alpha \leq 2, \quad (1_\lambda)$$

$$y(0) = \lambda y_0, \quad y(T) = \lambda y_T \qquad (2_\lambda)$$

which reduces to (1.1)-(1.2) for $\lambda = 1$. For $0 \leq \lambda \leq 1$, we define the operator $\mathcal{F}_\lambda : C(J, \mathbb{R}^n) \to \mathcal{P}(L^1(J, \mathbb{R}^n))$ by $(\mathcal{F}_\lambda y)(t) = \lambda F(t, y(t))$.

Step 1 shows that $\mathcal{F}_\lambda$ is usc, has convex values and sends bounded subsets of $C(J, \mathbb{R}^n)$ into bounded subsets of $L^1(J, \mathbb{R}^n)$ and if y is a solution of $(1_\lambda)-(2_\lambda)$ for some $\lambda \in [0, 1]$, then $\|y\|_{AC^{1,\alpha}} \leq R^*$, where R^* does not depend on λ.

For $\lambda \in [0, 1]$, we define the operators

$$\mathcal{J} : AC^{1,\alpha}(J, \mathbb{R}^n) \to C(J, \mathbb{R}^n) \text{ by } (\mathcal{J}y)(t) = y(t),$$

$$L : AC^{1,\alpha}(J, \mathbb{R}^n) \to L^1(J, \mathbb{R}^n) \text{ by } (Ly)(t) = {}^c D^\alpha y(t).$$

It is clear that $\mathcal{J}$ is continuous and completely continuous and L is linear, continuous and has a bounded inverse denoted by L^{-1}. Let

$$V := \{y \in AC^{1,\alpha}(J, \mathbb{R}^n); \|y\|_{AC^{1,\alpha}} < R^* + 1\}.$$

Define a map $H : [0, 1] \times \overline{V} \to AC^{1,\alpha}(J, \mathbb{R}^n)$ by

$$H(\lambda, y) = (L^{-1} \circ \mathcal{F}_\lambda \circ \mathcal{J})(y).$$

We can show that the fixed points of $H(\lambda,\cdot)$ are solutions of $(1_\lambda)-(2_\lambda)$. Moreover, H is a compact homotopy between $H(0,\cdot)\equiv 0$ and $H(1,\cdot)$. In fact, H is compact since $\mathcal{J}$ is completely continuous, $\mathcal{F}_\lambda$ is continuous and L^{-1} is continuous. Since solutions of (1_λ) satisfy

$$\|y\|_{AC^{1,\alpha}} \leq R^*$$

we see that $H(\lambda,.)$ has no fixed points on ∂V. Now $H(0,\cdot)$ is essential by Theorem 7. Hence by Theorem 8, $H(1,\cdot)$ is essential. This implies that $L^{-1}\circ\mathcal{F}_1\circ\mathcal{J}$ has a fixed point which is a solution to problem (1.1)-(1.2).

4 An Example

As an application of our results we consider the following boundary value problem

$$^cD^\alpha y(t) \in F(t,y), \quad t\in J := [0,1], \quad 1<\alpha\leq 2, \tag{4.1}$$

$$y(0) = 1, \quad y(1) = 2, \tag{4.2}$$

where $^cD^\alpha$ is the Caputo fractional derivative. Set

$$F(t,y) = \{v\in \mathbb{R} : f_1(t,y)\leq v\leq f_2(t,y)\},$$

where $f_1, f_2 : J\times\mathbb{R}\to\mathbb{R}$ are measurable in t. We assume that for each $t\in J$, $f_1(t,\cdot)$ is lower semi-continuous (i.e, the set $\{y\in\mathbb{R} : f_1(t,y)>\mu\}$ is open for each $\mu\in\mathbb{R}$), and assume that for each $t\in J$, $f_2(t,\cdot)$ is upper semi-continuous (i.e the set $\{y\in\mathbb{R} : f_2(t,y)<\mu\}$ is open for each $\mu\in\mathbb{R}$). Assume that there exists $p\in L^1(J,\mathbb{R}^+)$ and $\delta\in(0,1)$ such that

$$\max(|f_1(t,y)|,\ |f_2(t,y)|) \leq p(t)|y|^\delta, \quad t\in J, \text{ and all } y\in\mathbb{R}.$$

It is clear that F is compact and convex valued, and it is upper semi-continuous (see [13]). Since assumptions $(A_1)-(A_3)$ hold, Theorem 11 implies that the BVP (4.1)-(4.2) has at least one solution.

Received: November 2009. Revised: July 2010.

References

[1] R.P Agarwal, M. Benchohra and S. Hamani, *A survey on existence result for boundary value problems of nonlinear fractional differential equations and inclusions, Acta. Appl. Math. 109 (3) (2010), 973-1033.*

[2] J. P. Aubin and A. Cellina, *Differential Inclusions, Springer-Verlag, Berlin-Heidelberg, New York, 1984.*

[3] A. Belarbi, M. Benchohra, S. Hamani and S.K. Ntouyas, *Perturbed functional differential equations with fractional order, Commun. Appl. Anal. 11 (3-4) (2007), 429-440.*

[4] A. Belarbi, M. Benchohra and A. Ouahab, *Uniqueness results for fractional functional differential equations with infinite delay in Fréchet spaces, Appl. Anal. 85 (2006), 1459-1470.*

[5] M. Benchohra, J. R. Graef and S. Hamani, *Existence results for boundary value problems of nonlinear fractional differential equations with integral conditions, Appl. Anal. 87 (7) (2008), 851-863.*

[6] M. Benchohra and S. Hamani, *Nonlinear boundary value problems for differential inclusions with Caputo fractional derivative, Topol. Methods Nonlinear Anal. 32 (1) (2008), 115-130.*

[7] M. Benchohra, S. Hamani and S.K. Ntouyas, *Boundary value problems for differential equations with fractional order, Surv. Math. Appl. 3 (2008), 1-12.*

[8] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, *Existence results for fractional order functional differential equations with infinite delay, J. Math. Anal. Appl. 338 (2) (2008), 1340-1350.*

[9] A. Boucherif, and N. Chiboub-Fellah Merabet, *Boundary value problems for first order multivalued differential systems. Arch. Math. (Brno) 41 (2005), 187–195.*

[10] Y.-K. Chang and J.J. Nieto, *Some new existence results for fractional differential inclusions with boundary conditions, Math. Comput. Model. 49 (2009), 605-609.*

[11] K. Diethelm and A.D. Freed, *On the solution of nonlinear fractional order differential equations used in the modeling of viscoplasticity, in "Scientifice Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties" (F. Keil, W. Mackens, H. Voss, and J. Werther, Eds), pp 217-224, Springer-Verlag, Heidelberg, 1999.*

[12] K. Diethelm and N. J. Ford, *Analysis of fractional differential equations, J. Math. Anal. Appl. 265 (2002), 229-248.*

[13] K. Deimling, *Multivalued Differential Equations, Walter De Gruyter, Berlin-New York, 1992.*

[14] P. W. Eloe and J. Henderson, *Nonlinear boundary value problems and a priori bounds on solutions. SIAM J. Math. Anal. 15 (1984), 642–647.*

[15] M. Frigon, M., *Application de la transversalite topologique a des problemes non lineaires pour des equations differentielles ordinaires, Dissertationnes Math. 292, PWN, Warsaw 1990.*

[16] W. G. Glockle and T. F. Nonnenmacher, *A fractional calculus approach of self-similar protein dynamics, Biophys. J. 68 (1995), 46-53.*

[17] A. Granas, *Sur la méthode de continuité de Poincaré. C. R. Acad. Sci. Paris Sr. A-B 282 (1976), 983–985.*

[18] A. Granas and J. Dugundji, *Fixed Point Theory, Springer Verlag, New York, 2003.*

[19] J. Henderson and C.C. Tisdell, *Topological transversality and boundary value problems on time scales. J. Math. Anal. Appl. 289 (2004), 110–125.*

[20] R. Hilfer, *Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.*

[21] A.A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.*

[22] *V. Lakshmikantham, S. Leela and J. Vasundhara, Theory of Fractional Dynamic Systems, Cambridge Academic Publishers, Cambridge, 2009.*

[23] *A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. 13 (1965), 781-786.*

[24] *F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, J. Chem. Phys. 103 (1995), 7180-7186.*

[25] *K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.*

[26] *A. Ouahab, Some results for fractional boundary value problem of differential inclusions, Nonlinear Anal. 69 (11) (2008), 3877-3896.*

[27] *K.B. Oldham and J. Spanier, The Fractional Calculus, Academic Press, New York, London, 1974.*

[28] *I. Podlubny, Fractional Differential Equations, Academic Press, San Diego, 1999.*

[29] *S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives. Theory and Applications, Gordon and Breach, Yverdon, 1993.*

[30] *C. Yu and G. Gao, Existence of fractional differential equations, J. Math. Anal. Appl. 310 (2005), 26-29.*

[31] *S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional diffrential equations, Electron. J. Differential Equations 2006, No. 36, 12 pp.*

CUBO A Mathematical Journal
Vol.13, $N^{\underline{o}}$02, (151–161). June 2011

Some New Characterizations for $\mathsf{PGL}(2, \mathsf{q})$

B. KHOSRAVI [1 2], M. KHATAMI[2] AND Z. AKHLAGHI[2]
[1] *School of Mathematics,*
Institute for Research in Fundamental Sciences (IPM),
P.O. Box: 19395-5746, Tehran, Iran.
email: `khosravibbb@yahoo.com`
and
[2] *Dept. of Pure Math., Faculty of Math. and Computer Sci.,*
Amirkabir University of Technology (Tehran Polytechnic),
424, Hafez Ave., Tehran 15914, IRAN.

ABSTRACT

Many authors introduced some characterizations for finite groups. In this paper as the main result we prove that the finite group $\mathsf{PGL}(2, \mathsf{q})$ is uniquely determined by its noncommuting graph. Also we prove that $\mathsf{PGL}(2, \mathsf{q})$ is characterizable by its noncyclic graph. Throughout the proof of these results we prove that $\mathsf{PGL}(2, \mathsf{q})$ is uniquely determined by its order components and using this fact we give positive answer to a conjecture of Thompson and another conjecture of Shi and Bi for the group $\mathsf{PGL}(2, \mathsf{q})$.

RESUMEN

Muchos autores introdujeron algunas caracterizaciones de los grupos finitos. En este trabajo como principal resultado se demuestra que grupo finito $\mathsf{PGL}(2, \mathsf{q})$ es determinado nicamente por su gráfica no conmutativa. También se demuestra que $\mathsf{PGL}(2, \mathsf{q})$

[1]The First author was supported in part by a grant from IPM (no. 89200113).

es caracterizable por su gráfico no cíclico. A lo largo de la prueba de estos resultados se demuestra que $PGL(2, Q)$ es determinado únicamente por los componentes de su orden y con ello damos respuesta positiva a una conjetura de Thompson y otra conjetura de Shi Bi y para el grupo $PGL(2, q)$.

Keywords and phrases: Noncommuting graph, prime graph, noncyclic graph, order components.

Mathematics Subject Classification: 20D05, 20D60.

1 Introduction

If n is an integer, then we denote by $\pi(n)$ the set of all prime divisors of n. If G is a finite group, then $\pi(|G|)$ is denoted by $\pi(G)$. We construct the *prime graph* of G which is denoted by $\Gamma(G)$ as follows: the vertex set is $\pi(G)$ and two distinct primes p and q are joined with an edge if and only if G contains an element of order pq. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and let $\pi_1, \pi_2, ..., \pi_{t(G)}$ be the connected components of $\Gamma(G)$. If $2 \in \pi(G)$, then we assume that $2 \in \pi_1$.

Now we can express $|G|$ as a product of coprime natural numbers m_i, such that $1 \leq i \leq t(G)$ and $\pi(m_i) = \pi_i$. These integers are called *order components* of G. The set of order components of G is denoted by $OC(G)$.

One of the other graphs which associated with a non-abelian group G is the noncommuting graph that is denoted by $\nabla(G)$ and is constructed as follows: the vertex set of $\nabla(G)$ is $G \setminus Z(G)$ with two vertices x and y are joined by an edge whenever the commutator of x and y is not identity. In [1] the authors put forward the following conjecture:

Conjecture A. Let S be a finite non-abelian simple group and G be a finite group such that $\nabla(G) \cong \nabla(S)$. Then $G \cong S$.

The validity of this conjecture has been proved for all simple groups with non-connected prime graphs. Also it is proved that some finite simple groups with connected prime graphs, say A_{10}, $U_4(7)$, $L_4(8)$, $L_4(4)$ and $L_4(9)$, can be uniquely determined by their noncommuting graghs (see [19, 20, 21, 22]).

In this paper as the main result we prove that the almost simple group $PGL(2, q)$, where $q = p^n$ for a prime number p and a natural number n, is characterizable by its noncommuting graph. As a consequence of our results we prove the validity of a conjecture of Thompson and another conjecture of Shi and Bi for the group $PGL(2, q)$.

Let G be a noncyclic group and $Cyc(G) = \{x \in G | \langle x, y \rangle \text{ is cyclic for all } y \in G\}$. In [2], the authors introduced the cyclic graph of G, which is denoted by $\Gamma_1(G)$ as follows: take $G \setminus Cyc(G)$ as the vertex set and join two vertices if they do not generate a cyclic subgroup. In this graph the degree of each vertex x is equal to $|G| \setminus |Cyc_G(x)|$, where $Cyc_G(x) = \{y \in G | \langle x, y \rangle \text{ is cyclic}\}$. It is proved that some finite simple groups, S_n, D_{2^k}, D_{2n} ,where n is odd, are characterizable by the noncyclic graph. We show that $PGL(2, q)$ is uniquely determined by its noncyclic graph.

In this paper, all groups are finite and by simple groups we mean non-abelian simple groups. All further unexplained notations are standard and refer to [6], for example.

2. Preliminary results

In this section we bring some preliminary lemmas which are necessary in the proof of the main theorem.

Remark 2.1. Let N be a normal subgroup of G and p, q be incident vertices of $\Gamma(G/N)$. Then p, q are incident in $\Gamma(G)$. In fact if xN is of order pq, then there exists a power of x which is of order pq.

Definition 2.2. ([8]) A finite group G is called a 2-Frobenius group if it has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, where K and G/H are Frobenius groups with kernels H and K/H, respectively.

Lemma 2.3. Let G be a Frobenius group of even order and let H, K be Frobenius complement and Frobenius kernel of G, respectively. Then $t(G) = 2$, and the prime graph components of G are $\pi(H)$, $\pi(K)$ and G has one of the following structures:
(a) $2 \in \pi(K)$ and all Sylow subgroups of H are cyclic;
(b) $2 \in \pi(H)$, K is an abelian group, H is a solvable group, the Sylow subgroups of odd order of H are cyclic groups and the 2-Sylow subgroups of H are cyclic or generalized quaternion groups;
(c) $2 \in \pi(H)$, K is an abelian group and there exists $H_0 \leq H$ such that $|H : H_0| \leq 2$, $H_0 = Z \times SL(2,5)$, $\pi(Z) \cap \{2, 3, 5\} = \emptyset$ and the Sylow subgroups of Z are cyclic.

Also the next lemma follows from [8] and the properties of Frobenius groups [9]:
Lemma 2.4. Let G be a 2-Frobenius group, i.e., G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K and G/H are Frobenius groups with kernels H and K/H, respectively. Then

(a) $t(G) = 2$, $\pi_1 = \pi(G/K) \cup \pi(H)$ and $\pi_2 = \pi(K/H)$;

(b) G/K and K/H are cyclic, $|G/K| \mid (|K/H| - 1)$ and $G/K \leq Aut(K/H)$;

(c) H is nilpotent and G is a solvable group.

Lemma 2.5. ([4, Lemma 8]) Let G be a finite group with $t(G) \geq 2$ and let N be a normal subgroup of G. If N is a π_i-group for some prime graph component of G and $m_1, m_2, \ldots, m_r$ are some order components of G but not π_i-numbers, then $m_1 m_2 \cdots m_r$ is a divisor of $|N| - 1$.

Lemma 2.6. ([3, Lemma 1.4]) Suppose G and M are two finite groups satisfying $t(M) \geq 2$, $N(G) = N(M)$, where $N(G)=\{n \mid G$ has a conjugacy class of size n $\}$, and $Z(G) = 1$. Then

$|G| = |M|$.

Lemma 2.7. ([3, Lemma 1.5]) Let G_1 and G_2 be finite groups satisfying $|G_1| = |G_2|$ and $N(G_1) = N(G_2)$. Then $t(G_1) = t(G_2)$ and $OC(G_1) = OC(G_2)$.

Lemma 2.8. ([11]) Let G be a finite group and M be a finite group with $t(M) = 2$ satisfying $OC(G) = OC(M)$. Let $OC(M) = \{m_1, m_2\}$. Then one of the following holds:
(a) G is a Frobenius or 2-Frobenius group;
(b) G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/K is a π_1-group, H is a nilpotent π_1-group, and K/H is a non-abelian simple group. Moreover $OC(K/H) = \{m'_1, m'_2, \ldots, m'_s, m_2\}$, where $m'_1 m'_2 \ldots m'_s | m_1$. Also $G/K \leq Out(K/H)$.

Lemma 2.9. ([1]) Let G be a finite non-abelian group. If H is a group such that $\nabla(G) \cong \nabla(H)$, then H is a finite non-abelian group such that $|Z(H)|$ divides

$$\gcd(|G| - |Z(G)|, |G| - |C_G(x)|, |C_G(x)| - |Z(G)| : x \in G \setminus Z(G)).$$

Lemma 2.10. ([18]) Let G be a non-abelian group such that $\nabla(G) \cong \nabla(PSL(2, 2^n))$, where n is a natural number. Then $G \cong PSL(2, 2^n)$.

Lemma 2.11.([7, Remark 1]) The equation $p^m - q^n = 1$, where p and q are primes and $m, n > 1$ has only one solution, namely $3^2 - 2^3 = 1$.

Lemma 2.12. ([2]) Let G be a finite noncyclic group. If H is a group such that $\Gamma_1(G) \cong \Gamma_1(H)$, then H is a finite noncyclic group such that $|Cyc(H)|$ divides

$$\gcd(|G| - |Cyc(G)|, |G| - |Cyc_G(x)|, |Cyc_G(x)| - |Cyc(G)| : x \in G \setminus Cyc(G)).$$

Lemma 2.13. ([2]) Let G and H be two finite noncyclic groups such that $\Gamma_1(G) \cong \Gamma_1(H)$. If $|G| = |H|$, then $\pi_e(G) = \pi_e(H)$.

3. Main Results

We note that if $q = 2^n$, then $PGL(2, q) = PSL(2, q)$ and we know that $PSL(2, q)$ is characterizable by its noncommuting graph (see [18]). Therefore throughout this section we suppose M is the almost simple group $PGL(2, q)$, where $q = p^n$ for an odd prime number p and a natural number n.

Theorem 3.1. Let G be a group such that $\nabla(G) \cong \nabla(M)$. Then $|G| = |M|$.

Proof. First note that G is a finite non-abelian group. Since $\nabla(G) \cong \nabla(M)$, we have $|G| - |Z(G)| = |M| - |Z(M)|$. Then it is enough to prove that $|Z(G)| = |Z(M)|$.

By Lemma 2.9, $|Z(G)|$ divides $|M| - |Z(M)|$. Since $|Z(M)| = 1$, we have $|Z(G)|$ divides $q(q^2 - 1) - 1$. Let P be a Sylow p-subgroup of M. We know that $Z(P) \neq 1$. So there exists $1 \neq x \in Z(P)$.

We claim that $C_M(x) = P$. It is obvious that $P \leq C_M(x)$, since $x \in Z(P)$. On the contrary we suppose that $y \in C_M(x) \setminus P$. So we can conclude that $o(xy) = o(x)o(y)$. Without lose of generality we suppose $|y| = r$, where $r \neq p$ is a prime number. Then M has an element of order rp. But p is an isolated vertex in $\Gamma(M)$, a contradiction. Therefore our claim is proved.

By Lemma 2.9 we have $|Z(G)|$ divides $|C_M(x)| - |Z(M)|$. Then $|Z(G)|$ divides $q - 1$. We know that $Z(G)$ divides $q(q^2 - 1) - 1$, which implies that $|Z(G)| = 1$ and so $|G| = |M|$. □

Theorem 3.2. Let G be a group such that $\nabla(G) \cong \nabla(M)$, where $M = PGL(2, q)$. Then $OC(G) = OC(M)$.

Proof. Since $\nabla(G) \cong \nabla(M)$, the set of vertex degrees of two graphs are the same. Therefore

$$\{|G| - |C_G(x)| \mid x \in G\} = \{|M| - |C_M(y)| \mid y \in M\}.$$

On the other hand Theorem 3.1 implies that $|G| = |M|$, and so $N(G) = N(M)$. Now using Lemma 2.7 we have $OC(G) = OC(M)$. □

Theorem 3.3. Let G be a finite group and $OC(G) = OC(M)$. If $q = p^n \neq 3$ then G is neither a Frobenius group nor a 2-Frobenius group. If $q = 3$ and G is a 2-Frobenius group, then $G \cong S_4$.

Proof. If G is a Frobenius group, then by Lemma 2.3, $OC(G) = \{|H|, |K|\}$ where K and H are Frobenius kernel and Frobenius complement of G, respectively. Therefore $OC(G) = \{q, q^2 - 1\}$ and since $|H| \mid (|K| - 1)$ it follows that $|H| < |K|$ and so $|H| = q$ and $|K| = q^2 - 1$. Also $q \mid (q^2 - 2)$ implies that $q = 2$, which is a contradiction, since q is odd. Therefore G is not a Frobenius group.

Let G be a 2-Frobenius group. Hence $G = ABC$, where A and AB are normal subgroups of G; AB and BC are Frobenius groups with kernels A, B and complements B, C, respectively. By Lemma 2.4, we have $|B| = q$ and $|A||C| = q^2 - 1$. Also $|B| \mid (|A| - 1)$ and so $|A| = qt + 1$, for some $t > 0$. On the other hand, $|A| \mid (q^2 - 1)$, which implies that $q^2 - 1 = k(qt + 1)$, for some $k > 0$. Therefore $q \mid (k + 1)$ and so $q - 1 \leq k$. If $t > 1$, then $q^2 - 1 = k(qt + 1) \geq (q - 1)(qt + 1) > (q - 1)(q + 1)$, which is a contradiction. Hence $t = 1$ and $|A| = q + 1$ and $|C| = q - 1$.

If there exists an odd prime r such that $r \mid (q + 1)$, then let R be a Sylow r-subgroup of A. Since A is a nilpotent group, it follows that R is a normal subgroup of G. Now Lemma 2.5, implies that $q \mid (|R| - 1)$ and $|R| \mid (q + 1)/2$, which is a contradiction. Therefore $q + 1 = 2^\alpha$, for some $\alpha > 0$. Similarly $Z(A) \neq 1$ is a characteristic subgroup of A and hence A is abelian. Let $X = \{x \in A | o(x) = 2\} \cup \{1\}$. Then X is a non-identity characteristic subgroup of A. Therefore A is an elementary abelian 2-subgroup of G and $|A| = 2^\alpha = q + 1$. By Lemma 2.11, if $q = p^n$ such that $n > 1$, then the equation $2^\alpha - q = 1$ does not have any solution.

Now let $n = 1$. Suppose $F = GF(2^\alpha)$ and so A is the additive group of F. Also $|B| = q = p = 2^\alpha - 1$ and so B is the multiplicative group of F. Now C acts by conjugation on A and similarly C acts by conjugation on B and this action is faithful. Therefore C keeps the structure of the field F and so C is isomorphic to a subgroup of the automorphism group of F. Hence $|C| = 2^\alpha - 2 \leq |Aut(F)| = \alpha$. Therefore $\alpha \leq 2$. If $\alpha = 2$, then $G = S_4$, the symmetric group on 4 letters. □

Lemma 3.4. Let G be a finite group and $M = PGL(2, q)$, where $q > 3$ or $q = 3$ and M is not a 2-Frobenius group. If $OC(G) = OC(M)$, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a simple group. Moreover the odd order component of M is equal to an odd order component of K/H. In particular, $t(K/H) \geq 2$. Also $|G/H|$ divides $|Aut(K/H)|$, and in fact $G/H \leq Aut(K/H)$.

Proof. The first part of the lemma follows from Lemma 2.8 and Theorem 3.3, since the prime graph of G has two components. If K/H has an element of order pq, where p and q are primes, then by Remark 2.1, K has an element of order pq. Therefore G has an element of order pq. So by the definition of prime graph component, the odd order component of G is equal to an odd order component of K/H. Also $K/H \trianglelefteq G/H$ and $C_{G/H}(K/H) = 1$, which implies that

$$G/H = \frac{N_{G/H}(K/H)}{C_{G/H}(K/H)} \cong T \quad , \quad T \leq Aut(K/H). \ \square$$

Theorem 3.5. Let G be a finite group such that $OC(G) = OC(M)$, where $M = PGL(2, q)$. Then $G \cong PGL(2, q)$.

Proof. If $q = 3$ and G is a 2-Frobenius group, then Theorem 3.3 implies that $G = S_4 \cong PGL(2, 3)$, as desired. Otherwise Lemma 3.4 implies that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups and K/H is a simple subgroup and $t(K/H) \geq 2$.

Now using the classification of finite simple groups and the results in Tables 1-3 in [10], we consider the following cases.

Case 1. Let $K/H \cong A_m$, where $m = p', p' + 1$ or $p' + 2$ and $p' \geq 5$ is a prime number and m and $m - 2$ are not primes at the same time.

Then $q = p'$, and consequently $n = 1$ and $q = p = p'$. On the other hand, $|A_m| \mid |G| = p(p^2 - 1)$. If $m > p$, then $|A_m| > (p + 1)p(p - 1)$, which is a contradiction. Therefore $m = p$ and $|A_p| \mid |G| = p(p^2 - 1)$, and so $|A_p| = p!/2 \leq p(p^2 - 1)$. Hence $(p - 2)!/2 \leq p + 1$. But $p \geq 7$, since $p - 2$ is not a prime. So $(p - 2)(p - 3) < (p - 2)!/2 \leq p + 1$, which is a contradiction. This completes the proof.

Case 2. Let $K/H \cong A_{p'}$, where p' and $p' - 2$ are primes.

If $p = p'$, for $p' \geq 7$, then we can get a contradiction similarly to the previous case. So $p = 5$ and $K/H \cong A_5 \cong PSL(2, 5)$. Since $K/H \leq G/H \leq Aut(K/H)$, we have $PSL(2, 5) \leq G/H \leq PGL(2, 5)$. Hence G/H is isomorphic to $PSL(2, 5)$ or $PGL(2, 5)$. If $G/H \cong PSL(2, 5)$, then $|H| = 2$. But $H \trianglelefteq G$, which implies that $H \subseteq Z(G)$ and we get a contradiction. So $G/H \cong PGL(2, 5)$, which implies that $H = 1$ and $G \cong PGL(2, 5)$.

Let $p = p' - 2$. Since $p' \mid |A_{p'}|$, we have $p' \mid |G| = p(p^2 - 1)$. But we know that $p = p' - 2$ is the greatest prime divisor of $|G|$, which is a contradiction.

Case 3. Let K/H be a sporadic simple group.

Using the tables in [10] we see that the odd order components of sporadic simple groups are prime.

Let S be a sporadic simple group and $K/H \cong S$. Since q is equal to the greatest odd order component of K/H, we have $q = m_i$, such that $m_i = max\{m_2, m_3, ..., m_{t(S)}\}$. So q is a prime number.

If $S = J_4$, then $q = p = 43$. Since $11^2 \mid |K/H|$, we have $11^2 \mid (p^2 - 1) = 43^2 - 1$, which is a contradiction.

If $S = Co_2$, then $q = p = 23$. Since $7 \mid |K/H|$, we have $7 \mid (23^2 - 1)$, which is a contradiction.

The proof of other cases are similar and we omit them for convenience.

If K/H is isomorphic to ${}^2A_3(2)$, ${}^2F_4(2)'$, $A_2(4)$, ${}^2A_5(2)$, $E_7(2)$, $E_7(3)$ or ${}^2E_6(2)$, then similarly we get a contradiction.

In the sequel of the proof we consider simple groups of Lie type. Since the proofs of these cases are similar we state only a few of them.

In all of the following cases p' is an odd prime number and q' is a prime power.

Case 4. Let $K/H \cong A_{p'-1}(q')$, where $(p', q') \neq (3, 2), (3, 4)$. By hypothesis we have $q = (q'^{p'} - 1)/((q' - 1)(p', q' - 1))$. Hence $q < q'^{p'} - 1 < q'^{p'}$. Then $q^2 - 1 < q'^{2p'}$. On the other hand, we know $q'^{p'(p'-1)/2} \mid (q^2 - 1)$ and therefore $q'^{p'(p'-1)/2} < q'^{2p'}$. So $p'(p' - 1)/2 < 2p'$ and hence $p' < 5$. So $p' = 3$ and $q = (q'^2 + q' + 1)/(3, q' - 1)$, which implies that $q < 2q'^2$. Therefore $q^2 - 1 < 4q'^4 - 1$. On the other hand, $q'^3(q'^2 - 1)(q' - 1) \mid (q^2 - 1)$ and consequently $q'^3(q'^2 - 1)(q' - 1) < 4q'^4 - 1$. So $q' = 2, 3$ or 4. Since $(p', q') \neq (3, 2), (3, 4)$, we have $q' = 3$ and $q = 13$. Then $3^3(3^2 - 1)(3 - 1) \mid (13^2 - 1)$, which is a contradiction.

Case 5. Let $K/H \cong {}^2A_{p'}(q')$, where $(q' + 1) \mid (p' + 1)$ and $(p', q') \neq (3, 3), (5, 2)$. In this case we have $q = (q'^{p'} + 1)/(q' + 1)$. Therefore $q < q'^{p'} + 1 < 2q'^{p'} \leq q'^{p'+1}$ and hence $q^2 - 1 < q'^{2(p'+1)}$. On the other hand, we have $q'^{p'(p'+1)/2} \mid (q^2 - 1)$. So we conclude that $q'^{p'(p'+1)/2} < q'^{2(p'+1)}$. Hence $p'(p' + 1)/2 < 2(p' + 1)$, which implies that $p' = 3$. Then $(q' + 1) \mid 4$ and hence $q' = 3$. So $(p', q') = (3, 3)$, which is impossible.

Case 6. Let $K/H \cong B_n(q')$, where $n = 2^m \geq 4$ and q' is odd. Therefore $q = (q'^n + 1)/2$. So $q < 2q'^n < q'^{n+1}$. Therefore $q^2 - 1 < q'^{2(n+1)}$. On the other hand, we have $q'^{n^2} \mid (q^2 - 1)$ and consequently $q'^{n^2} < q'^{2(n+1)}$. So $n^2 < 2(n+1)$, which implies that $n = 2$, and this is a contradiction.

Case 7. Let $K/H \cong C_n(q')$, where $n = 2^m \geq 2$. Then $q = (q'^n + 1)/(2, q' - 1)$. Therefore $q \leq q'^n + 1 < 2q'^n \leq q'^{n+1}$, which implies that $q^2 - 1 < q'^{2(n+1)}$. On the other hand, we have $q'^{n^2} \mid (q^2 - 1)$, which implies that $q'^{n^2} < q'^{2(n+1)}$. So we have $n^2 < 2(n+1)$ and hence $n = 2$. Therefore $q < 2q'^2$ and so $q'^4(q'^2 - 1) < q^2 - 1 < 4q'^4$, which is impossible.

Case 8. Let $K/H \cong {}^2D_{p'}(3)$, where $p' = 2^n + 1 \geq 5$. So we have $q = (3^{p'} + 1)/4$ or $q = (3^{p'-1} + 1)/2$.

If $q = (3^{p'} + 1)/4$, then $q < 3^{p'+1}$. On the other hand, we have $3^{p'(p'-1)} \mid (q^2 - 1)$, which implies that $3^{p'(p'-1)} \leq q^2 - 1 < 3^{2(p'+1)}$. Therefore $p'(p' - 1) < 2(p' + 1)$, and hence $p' \leq 3$, which is impossible.

If $q = (3^{p'-1} + 1)/2$, then $q < 3^{p'}$. On the other hand, $3^{p'(p'-1)} \mid (q^2 - 1)$, which implies that $3^{p'(p'-1)} < 3^{2p'}$, and so $p'(p' - 1) < 2p'$, which is impossible.

Case 9. Let $K/H \cong {}^2B_2(q')$, where $q' = 2^{2n+1} > 2$. In this case we have $q = q' \pm \sqrt{2q'} + 1$ or $q = q' - 1$.

If $q = q' \pm \sqrt{2q'} + 1$, then $q^2 - 1 = q'^2 + 4q' \pm 2\sqrt{2q'}(q' + 1)$. On the other hand, we have $q'^2 \mid (q^2 - 1)$ and so $q' \mid (q'^2 + 4q' \pm 2\sqrt{2q'}(q' + 1))$, which implies that $q' \leq 2\sqrt{2q'}$. Hence $q'^2 \leq 8q'$. Therefore $q' = 8$ and so $q = 5$ or 13, which is a contradiction by $q'^2 \mid (q^2 - 1)$.

If $q = q' - 1$, then $q'^2 | (q'^2 - 2q')$, which is a contradiction.

Case 10. Let $K/H \cong {}^2F_4(q')$, where $q' = 2^{2n+1} > 2$. In this case we have $q = q'^2 \pm \sqrt{2q'^3} + q' \pm \sqrt{2q'} + 1$. Therefore $q < 4q'^2 < q'^3$ and so $q^2 - 1 < q'^6$. On the other hand, $q'^{12} \mid (q^2 - 1)$, which is a contradiction.

Case 11. Let $K/H \cong A_1(q')$, where $4|q'$. By hypothesis we have $q = q' - 1$ or $q = q' + 1$.

If $q = q' - 1$, then $q^2 - 1 = q'^2 - 2q'$. But we know $q'(q' + 1) \mid (q^2 - 1)$, which is a contradiction.

If $q = q' + 1$, then $q^2 - 1 = q'^2 + 2q'$. Since $q'(q' - 1) \mid (q^2 - 1)$, we conclude that $(q' - 1) \mid 3$. So $q' = 4$ and hence $K/H \cong A_1(4) \cong A_5$. By the proof of Case 2 we have $K/H \cong PGL(2, 5)$.

Case 12. If $K/H \cong A_1(q')$, where $4|(q' - 1)$, then $q = (q' + 1)/2$ or $q = q'$.

If $q = (q' + 1)/2$, then $q^2 - 1 = (q'^2 - 3 + 2q')/4$. On the other hand, $q'(q' - 1) \mid (q^2 - 1)$

and hence $q'(q'-1) \leq (q'^2 - 3 + 2q')/4$. So $q'^2 - 2q' + 1 \leq 0$, which is a contradiction.

If $q = q'$, then $K/H \cong A_1(q) = PSL(2,q)$. Since $K/H \leq G/H$ and $|G| = 2|PSL(2,q)|$, we conclude that $|H| = 1$ or 2. Let $|H| = 2$. Since $H \trianglelefteq G$ we have $H \subseteq Z(G)$, which is a contradiction. So $H = 1$.

By Lemma 2.8, $G/K \leq Out(K/H)$ and $|G/K| = 2$. If G/K contains a field automorphism, then $2p \in \pi_e(G)$, which is a contradiction. If G/K contains a diagonal-field automorphism, then G is the non-split extension of $PSL(2,q)$ by $\mathbb{Z}_2$ and we know that the prime graph of G is the prime graph of $PSL(2,q)$ (see [12]), which is a contradiction. So a diagonal automorphism generates G/K and consequently $G \cong PGL(2,q)$.

If $K/H \cong A_1(q')$, where $4|(q'+1)$, then similarly we conclude that $G \cong PGL(2,q)$. □

Theorem 3.6. Let G be a group such that $\nabla(G) \cong \nabla(M)$, where $M = PGL(2,q)$ and q is a prime power. Then $G \cong M$.

Proof. If $q = 2^n$, where n is an integer, then $PGL(2,q) \cong PSL(2,q)$ and so Lemma 2.10 implies that $G \cong M$. If q is odd, then obviously the theorem follows from Theorems 3.2 and 3.5. □

Remark 3.7. It is a well known conjecture of J. G. Thompson that if G is a finite group with $Z(G) = 1$ and M is a non-abelian simple group satisfying $N(G) = N(M)$, then $G \cong M$.

We can give a positive answer to this conjecture for the group $PGL(2,q)$ by our characterization of this group.

Corollary 3.8. Let G be a finite group with $Z(G) = 1$ and $M = PGL(2,q)$, where q is a prime power. If $N(G) = N(M)$, then $G \cong M$.

Proof. By Lemmas 2.6 and 2.7, if G and M are two finite groups satisfying the conditions of Corollary 3.8, then $OC(G) = OC(M)$. So using Theorem 3.5 we get the result. □

Remark 3.9. W. Shi and J. Bi in [16] put forward the following conjecture:
Conjecture. Let G be a group and M be a finite simple group. Then $G \cong M$ if and only if
(i) $|G| = |M|$, and,
(ii) $\pi_e(G) = \pi_e(M)$, where $\pi_e(G)$ denotes the set of orders of elements in G.

This conjecture is valid for sporadic simple groups [13], alternating groups [17], and some simple groups of Lie type [14, 15, 16]. As a consequence of Theorem 3.5, we prove the validity of this conjecture for the almost simple group $PGL(2,q)$, where q is a prime power.

Corollary 3.10. Let G be a finite group and $M = PGL(2,q)$, where q is a prime power. If

$|G| = |M|$ and $\pi_e(G) = \pi_e(M)$, then $G \cong M$.

Proof. By assumption we have $OC(G) = OC(M)$. Thus the corollary follows from Theorem 3.5. □

Proposition 3.11. Let G be a group such that $\Gamma_1(G) \cong \Gamma_1(M)$, where $M = PGL(2, q)$ and q is a prime power. Then $G \cong M$.

proof. First we show that $|G| = |M|$. By Lemma 2.12 we have $|Cyc(G)|$ divides $|M| - |Cyc(M)|$. Since $Cyc(M) \leq Z(M) = 1$, it follows that $|Cyc(G)|$ divides $|M| - 1$. On the other hand, by Lemma 2.12, $|Cyc(G)|$ divides $|Cyc_M(x)| - |Cyc(M)|$, where $x \in M \setminus Cyc(M)$. Let x be a p-element of M. We claim that $\langle x \rangle = Cyc_M(x)$. We know that $\langle x \rangle \subseteq Cyc_M(x)$ and so it is enough to prove that $Cyc_M(x) \subseteq \langle x \rangle$. On the contrary let $y \in Cyc_M(x) \setminus \langle x \rangle$ and hence $\langle y, x \rangle$ is cyclic. If y is a p-element, then we know that $\langle y, x \rangle$ has only one subgroup of order p and so $\langle x \rangle = \langle y \rangle$, which is a contradiction. Therefore y is not a p-element. So we have an element of order $po(y)$, which is a contradiction by the structure of $\Gamma(M)$. So $p = |\langle x \rangle| = |Cyc_M(x)|$. Therefore $|Cyc(G)|$ divides $p - 1$ and $p - 1$ divides $|M|$. We know that $|Cyc(G)|$ divides $|M| - 1$ and so $|Cyc(G)| = 1$ and $|G| = |M|$. Now using Lemma 2.13 we conclude that $\pi_e(G) = \pi_e(M)$ and by Corollary 3.10 the proof is complete. □

Remark 3.12. We note that in the main theorem of [5] it is proved that $PGL(2, q)$ is uniquely determined by $\pi_e(G)$.

Received: February 2009. Revised: August 2010.

References

[1] A. Abdollahi, S. Akbari and H. R. Maimani, *Non-commuting graph of a group*, J. Algebra, 298 (2) (2006), 468-492.

[2] A. Abdollahi and Mohammadi Hassanabadi, *Noncyclic graph of a group*, Comm. Algebra, 35 (2007), 1-25.

[3] G. Y. Chen, *On Thompson's conjecture*, J. Algebra, 185 (1) (1996), 184-193.

[4] G. Y. Chen, *Further reflections on Thompson's conjecture*, J. Algebra, 218 (1) (1999), 276-285.

[5] G. Y. Chen, V. D. Mazurov, W. J. Shi, A. V. Vasil'ev and A. Kh. Zhurtov, *Recognition of the finite almost simple groups* $PGL(2, q)$ *by their spectrum*, J. Group Theory, 10 (2007), 71-85.

[6] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *Atlas of Finite Groups, Oxford University Press*, Oxford (1985).

[7] P. Crescenzo, *A diophantine equation which arises in the theory of finite groups*, Advances in Math., 17 (1) (1975), 25-29.

[8] K. W. Gruenberg and K. W. Roggenkamp, *Decomposition of the augmentation ideal and of the relation modules of a finite group*, Proc. London Math. Soc. (3), 31 (2) (1975), 149-166.

[9] B. Huppert, *Endliche Gruppen I*, Springer Verlag, Berlin, 1967.

[10] A. Iranmanesh, S. H. Alavi and B. Khosravi, *A Characterization of* $\mathsf{PSL}(3, q)$ *where* q *is an odd prime power*, J. Pure Appl. Algebra, 170 (2-3) (2002), 243-254.

[11] A. Khosravi and B. Khosravi, *A new characterization of almost sporadic groups*, J. Algebra Appl., 1 (3) (2002), 267-279.

[12] M. S. Lucido and E. Jabara, *Finite groups with hall covering*, J. Aust. Math. Soc., 78 (1) (2005), 1-16.

[13] W. Shi, *A new characterization of the sporadic simple groups*, Group Theory, Proceeding of the 1987 Singapore Group Theory Conference, Walter de Gruyter, Berlin, New York, 1989, 531-540.

[14] W. Shi, *A new characterization of some simple groups of Lie type*, Contemp. Math., 82 (1989), 171-180.

[15] W. Shi, *Pure quantitative characterization of finite simple groups (I)*, Progr. Natur. Sci., 4 (3) (1994), 316-326.

[16] W. Shi and J. Bi, *A characteristic property for each finite projective special linear group*, Lecture Notes in Math., 1456 (1990), 171-180.

[17] W. Shi and J. Bi, *A new characterization of the alternating groups*, Southeast Asian Bull. Math., 16 (1) (1992), 81-90.

[18] L. Wang and W. J. Shi, *A new characterization of* $\mathsf{L}_2(q)$ *by its noncommuting graph*, Front. Math. China, 2 (1) (2007), 143-148.

[19] L. Wang and W. Shi, *A new characterization of* A_{10} *by its noncommuting graph*, Comm. Algebra, 36 (2) (2008), 523-528.

[20] L. C. Zhang, G. Y. Chen, S. M. Chen and X. F. Liu, *Notes on finite simple groups whose orders have three or four prime divisors*, J. Algebra Appl., 8 (3) (2009), 389–399.

[21] L. C. Zhang and W. J. Shi, *Noncommuting graph characterization of some simple groups with connected prime graphs*, Int. Electron. J. Algebra, 5 (2009), 169–181.

[22] L. C. Zhang, W. J. Shi and X. L. Liu, *A characterization of* $\mathsf{L}_4(4)$ *by its noncommuting graph*, Chinese Annals of Mathematics, 30A (4) (2009), 517–524. (in chinese)

CUBO
A Mathematical Journal

Submitting: By Submitting a paper to this journal, authors certify that the manuscript has not been previously published nor is it under consideration for publication by another journal or similar publication. Work submitted to **CUBO** will be refereed by specialist appointed by the Editorial Board of the journal. The acceptance (or not) of the paper will be communicated to the corresponding author by the Editor in Chief.

Manuscript: Manuscript should be written in English or French and submitted in duplicate to any one of the editors or to the Editorial Office. Authors should send two copies in their final form, printed on one side of the paper with generous margins. The first page should contain a short descriptive title, the name(s) of the author(s), the institutional affiliation and complete address (including e-mail) of each author. Paper should be accompanied by a short abstract, key words and phrases and Mathematical Subject Classification. References are indicated in the text by consecutive arabic numeral enclosed in square bracket. The full list should be collected and typed at the end of the paper in numerical order.

Press requirement: The abstract, no longer than 100 words, must be send in two languages, being english one of these. The **CUBO** Journal strongly encourages the use of LaTeX(or one of the TeXvariants: AMSTeX or AMSLaTeX) for manuscript preparation. References should be headed numerically, alphabetically organized and complete in all its details. Authors' initial should precede their name; journal title abbreviations should follow the style of Mathematical Reviews Index.

Reprint: Electronic reprints(PDF) will be supplied free of charge to the corresponding author.

Exchanges: The **CUBO** Journal accepts exchanges with other mathematical publications. More information can be obtained from the Editor.

For any technical question about **CUBO**, please send a e-mail to: José Labrin(jlabrin@ufro.cl).

CUBO
A MATHEMATICAL JOURNAL
Universidad de La Frontera
Universidade Federal de Pernambuco
Volume 13/Nº02 – JUNE 2011

CONTENTS